MATH 133 Calculus 1 with FUNdamentals

Section 1.4: Trigonometric Functions

Radians

Angles in trigonometry are measured in radians, which corresponds to an actual physical length (as opposed to degrees, which is based on the fact that 360 has many factors).

Definition 1: An angle of 1 **radian** is equal to the angle made by 1 unit of arc length along the unit circle. To convert between radians and degrees, use the formula

$$\boxed{180^\circ \;=\; \pi \text{ radians.}} \tag{1}$$

Exercise 1: Convert each of the following from degrees to radians or vice-versa.

a. $270^\circ \;=$ ______ **b.** $-135^\circ \;=$ ______ **c.** $5\pi/6 \;=$ ______ **d.** $-4\pi/3 \;=$ ______

The Sine and Cosine Functions

There are multiple definitions of the sine function $y = \sin\theta$, but one of the simplest to understand is that it represents the y-coordinate on the unit circle at an angle of θ radians.

Definition 2: The **sine of** θ, denoted by $\sin(\theta)$ or just $\sin\theta$, is the y-coordinate of the point of intersection between the unit circle and a ray emanating from the origin at an angle of θ radians. The **cosine of** θ, denoted by $\cos(\theta)$ or just $\cos\theta$, is the x-coordinate.

It is important to remember that the input into the functions $f(\theta) = \sin\theta$ or $g(\theta) = \cos\theta$ is an *angle* θ. Since the unit circle has a radius of one, the x- and y-coordinates of any point on the unit circle always lie between -1 and 1. Thus, the range of $\sin\theta$ and $\cos\theta$ is $[-1, 1]$. By the Pythagorean Theorem, we have a fundamental relationship between $\sin\theta$ and $\cos\theta$:

$$\boxed{\cos^2(\theta) + \sin^2(\theta) \;=\; 1 \quad \text{for any angle } \theta.} \tag{2}$$

An alternative but equivalent definition for the sine or cosine of an angle comes from using right triangles. This only works for finding the trig function of an angle between 0 and $\pi/2$. Recall the mnemonic phrase SOH-CAH-TOA, which reminds us of the following definitions:

$$\sin\theta \;=\; \frac{\text{opp.}}{\text{hyp.}}, \qquad \cos\theta \;=\; \frac{\text{adj.}}{\text{hyp.}}, \qquad \tan\theta \;=\; \frac{\text{opp.}}{\text{adj.}}.$$

Exercise 2: Without using a calculator, evaluate each of the following:

a. $\sin(3\pi/2) =$ _____ **b.** $\cos(15\pi) =$ _____ **c.** $\sin(7\pi/6) =$ _____ **d.** $\tan(3\pi/4) =$ _____

Other Trig Functions

There are four other trigonometric functions, each of which can be defined in terms of $\sin\theta$ and $\cos\theta$.

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}.} \tag{3}$$

Exercise 3: Use the fundamental identity given in equation (2) to derive the identity

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 4: Suppose that $\sin\theta = \dfrac{3}{5}$ and that $\pi/2 < \theta < \pi$. Find the values of $\cos\theta, \tan\theta$, and $\csc\theta$.

Unit Circle

Since the cosine and sine functions are defined in terms of the coordinates on the unit circle, it is absolutely critical to have a solid understanding of the unit circle and its key values.

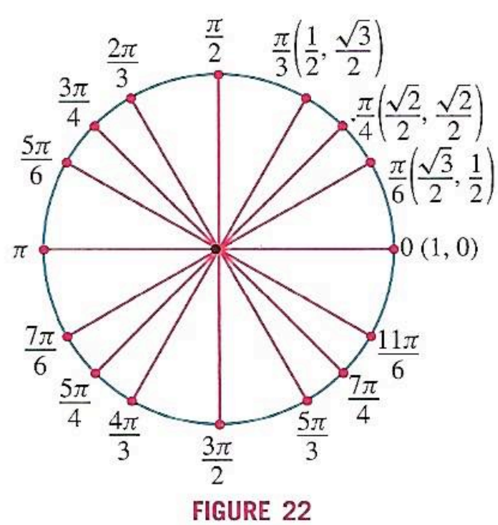


Figure 1: The unit circle. Fill in the x- and y-coordinates for all the angles shown. You should **memorize this figure!**

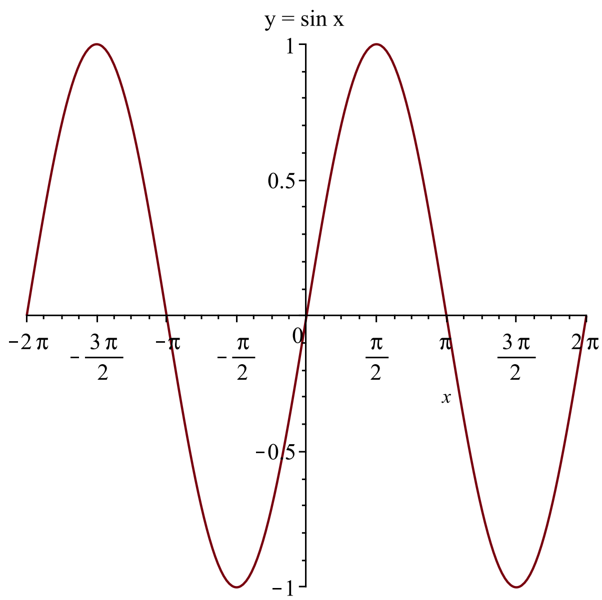
Exercise 5: Use the unit circle to find all solutions to the equations below, assuming that $0 \leq \theta \leq 2\pi$.

