MATH 133 Calculus 1 with FUNdamentals

Section 2.6: Trigonometric Limits

This section focuses on two key limits involving $\sin x$ and $\cos x$ that are important for finding the slope of the tangent line to each function. These limits are proven through an important and intuitive theorem called the **Squeeze Theorem**.

The Squeeze Theorem: Suppose that $l(x) \leq f(x) \leq u(x)$ when x is near c (except possibly at $x = c$) and that

$$\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

The Squeeze Theorem states that if one function is "squeezed" between two others having a common limit, then the inner function takes on the same limit. It is best understood visually (see Figure 1).

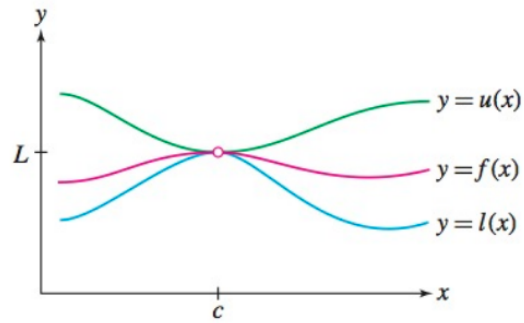

Figure 1: The function $f(x)$ is "squeezed" between $l(x)$ and $u(x)$ as $x \to c$, and thus has the same limit L as the two bounding functions.

Exercise 1: Suppose that $f(x)$ satisfies $\cos x \leq f(x) \leq x^2 + 1$ for x-values near $x = 0$. Use the Squeeze Theorem to find $\lim_{x\to 0} f(x)$.

Exercise 2: Use the Squeeze Theorem to verify that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.
Hint: What two numbers bound the output of the sine function?

Two important trigonometric limits are

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.} \tag{1}$$

These can be checked by using a calculator (set it to radians!) and plugging in values very close to 0. Note that each limit takes the form of $\frac{0}{0}$, an indeterminate form.

To prove the first limit in Equation (1), we use the fact that (see Figure 2)

$$\cos x \le \frac{\sin x}{x} \le 1 \quad \text{for } -\pi/2 < x < \pi/2.$$

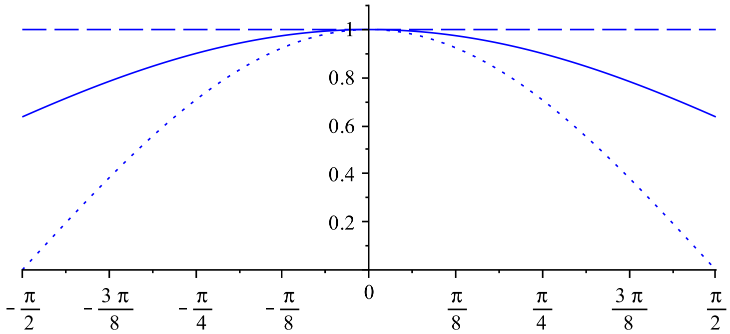

Figure 2: The graphs of the functions $y = 1$ (dashed), $y = \frac{\sin x}{x}$ (solid), and $y = \cos x$ (dotted).

Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, by the Squeeze Theorem, we have that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, as desired.

Exercise 3: Evaluate $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$. **Hint:** The limit of the product equals the product of the limits. Recall that $\sin^2 x = (\sin x)^2$.

Exercise 4: Use a calculator to evaluate $\lim_{t\to 0} \frac{\sin(7t)}{t}$. Then verify your answer by making the substitution $x = 7t$.

Hint: If t is tending toward 0, and $x = 7t$, then what is x approaching? Try and rewrite the limit using only the variable x so that the fraction $\frac{\sin x}{x}$ is present.

Exercise 5: Evaluate each limit.

(a) $\lim_{\theta\to 0} \frac{\sin(9\theta)}{5\theta}$

(b) $\lim_{x\to 0} \frac{-4x}{\tan(7x)}$