MATH 133 Calculus 1 with FUNdamentals

Section 1.5: Inverse Functions

The Inverse

One of the simplest ways to obtain a new function from an old one is to simply flip the domain and range. So if $f(2) = 7$, then the new function f^{-1}, called the **inverse of** f, has $f^{-1}(7) = 2$. The function f^{-1} simply maps each element in the range of f back to the element in the domain it came from; it "inverts" f.

However, there will be a problem if an element b in the range has two or more elements in the domain, say a_1 and a_2, that map to it (i.e., $f(a_1) = f(a_2) = b$). Then, the inverse of b would not be unique (it could be either a_1 or a_2), and f^{-1} would *not* be a function. To rectify this, we must assume that f is **one-to-one**, that is, each element in the range of f has one and only one pre-image that was sent to it in the domain.

For example, the function $f(x) = x^2$ is not one-to-one because both -2 and 2 are each sent to the same element in the range, $f(-2) = f(2) = 4$. This function fails the **horizontal line test** and is really two-to-one. However, if we restrict the domain to $x \geq 0$ (think of erasing the left half of the parabola), then f becomes one-to-one and now the inverse is actually a function. You know it already as $f^{-1}(x) = \sqrt{x}$. By definition (i.e., restricting the domain of x^2), $\sqrt{x}$ only spits out non-negative values.

Key Point: The only functions with well-defined inverses are those that are **one-to-one**. They must pass the **horizontal line test.**

Exercise 1: Which of the following functions are one-to-one on their full domains? Draw a few examples to illustrate the difference between a one-to-one function and a function that is not one-to-one.

a. $f(x) = \frac{2}{3}x - 7$ **b.** $F(x) = \frac{2}{3}$ **c.** $g(x) = (x-3)^2 + 5$ **d.** $G(x) = (x-3)^3 + 5$

e. $h(\theta) = 2\sin(3\theta)$ **f.** $H(\beta) = \tan(\beta)$ **g.** $i(t) = |t+3|$ **h.** $I(t) = \sqrt{t+3}$

Note that the notation for the inverse of f is *not* the usual exponent notation. In other words,

$$f^{-1} \neq \frac{1}{f}$$

The choice of -1 as the exponent is mathematical shorthand for **inverse**. Don't confuse this!

Based on the definition of the inverse of a function, the following formulas should make sense:

$$\boxed{f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.} \tag{1}$$

Simply put, the inverse of f reverses what f does to x. Likewise, f reverses the action of f^{-1}, that is, the inverse of f^{-1} is just f. Note that the **domain of f^{-1} equals the range of f,** and vice-versa.

Graphically, if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}. Thus, to obtain the graph of f^{-1} from the graph of f (or vice versa), just reflect the graph of f about the line $y = x$. This is also how one obtains an analytic formula for the inverse of a function: **interchange the variables x and y and solve for y.**

Exercise 2: Find the inverse of the function $f(x) = 1 + \sqrt{2 + 3x}$. What should the domain of f^{-1} be in order to have f as its inverse?

Inverse Trig Functions

The key to defining the inverse trig functions is to restrict the domains of the original trig functions in order to ensure that they are one-to-one. For example, the sine function is one-to-one on the domain $-\pi/2 \leq \theta \leq \pi/2$ (check the graph). By making this restriction, we then **define** the range of the inverse sine function (also called the **arcsine function**) to be $[-\pi/2, \pi/2]$. The domain of the inverse sine function is $[-1, 1]$ because this is precisely the range of the sine function.

Key Point: The inverse sine function, denoted $\sin^{-1}(x)$, inputs numbers between -1 and 1 and outputs angles between $-\pi/2$ and $\pi/2$. Thus, $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$. (Go backwards!)

The domains and ranges of the other inverse trig functions are given below: (the first two are the most important)

- $\cos^{-1}(x)$: Domain: $[-1, 1]$ Range: $[0, \pi]$
- $\tan^{-1}(x)$: Domain: $(-\infty, \infty)$ Range: $(-\pi/2, \pi/2)$
- $\cot^{-1}(x)$: Domain: $(-\infty, \infty)$ Range: $(0, \pi)$
- $\sec^{-1}(x)$: Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $[0, \pi/2) \cup (\pi/2, \pi]$
- $\csc^{-1}(x)$: Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $[-\pi/2, 0) \cup (0, \pi/2]$

Exercise 3: Evaluate each of the following without using a calculator:

a. $\sin^{-1}(1/\sqrt{2}) =$ ________ **b.** $\sin^{-1}(1) =$ ________ **c.** $\cos^{-1}(-\sqrt{3}/2) =$ ________

d. $\tan^{-1}(-1) =$ ________ **e.** $\sec^{-1}(2) =$ ________ **f.** $\csc^{-1}(-1) =$ ________

Exercise 4: Explain why $\cos^{-1}(\cos(17\pi)) = \pi$ and not 17π. What is $\sin^{-1}(\sin(11\pi/3)$?

Graph of $y = \tan^{-1} x$

One of the coolest graphs in calculus is that of the inverse tangent function (shown below). It is special because it is an example of a function with two different horizontal asymptotes. Because $y = \tan x$ has vertical asymptotes at $x = \pi/2$ and $x = -\pi/2$, $y = \tan^{-1} x$ has *horizontal asymptotes* at $y = -\pi/2$ and $y = \pi/2$. You should memorize this graph.

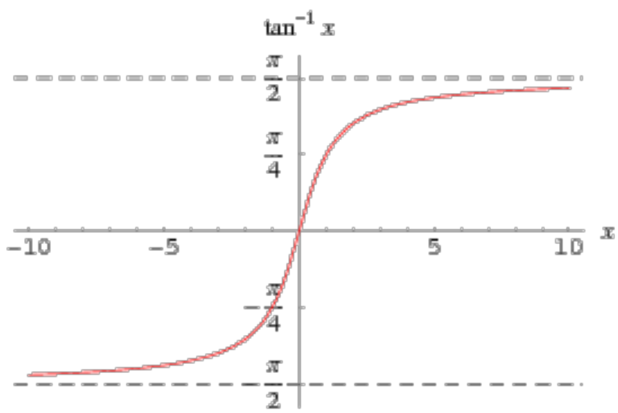

Figure 1: The graph of the inverse tangent function $y = \tan^{-1} x$ is cool. It maps the entire real line one-to-one and onto the open interval $(-\pi/2, \pi/2)$. Note the horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.