MATH 133 Calculus 1 with FUNdamentals

Section 3.7: The Chain Rule

One of the most important differentiation rules is the **Chain Rule**, used to take the derivative of the composition of two functions. The chain rule allows us to differentiate complicated functions that can be broken down into the composition of simpler functions.

Theorem (Chain Rule): If $f(x)$ and $g(x)$ are differentiable functions, then so is the composition $f(g(x))$. Its derivative is

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) \;=\; f'(g(x)) \cdot g'(x)\,.} \tag{1}$$

In words, formula (1) states that the derivative of the composition of two functions is given by the derivative of the outside function **evaluated at the inside function** times the derivative of the inside function.

Let us check the Chain Rule on a simple example. Suppose that $h(x) = (x^4+1)^2$. The outside function is $f(x) = x^2$, since the outer operation is squaring, and the inside function is $g(x) = x^4+1$. Then we have $h(x) = f(g(x))$. Since $f'(x) = 2x$ and $g'(x) = 4x^3$, the Chain Rule gives

$$h'(x) \;=\; 2(x^4+1)\cdot 4x^3 \;=\; 8x^3(x^4+1) \;=\; 8x^7+8x^3.$$

To check this, we can first expand $h(x)$ by writing

$$h(x) \;=\; (x^4+1)^2 \;=\; (x^4+1)(x^4+1) \;=\; x^8+2x^4+1.$$

By the Power Rule, we have $h'(x) = 8x^7+8x^3$, which confirms the calculation above via the Chain Rule.

Exercise 1: Use the Chain Rule to find $h'(x)$ if $h(x) = \sin(x^5-6x)$. What is the outside function $f(x)$ and the inside function $g(x)$?

Exercise 2: If $F(x) = \sqrt{x^4+3}$, use the Chain Rule to find and simplify $F'(x)$.

Leibniz Notation

It is instructive to write the Chain Rule using Leibniz notation. Suppose that $y = f(x)$ and $x = g(t)$ (y is a function of x and x, in turn, is a function of t). Then $y = f(g(t))$ and by the Chain Rule, $dy/dt = f'(g(t)) \cdot g'(t)$ or more simply

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}\ .} \tag{2}$$

The nice aspect of formula (2) is that we can visualize the dx's canceling out to help remember the Chain Rule. In this form of the Chain Rule, the derivative of the outside is given by dy/dx and the derivative of the inside is given by dx/dt.

Exercise 3: If $y = e^{x^2}$, calculate $\dfrac{dy}{dx}$. If $z = e^{\tan t}$, calculate $\dfrac{dz}{dt}$.
Hint: In both cases, the outside function is the same.

Exercise 4: Find a general formula for $\dfrac{dy}{dx}$ when $y = e^{u(x)}$.

Sometimes we need to apply the Chain Rule multiple times.

Exercise 5: Find and simplify $G'(x)$ if $G(x) = \sin(\sqrt{x^2 + 2}) + e^{\cos(4x)}$.

Exercise 6: Calculate $\dfrac{d}{dt}\big(\tan(\cos(e^{6t}))\big)$.

Exercise 7: Find and simplify $\dfrac{dy}{dt}$ if $y = \sqrt{\dfrac{t+1}{t-1}}$. What is the equation of the tangent line to the graph of y at the point $t = 2$?

Exercise 8: Find and simplify $F'(x)$ if $F(x) = \sin(x^2)\cos(x^2)$. Then find the equation of the tangent line at the point $x = \sqrt{\pi}$.

Exercise 9: Suppose that $h(x) = f(g(x))$ and that $f'(3) = 4, f(3) = 2, f'(6) = -1, g(3) = 6$ and $g'(3) = 7$. What is $h'(3)$?