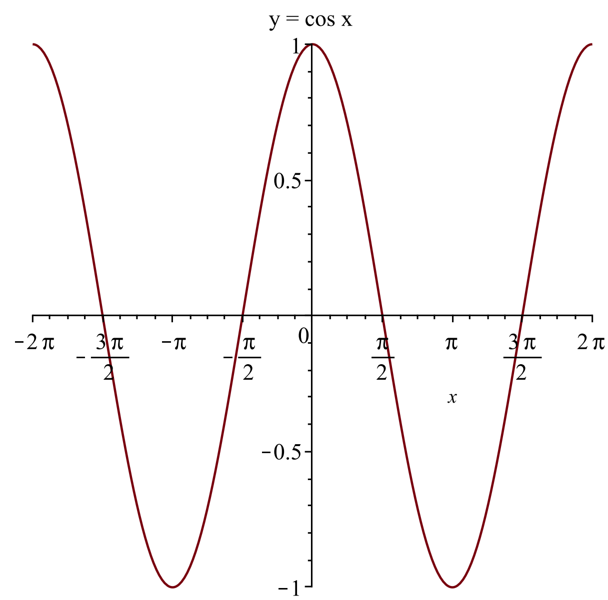
a. $\sin\theta = -\dfrac{1}{2}$

b. $\tan\theta = \sqrt{3}$

Graphing Trig Functions

You should know the graphs of $\sin x, \cos x$, and $\tan x$. These are excellent examples of **periodic** functions, that is, functions that repeat themselves after some time, called the **period**. The period of $\sin x$ and $\cos x$ are each 2π. For the function $y = a\sin(bx)$, the amplitude is $|a|$ and the period is $2\pi/b$ (assuming $b > 0$). The period of $\tan x$ is π; it has vertical asymptotes at $x = \pm\pi/2, \pm 3\pi/2, \cdots$, where the function is undefined. The graph of $y = \tan x$ is on the next page.

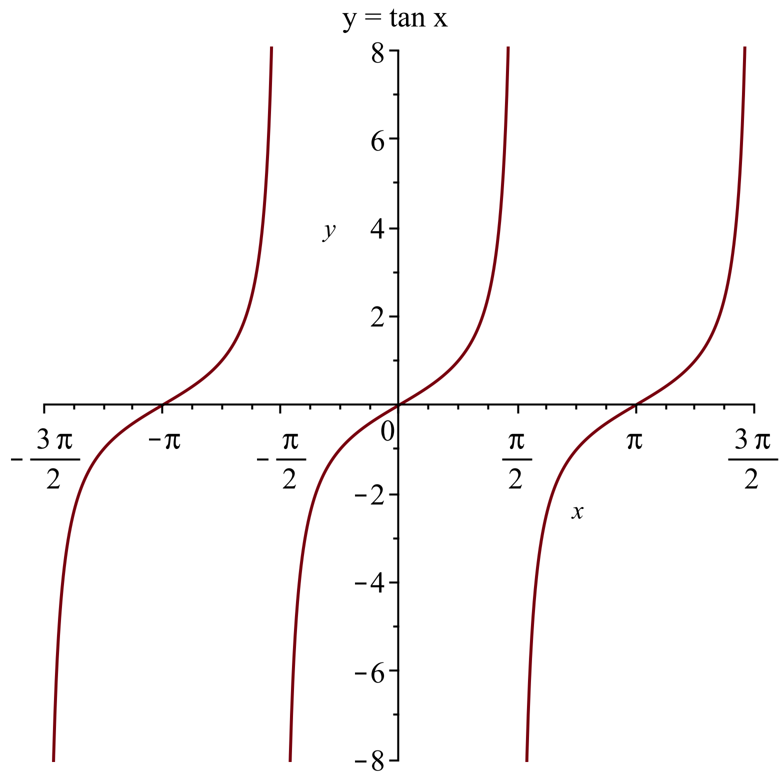




Exercise 6: Sketch a graph of the following trig functions. State the amplitude and period in each case.

a. $y = 2\sin(3\theta)$

b. $y = -3\cos\left(\dfrac{\pi}{2}\theta\right)$


y = tan x
8
6
y
4
2
0
-2
-4
-6
-8
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
x