MATH 133 Calculus 1 with FUNdamentals

Section 3.9: Derivatives of General Exponential and Logarithmic Functions

In this section we learn how to take the derivative of exponential functions, such as $f(x) = 2^x$, and logarithms, such as $g(x) = \ln x$. This is accomplished by using implicit differentiation and our knowledge of the derivative of the inverse function.

Example 1: Use implicit differentiation to show that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Answer: First we start by letting $y = \ln x$. If we raise both sides to the base e, we find

$$e^y = e^{\ln x} = x\,,$$

since e^x and $\ln x$ are inverses of each other. The advantage of this step is that we know the derivative of e^x. Differentiating the previous equation with respect to x, and thinking of $y = y(x)$ (using implicit differentiation), we find

$$e^y \cdot \frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

as desired.

This is an important formula (#11 on our list of differentiation rules).

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Exercise 1: If $f(x) = \ln(\cos x)$, find and simplify $f'(x)$.

Example 2: Use implicit differentiation to show that $\frac{d}{dx}(b^x) = \ln b \cdot b^x$. Here b is any positive constant (b for "base").

Answer: As with the previous example, we start by letting $y = b^x$. If we take the natural log of both sides we find

$$\ln y = \ln b^x = x \cdot \ln b,$$

using rules of logarithms. The advantage of this step is that we now know the derivative of $\ln x$. Differentiating the previous equation with respect to x, and thinking of $y = y(x)$ (using implicit differentiation), we find

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln b \quad \Longrightarrow \quad \frac{dy}{dx} = \ln b \cdot y = \ln b \cdot b^x$$

as desired.

This is formula #5 on our list of differentiation rules.

$$\boxed{\frac{d}{dx}\left(b^x\right) \;=\; \ln b \cdot b^x}$$

For example, if $g(x) = 2^x$, then $g'(x) = \ln 2 \cdot 2^x$. Note that the formula applies correctly to e^x (when choosing $b = e$), since $\ln e = 1$.

Exercise 2: If $f(x) = 3^{\sqrt{x}}$, find and simplify $f'(x)$.

The final differentiation rule (rule #12) can be derived in a similar fashion to the above examples.

$$\boxed{\frac{d}{dx}\left(\log_b x\right) \;=\; \frac{1}{\ln b \cdot x}}$$

Exercise 3: If $g(x) = \ln(\ln x)$, find and simplify $g'(x)$.

Exercise 4: If $F(x) = x^3 \log_3 x$, find and simplify $F'(x)$.

Exercise 5: Find the equation of the tangent line to $y = 2^{\sin x}$ at $x = \pi$.