

Symmetry in Music

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Topics in Mathematics: Math and Music

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Symmetry Operations in Music

How to get more music out of a little motif:

Translations (shifting graph vertically) \iff **Transpositions** (shifting notes up or down)

Example: **Stadium sports chants (organ)**

Vertical Reflection (symmetry between right and left) \iff
Retrograde (music same forward and backward)

Example: **Lean on Me**

Horizontal Reflection (symmetry between top and bottom) \iff
Inversion (what goes up, must come down)

Example: **Bach, Bach and more Bach**

Symmetry in Music: Transposition

Allegro con brio

The image shows two staves of musical notation for the opening of Beethoven's Fifth Symphony. The first staff contains measures 1 through 10, starting with a fortissimo (*ff*) dynamic and a piano (*p*) dynamic. The second staff, starting at measure 11, continues the piece with a crescendo leading to a fortissimo (*f*) dynamic. The key signature is three flats (B-flat, E-flat, A-flat) and the time signature is 4/4.

Figure: The opening measures of Beethoven's famous fifth symphony.

Molto adagio

The image shows the opening of Samuel Barber's Adagio for Strings. It features a single melodic line on a treble clef staff in a key signature of three flats (B-flat, E-flat, A-flat) and a 4/4 time signature. The tempo is marked 'Molto adagio' and the dynamic is pianissimo (*pp*). A long, sweeping slur covers the entire phrase, which ends with a change in time signature to 5/2.

Figure: The hauntingly sublime opening melody of Samuel Barber's *Adagio for Strings*.

Symmetry in Music: Retrograde – Haydn

The image displays the musical score for the first 20 measures of the 'Menuetto al Rovescio' from Joseph Haydn's Piano Sonata in A Major. The score is written for piano in 3/4 time with a key signature of three sharps (F#, C#, G#). The title 'Menuetto al Rovescio' is written above the first staff. The music is presented in three systems, each with a treble and bass staff. Measure numbers 5, 10, 15, and 20 are indicated above the treble staves. The piece is a musical palindrome, meaning it is its own reverse.

Figure: Joseph Haydn, *Piano Sonata in A Major*, (Hob. XVI/26; Landon 41) “Minuet in Reverse.” Both the minuet and trio are exact musical palindromes.

Symmetry in Music: Retrograde and Transposition

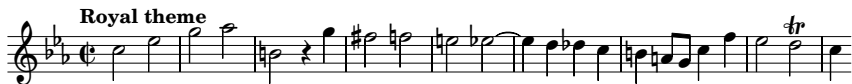
The image shows two staves of musical notation. The top staff is in treble clef with a key signature of one sharp (F#) and a common time signature (C). It contains a melody with the lyrics "for the Lord God Om-ni - po-tent reign - eth" written below it. The bottom staff is also in treble clef with a key signature of one sharp (F#) and a common time signature (C). It contains a melody that is a loose retrograde of the top staff, with the notes mirrored across the staff.

George F. Handel, *Messiah, Hallelujah chorus* (loose retrograde – form of **tone painting**)

The opening minute of the piece features just two motifs, the famous “Hallelujah” motif and the excerpt above.

Johann Sebastian Bach, *Musical Offering*

- Written in 1747, three years before Bach's death, for **Frederick the Great** (King of Prussia).
- Upon visiting the King's palace, Bach was challenged by the King to improvise three-part and six-part fugues based on the "Royal theme."



- Bach succeeded in improvising a three-part fugue. Although he could not do a six-part fugue based on the Royal theme, he stunned the court audience by improvising a six-part fugue based on a theme of his own choosing.

Bach's *Musical Offering*, cont.

- Bach returned home to compose the six-part fugue, a **ricercar**, as well as several other pieces, all based on the Royal theme, and sent it to the King as his *Musical Offering*.
- The work contains 13 pieces, organized symmetrically as follows:



- A **canon** is a sophisticated type of **round**, where a main theme is imitated in some form and played by a different part after the main theme has begun. The imitations can be direct repetition; repetition at a different interval (transposition); in inversion; or in retrograde. Sometimes the theme and its imitation begin together.

Bach's *Musical Offering* – A Musical Puzzle

- Bach used all of the different symmetry types in his canons. However, to make things interesting, Bach only wrote out the full parts for one of the 10 canons. The others were left as musical puzzles, where Bach left clues to indicate how the remaining parts were to be determined.
- **Quaerendo invenietis** (“Seek and ye shall find”) was inscribed on certain canons, particularly those without titles.
- The “puzzle” offered by Bach was solved and first published by Bach’s student **Johann Philipp Kirnberger**.

Bach's *Musical Offering* – Crab Canon

Canones diversi

super thema regium. **

Canon a 2.

The image shows a musical score for 'Canon a 2' from Bach's 'Musical Offering'. It consists of three staves of music. The first staff begins with a treble clef, a key signature of two flats (B-flat and E-flat), and a common time signature (C). The second and third staves continue the piece. At the very end of the third staff, there is a reflected clef (a bass clef) and a key signature change to one flat (B-flat), indicating a reflection of the piece.

Figure: The unsolved version of one of Bach's canons from the *Musical Offering*. Notice the reflected clef and key signature at the very end of the piece.

