MATH 133 Calculus 1 with FUNdamentals

Section 1.2: Linear and Quadratic Functions

Linear Functions

A **linear function** is one of the form $f(x) = mx + b$, where m and b are arbitrary constants. It is "linear" in x (no exponents, fractions, trig, etc.). The graph of a linear function is a line. The constant m is the **slope** of the line and this number has the same value **everywhere** on the line. If (x_1, y_1) and (x_2, y_2) are any two points on the line, then the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \qquad (\Delta = \text{"Delta" means change}).$$

This number will be the same no matter which two points on the line are chosen.

Two important equations for a line are:

1. Slope-intercept form: $y = mx + b$ (m is the slope and b is the y-intercept.)
2. Point-slope form: $y - y_0 = m(x - x_0)$ (m is the slope and (x_0, y_0) is any point on the line.)

If $m > 0$, then the line is increasing while if $m < 0$, the line is decreasing. When $m = 0$, the line has zero slope and is horizontal (a constant function). Two lines are **parallel** when they have the same slope, while two lines are **perpendicular** if the product of their slopes is -1 (negative reciprocals).

Exercise 1: Find the equation of the line with the given information.

(a) The line passing through the points $(-2, 3)$ and $(4, 1)$.

(b) The line parallel to $2y + 5x = 0$, passing through $(2, 5)$.

(c) The line perpendicular to $2y + 5x = 0$, passing through $(2, 5)$.

Quadratic Functions

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are arbitrary constants and $a \neq 0$. The graph of a quadratic function is a **parabola**, an important curve that arises in many fields (e.g., physics, acoustics, astronomy). The parabola opens up when $a > 0$ and down for $a < 0$. The x-coordinate of the vertex of the parabola is located at $x = -b/(2a)$.

A quadratic function may have either two, one, or zero real roots, found by solving the equation $ax^2 + bx + c = 0$. The number of roots is determined by the **discriminant** $D = b^2 - 4ac$. If $D > 0$, then there are two distinct real roots. If $D = 0$, then there is one root, called a **repeated root**. If $D < 0$, then there are no real roots. The roots can be found either by factoring $ax^2 + bx + c = 0$ (in special cases) or by using the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}.$$

Note the appearance of D under the square root. If $D < 0$, then the roots are not real. In this case, the parabola does *not* cross the x-axis. If $D > 0$, then there are two real roots and these are equidistant from the x-coordinate of the vertex of the parabola.

Exercise 2: Find the roots of the quadratic function $f(x) = -3x^2 + 9x + 12$ in two different ways: **(a)** by factoring, and **(b)** by using the quadratic formula. Use this information to sketch a graph of $f(x)$.

Completing the Square

One important algebraic technique for understanding quadratic functions is **completing the square**. This means to write the function as a multiple of a perfect square plus a constant. Here is an example:

$$\begin{aligned}
2x^2 + 6x + 7 &= 2(x^2 + 3x + \underline{\qquad}) + 7 && \text{(factor out the 2 from the first two terms)} \\
&= 2\left(x^2 + 3x + \frac{9}{4}\right) + 7 - \frac{9}{2} && \text{(add } 9/4 = (3/2)^2 \text{ inside the parentheses)} \\
&= 2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2} && \text{(write the parentheses as a perfect square)}
\end{aligned}$$

The key to completing the square is to add and subtract the correct constant to make the factorization into a perfect square. Above we added $9/4$ inside the parentheses but $9/2$ outside because it is multiplied by 2. After factoring out the leading coefficient (on the x^2 term), the missing constant is found by cutting the coefficient of the linear term in half, and then squaring.

Exercise 3: Complete the square for the function $f(x) = x^2 - 7x + 15$. In other words, write it in the form $f(x) = (x - h)^2 + k$. (You have to figure out what the constants h and k are.)

Exercise 4: By completing the square, find the range of the quadratic function $f(x) = 4x^2 - 24x + 31$. What are the coordinates of the vertex of the corresponding parabola?

Exercise 5: **(Challenge)** Find the quadratic function that is even and passes through the points $(-1, 1)$ and $(2, 13)$.