MATH 133 Calculus 1 with FUNdamentals

Section 2.3: Basic Limit Laws

This section focuses on some intuitive properties of limits that are very useful in practice.

Basic Limit Laws

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (*limit of the sum = sum of the limits*)
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ (*limit of the difference = difference of the limits*)
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ for any constant c (*constants pull out*)
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ (*limit of the product = product of the limits*)
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ (*limit of the quotient = quotient of the limits*)
6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is any positive integer (*this follows from 4.*)
7. $\lim_{x\to a} c = c$ for any constant c (*the limit of a constant is itself*)
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. (If n is even, we assume that $a > 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer. (If n is even, we assume that $\lim_{x\to a} f(x) > 0$.)
12. $\lim_{x\to a}[f(x)]^{p/r} = [\lim_{x\to a} f(x)]^{p/r}$, where p and r are integers with $r \neq 0$.

Example 1: Compute $\lim_{x\to 2} \dfrac{3x-7}{5x^4}$ using the limit laws above.

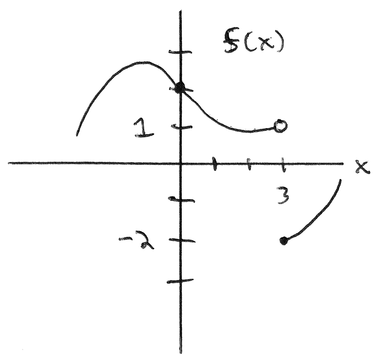
Solution:

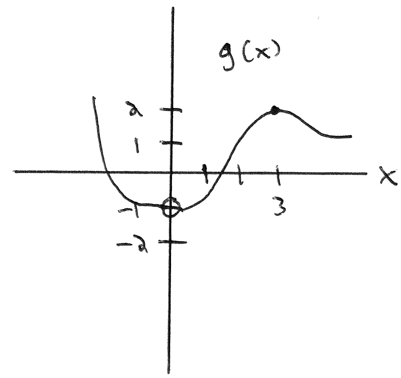
$$\begin{aligned}
\lim_{x\to 2} \frac{3x-7}{5x^4} &= \frac{\lim_{x\to 2} 3x-7}{\lim_{x\to 2} 5x^4} && \text{(Rule 5)} \\
&= \frac{\lim_{x\to 2} 3x - \lim_{x\to 2} 7}{5\lim_{x\to 2} x^4} && \text{(Rules 2 and 3)} \\
&= \frac{3\lim_{x\to 2} x - 7}{5\cdot 16} && \text{(Rules 3, 7, and 9)} \\
&= \frac{3\cdot 2-7}{80} = -\frac{1}{80} && \text{(Rule 8).}
\end{aligned}$$

Exercise 1: Using the basic limit laws, compute each of the following limits:

(a) $\lim\limits_{x\to 3} \dfrac{6-4x^2}{4x-7}$ **(b)** $\lim\limits_{x\to 0} \sqrt[3]{3x^7+8}+4\sqrt{x}$

Exercise 2: Use the graphs of $f(x)$ and $g(x)$ below to evaluate each of the following limits, if they exist.





(a) $\lim\limits_{x\to 0} (f(x)+g(x))$ **(b)** $\lim\limits_{x\to 0} 3f(x)\cdot g(x)$

(c) $\lim\limits_{x\to 3} \dfrac{f(x)}{g(x)}$ **(d)** $\lim\limits_{x\to 3^-} \dfrac{f(x)}{g(x)}$ **(e)** $\lim\limits_{x\to 3^+} \sqrt{2f(x)+5g(x)}$

Other Properties of Limits

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then $\lim\limits_{x\to a} f(x)=f(a)$. *Just plug it in!*

Exercise 3: Evaluate the limit below by direct substitution.

$$\lim_{x\to -2} \frac{3x^2-4x+1}{x^3-1}$$

Limit Existence Theorem $\lim\limits_{x\to a} f(x)=L$ if and only if $\lim\limits_{x\to a^-} f(x)=L=\lim\limits_{x\to a^+} f(x)$.

The left- and right-hand limits must both exist and be equal for the general limit to exist.