MATH 133 Calculus 1 with FUNdamentals

Section 1.3: Important Types of Functions

Below is a brief catalog of the standard functions that we will be studying this semester. It is important to understand the properties of each function: defining equation, typical graph, domain and range, when it is used, etc.

Linear: $L(x) = mx + b$ Examples: $L(x) = 3x - 1,\ L(x) = -2x + \sqrt{3},\ L(x) = -7$.

Linear functions have a **constant** rate of change (determined by the slope m). The graph of a linear function is a line. It moves upwards from left to right if $m > 0$, downwards from left to right if $m < 0$, and is horizontal when $m = 0$. Remember that a vertical line $x = c$ is **not** a function! Linear functions are often used as a first approximation to a graph. This is called the **tangent line**, the primary focus of Calc 1.

Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0$ Example: $p(x) = 3x^5 - 2x^2 + 14$.

The constant a_n is called the **leading coefficient** and n is the **degree** of the polynomial. In general, an nth degree polynomial has n roots (zeros), although some of these may be complex (non-real). If $n = 2$, the polynomial is a **quadratic** function; if $n = 3$, it is called a **cubic**; if $n = 4$, it is called a **quartic**, etc. The domain of a polynomial function is $\mathbb{R}$, the set of all real numbers. Typically, the graph of an nth degree polynomial has $n - 1$ humps (facing up or down). Polynomials are often used to approximate more complicated functions. They are particularly nice because the derivative and integral are easy to calculate using the power rule (to be discussed later).

Rational: $R(x) = \dfrac{p(x)}{q(x)}$ Example: $R(x) = \dfrac{2x^3 - 5x^2 + 12}{x^2 - 2x - 3}$

A rational function is the **ratio** of two polynomials. The domain of a rational function is all real numbers except for the roots of $q(x)$, since a root of the denominator would make the function undefined (can't divide by 0). Typically, $R(x)$ has a **vertical asymptote** at the x-values which are roots of $q(x)$. A vertical asymptote is a dashed vertical line which the graph of the function approaches, either upwards (toward $+\infty$) or downwards (toward $-\infty$).

Exercise 1: Find the domain of the function

$$R(x) = \frac{3x^4 - 7x^3 + \pi}{x^2 - 16}.$$

Exponential: $f(x) = b^x$, where b is some positive constant. Examples: $2^x, (1/2)^x, e^x, 1.003^x$.

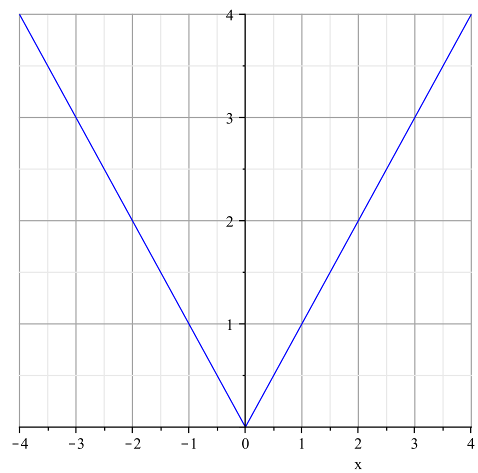
Exponential functions are very important in fields such as economics, biology, physics, mathematical modeling, and finance, to name a few. Any quantity that grows or decays based on how much of that quantity is present is described by an exponential function.

Note: The variable in an exponential function is an **exponent**. There is a huge difference between x^2 (squaring function) and 2^x (doubling function). Exponential functions grow very, very fast. Their domains are all real numbers. The base of an exponential function $f(x) = b^x$ is the constant b, which is always assumed to be positive. If $b > 1$, we have exponential **growth**, while if $b < 1$, we have exponential **decay.** We will discuss these important functions further in Section 1.6.

Piecewise: A function that has multiple parts defined on different domains.

Sometimes a function is split into separate pieces, with a different definition used on each domain. The most familiar example is the V graph of the absolute value function (see figure below):

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$



We graph the line $y = x$ (positive slope) over the domain $x \geq 0$ (the right-hand side of the graph). Then we graph the line $y = -x$ (negative slope) over the domain $x < 0$ (the left-hand side of the graph). Since both lines meet at the point $(0, 0)$, we obtain a V-shaped graph with the vertex at $(0, 0)$.

Exercise 2: Carefully draw the graph of

$$g(x) = \begin{cases} (x-3)^2 & \text{if } x \geq 3 \\ 2x - 3 & \text{if } -1 < x < 3 \\ -5 & \text{if } x \leq -1\,. \end{cases}$$

Composing Functions: Example: $f(x) = 2^x, g(x) = -3x + 1$ yields $f(g(x)) = 2^{-3x+1}$.

One way to create a new function is to take two other functions and **compose** them together. The notation for composition of functions is $f \circ g$ which means the function $f(g(x))$, pronounced "f of g of x." In this case, x is first plugged into the function g, and then the output $g(x)$ is plugged into f.

For example, suppose that we define the function $h(x) = f(g(x))$. If $g(2) = 7$, and $f(7) = -3$, then $h(2) = -3$ because

$$h(2) \;=\; f(g(2)) \;=\; f(7) \;=\; -3.$$

If we flip the order of f and g, we usually obtain a new function, that is, $f(g(x)$ and $g(f(x))$ are **different** functions. The domain of the function $f \circ g$ is all x in the domain of g that map into the domain of f.

Exercise 3: Suppose that $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$. Find $f(g(x))$ and $g(f(x))$ and their respective domains.