MATH 133 Calculus 1 with FUNdamentals

Section 3.3: Product and Quotient Rules

There are two useful rules for computing the derivative of a product and quotient of two functions. Interestingly, Leibniz (one of the scholars credited with inventing Calculus) messed up the product rule in an early draft of his manuscript on the subject.

Product Rule: If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \;=\; f'(x)\,g(x) + f(x)\,g'(x)\,.} \tag{1}$$

The shorthand notation for the product rule is $(fg)' = f'g + fg'$. Notice the symmetry in formula (1) and that it is **not** the case that the derivative of the product equals the product of the derivatives. For instance, suppose we applied the Product Rule to take the derivative of $x \cdot x$. If we just multiplied the product of the derivatives, we would get

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) \;=\; 1 \cdot 1 \;=\; 1\,???$$

But this is clearly incorrect since $x \cdot x = x^2$ and the derivative of x^2 is $2x$ by the Power Rule. A correct application of the Product Rule is as follows:

$$\frac{d}{dx}(x \cdot x) \;=\; \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) \;=\; 1 \cdot x + x \cdot 1 \;=\; 2x.$$

Exercise 1: Use the Product Rule to find $f'(x)$ where $f(x) = (3x^2 + 1)e^x$. Simplify your answer.

Quotient Rule: If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ as long as $g(x) \neq 0$. The derivative of the quotient is given by

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \;=\; \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,.} \tag{2}$$

Note: The Quotient Rule can be derived from the Product Rule. Start by letting $Q(x) = \dfrac{f(x)}{g(x)}$. Then cross multiply and differentiate both sides with respect to x using the Product Rule. Solving for $Q'(x)$ leads to formula (2).

Exercises:

2. Use the Quotient Rule to calculate the derivative of $\frac{1}{x^4}$ and check your answer against the result obtained from the Power Rule.

3. If $g(x) = \frac{3x+1}{2x-5}$, find and simplify $g'(x)$.

4. If $h(x) = \frac{e^x}{x^2+1}$, find and simplify $h'(x)$.

5. Suppose that $f(3) = 5, f'(3) = -7, g(3) = 2$ and $g'(3) = 1/2$. If $H(x) = \frac{f(x)}{xg(x)}$, find $H'(3)$.