MATH 133 Calculus 1 with FUNdamentals

Section 2.8: Intermediate Value Theorem

This section deals with a key theorem from Calculus, namely the Intermediate Value Theorem. The basic idea is that a continuous function can not skip over any values; it passes through all the intermediate values between any two points in the range.

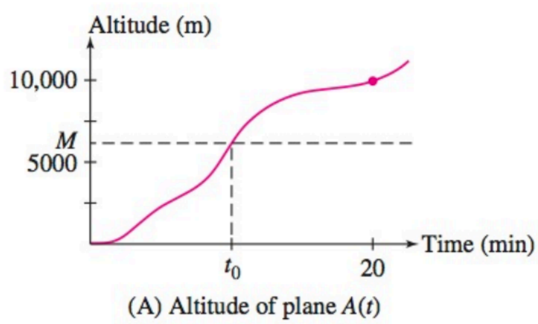

Figure 1: The function in (A) is continuous and takes on every value between 0 and 10,000, while the function in (B) is not continuous and misses many, many numbers between -1 and 3.

Consider the two graphs in Figure 1. The left function $A(t)$ measures the altitude of a plane as a function of time t. This is a continuous function and every height between 0 and 10,000 is attained at some time t_0. In other words, for any height M between 0 and 10,000, there exists a time t_0 such that $A(t_0) = M$. For example, $A(t_0) = 7{,}256$ means the plane was flying at an altitude of 7,256 meters at time t_0. We don't know the specific value of t_0, but we know it exists and that it must lie between 0 and 20 minutes.

In contrast, the function f shown on the right in Figure 1 is discontinuous. It is an example of a step function. It misses lots and lots of values on its way from -1 to 3. In fact, the range of this function is simply $\{-1, 0, 1, 2, 3\}$ (5 values). Any number not equal to one of these values will be completely missed by the function. (This would correspond to an absolutely terrifying and physically impossible plane ride!) The Intermediate Value Theorem states that continuous functions never miss any values between two points in the range.

The Intermediate Value Theorem (IVT): Suppose that $f(x)$ is a continuous function on a closed interval $[a, b]$. If M is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = M$ (see Figure 2).

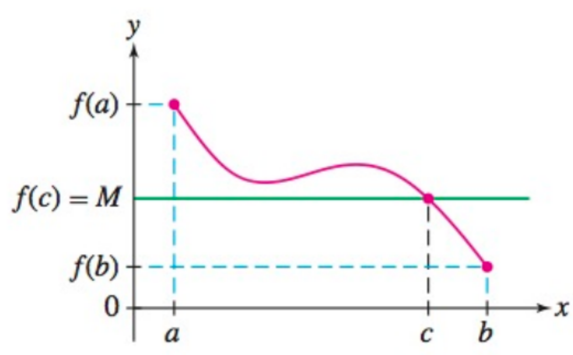

Figure 2: If f is a continuous function, then every height M between $f(a)$ and $f(b)$ has a value c in (a, b) that maps to it, that is, $f(c) = M$. No number between $f(a)$ and $f(b)$ is skipped by the function.

Exercise 1: Suppose that f is a continuous function on $[-2, 5]$ and that $f(-2) = 3$ and $f(5) = -0.5$. Which of the following values M must be attained by the function over the interval $[-2, 5]$? In other words, which M have a corresponding c in $(-2, 5)$ such that $f(c) = M$.

(a) -2 **(b)** 0 **(c)** 1.7 **(d)** $\sqrt{2}$ **(e)** π **(f)** e

Exercise 2: Draw the graph of a **discontinuous** function $f(x)$ over an interval $[a, b]$ that misses **all** of the values between $f(a)$ and $f(b)$.

Exercise 3: Use the Intermediate Value Theorem to show that $e^x - 3x = 0$ has a solution between 0 and 1. Find another interval $[a, b]$ containing a solution to the equation.

Hint: Let $f(x) = e^x - 3x$ and apply IVT on the interval $[0, 1]$.

Exercise 4: Use the Intermediate Value Theorem to show that $\sqrt{x} + \sqrt{x+5} = 4$ has a solution. Then use algebra to find the exact solution.