MATH 133 Calculus 1 with FUNdamentals

Section 1.6: Exponential and Logarithmic Functions

Exponential Functions

Exponential functions are very important in fields such as economics, biology, physics, ecology, mathematical modeling, and finance, to name a few. Any quantity that grows or decays based on how much of that quantity is present is described by an exponential function.

The general form of an exponential function is $f(x) = c \cdot b^x$, where $b > 0$ is some positive constant called the **base**. The constant $c \neq 0$ can be positive or negative. Some examples of exponential functions are:

$$2^x, \quad (1/2)^x, \quad 3e^x, \quad 800(1/2)^x = 800 \cdot 2^{-x}.$$

Key Point: The variable in an exponential function is an **exponent**. There is a huge difference between x^2 (squaring function) and 2^x (doubling function). For instance, if $x = 10$, then $10^2 = 100$ is much smaller than $2^{10} = 1024$. Exponential functions grow very, very fast.

The domain of $y = b^x$ is all real numbers, but the range is $(0, \infty)$. If $b > 1$, we have exponential **growth**, while if $b < 1$, we have exponential **decay.** (See the graphs below.)

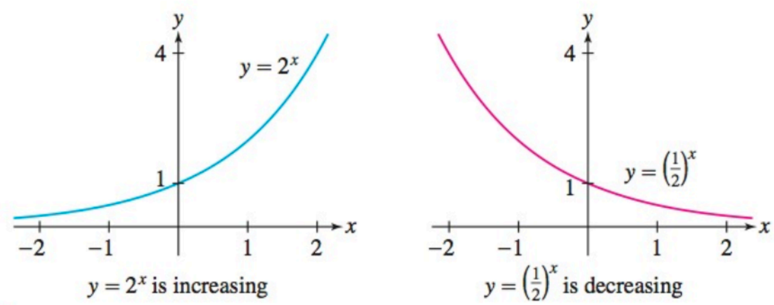

Figure 1: The graph of an exponential function $y = b^x$ with $b > 1$ (left figure) is increasing and concave up. It grows very, very fast. If $0 < b < 1$, the graph is decreasing and concave up (right figure). It decays to zero very, very fast (horizontal asymptote).

Laws of Exponents

1. $b^0 = 1$
2. $b^x \cdot b^y = b^{x+y}$ (e.g., $2^2 \cdot 2^3 = 2^5$)
3. $\dfrac{b^x}{b^y} = b^{x-y}$
4. $b^{-x} = \dfrac{1}{b^x}$ (e.g., $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$)
5. $(b^x)^y = b^{xy}$
6. $b^{p/r} = \sqrt[r]{b^p} = (\sqrt[r]{b})^p$ (power/root)

Exercise 1: Simplify each of the following expressions:

a. $27^{2/3}$ **b.** $\dfrac{2^5}{1024}$ **c.** 100^0 **d.** $100^{-1/2}$

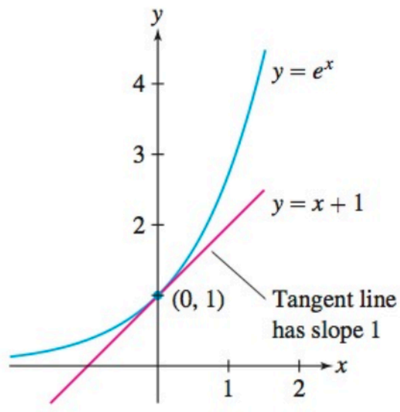

Figure 2: The graph of the important function $y = e^x$. The **special number** $e \approx 2.71828\ldots$ is defined to be the unique base which makes the slope of the tangent line at $(0, 1)$ equal to 1. Different bases give different slopes at $(0, 1)$. Only the number e, which is irrational, has a slope of 1.

Logarithms

Definition: A **logarithm** is the inverse of an exponential function. The function $f(x) = \log_b x$ is the inverse of the function $g(x) = b^x$ (b is still called the base and is assumed to be positive). The output of a logarithm is an *exponent*.

Example 1: The function $y = \log_2 x$ is the inverse of $y = 2^x$. We have

$$\log_2 8 \;=\; 3 \quad \text{since} \quad 2^3 \;=\; 8, \qquad \text{and} \qquad \log_2 \frac{1}{2} \;=\; -1 \quad \text{since} \quad 2^{-1} \;=\; \frac{1}{2}.$$

The point $(3, 8)$ is on the graph of $y = 2^x$ while the point $(8, 3)$ is on the graph of its *inverse* $y = \log_2 x$.

Key Point: When computing a logarithm, ask what *power* do I raise the base in order to obtain the given number.

Exercise 2: Evaluate each of the following logarithms:

a. $\log_2 32$ **b.** $\log_2 \frac{1}{16}$ **c.** $\log_{10} 1000$ **d.** $\log_4 0$

The domain and range of b^x and $\log_b x$ are opposites of each other:

- $y = b^x$: Domain: $(-\infty, \infty)$ Range: $(0, \infty)$
- $y = \log_b x$: Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

Definition: The **natural logarithm**, denoted $\ln x$, is the inverse of the function $y = e^x$. In other words, $y = \ln x$ is equivalent to $y = \log_e x$, the logarithm to the base $b = e$. By definition, we have

$$\boxed{\ln(e^x) \;=\; x \qquad \text{and} \qquad e^{\ln x} \;=\; x\,.} \tag{1}$$

Exercise 3: Evaluate each of the following logarithms:

a. $\ln(e^4)$ **b.** $\ln\left(\dfrac{1}{e}\right)$ **c.** $\ln(\sqrt{e}\,)$

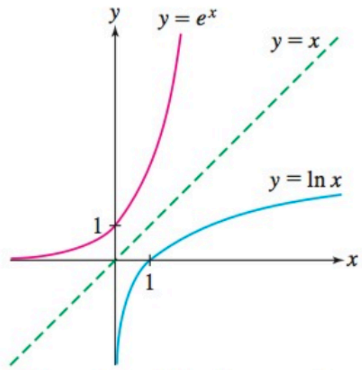

FIGURE 5 $y = \ln x$ is the inverse of $y = e^x$.

Figure 3: The graph of $y = \ln x$ is obtained by reflecting the graph of $y = e^x$ about the diagonal line $y = x$. Since e^x goes to ∞ very, very fast, $\ln x$ goes to ∞ very, very *slowly* because it is the inverse. **Note:** $y = \ln x$ has a vertical asymptote at $x = 0$ because $y = e^x$ has a horizontal asymptote at $y = 0$.

Laws of Logarithms

1. $\log_b(1) = 0$ (because $b^0 = 1$)
2. $\log_b(x \cdot y) = \log_b x + \log_b y$ (log of the product equals the *sum* of the logs)
3. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$ (log of the quotient equals the *difference* of the logs)
4. $\log_b(x^y) = y \log_b x$ (exponents come out front for logs — a very useful property!)

Exercise 4: The price of a bleacher seat at Fenway Park is modeled by $P(t) = 0.75(1.074)^t$, where t is measured in years and $t = 0$ corresponds to the year 1960.

a. What was the price of a bleacher seat in 1960?

b. What is the price of a bleacher seat today?

c. According to the model, when will the price reach \$100?