Solution for the Crab Canon

Canon a 2 (Crab Canon)

The image displays a musical score for 'Canon a 2 (Crab Canon)' in B-flat major, 3/4 time. It consists of four systems of two staves each. The first system shows the beginning of the piece. The second system is marked with a '5' at the start of the first staff. The third system is marked with a '10' at the start of the first staff. The fourth system is marked with a '15' at the start of the first staff. The score illustrates the original melody in the upper voice and its retrograde in the lower voice, which is the defining characteristic of a crab canon.

Figure: The *Crab Canon* from Bach's *Musical Offering*

Bach's Crab Canon – Analysis

- The primary theme (first part) consists of the Royal theme followed by an eighth-note countermelody. The entire part sounds perfectly fine in retrograde (played backwards). Thus, the second part plays the primary theme backwards **simultaneously** as the first part plays it forwards.
- Alternatively, a vertical reflection (retrograde) occurs at the end of measure nine. Each part moves in retrograde, but the parts are **interchanged**; the first part plays the second part backwards and vice-versa.
- Mathematically, this last interpretation can be visualized on a **Möbius strip!** Take the primary theme and cut it in half. Glue the two parts together, but make a twist before gluing. Each player now travels in opposite directions around the strip, with the vertical reflection taking place when the two parts pass each other after one “loop.” The twist represents the interchanging of the parts.

▶ Bach on a Möbius strip

Inversions

- An **inversion** occurs when the main theme is reflected horizontally about some note. If the melody goes up by a fourth, then the inversion goes down by a fourth, etc.
- Two types of inversions: **tonal** and **exact**. A tonal inversion is one where the inversion remains in the given key; an exact inversion requires all intervals to be reflected precisely.
- For example, in the key of C major, a melody that begins on a C and goes up a major third to E, would be reflected in a tonal inversion about C to the notes C and A (down a **minor** third), in order to avoid any accidentals. If the inversion were exact, then it would be C to A \flat (down a major third).



Figure: A simple melody along with its tonal and exact inversions. Here the horizontal reflection is about C, as can be viewed clearly in the lower-right excerpt.

Excerpt	Sequence of Intervals
Original Melody	↑ P4, ↓ m3, ↑ whole step, ↑ m3, ↓ P5
Exact Inversion	↓ P4, ↑ m3, ↓ whole step, ↓ m3, ↑ P5
Tonal Inversion	↓ P4, ↑ M3 , ↓ whole step, ↓ M3 , ↑ P5

Table: The interval sequences for the excerpts in the above figure.

Symmetry in Music: Inversion – Bartók

Allegro



Béla Bartók, *Mikrokosmos, No. 141, Subject and Reflection* – exact inversion about B \flat .

Allegro



Symmetry in Music: Inversion – Sousa



John Philip Sousa, opening of the march *The Thunderer*. (Analyze for HW.)

Symmetry in Music: Inversion – Bach

The image displays three staves of musical notation in treble clef, common time (C), and the key of D# minor (F# C# G# D# A# E#). The first staff, labeled "Subject", shows the original melodic line. The second staff, labeled "Inverted subject" and starting at measure 45, shows the subject inverted (mirrored across a horizontal axis), with some notes marked with an 'x' to indicate alterations. The third staff, also labeled "Inverted subject" and starting at measure 30, shows another inversion of the subject, demonstrating a different symmetrical transformation.

Figure: The subject and two different inversions of the subject in Bach's *Fugue No. 8 in D# minor* from the *Well-Tempered Clavier*, vol. I

Bach: The Well-Tempered Clavier, Fugue No. 8 in D \sharp minor

Handbook for Keyboard Teacher + Performer Chapter Five A Proposed Course of Study and Analysis
by Laurette Adberg Example 5.2 Fugue No. 8 in d \sharp minor from WTC I by J.S. Bach ABA' form (Sonata-form)

The image displays a musical score for Fugue No. 8 in D \sharp minor from the Well-Tempered Clavier, Part I by J.S. Bach. The score is presented in a three-staff format (treble, alto, and bass clefs). The key signature is D \sharp minor (three sharps: F \sharp , C \sharp , G \sharp). The time signature is 4/4. The score is divided into sections: Exposition (measures 1-11), Development (measures 15-24), and a final section (measures 25-35). Annotations include '3 Tonal answer' above measure 3, '8' above measure 8, '12a' above measure 12, 'S (redundant statement)' below measure 12, 'Development' above measure 15, '20 (rhythmic sync.)' above measure 20, '24' above measure 24, 'S' below measure 24, '25' above measure 25, and 'S' below measure 25. A vertical bar on the left side of the page contains a series of small, identical icons.

30 **Inverted subject**

75 76

40 **(Inverted and ornamented opening)**

45 **S-inverted** **Inverted rhythmic modification**

50 **S 1/2** **S 1/2** **S 1/2** **S 1/2**

55

60 ^{6a} **Recapitulation**

chrom. variation

S-augmented

64 S-inverted F# Major S-aug.

69 S

73 Coda (stretto) S-aug. S

79 rhythmic orn. (orn.) S (orn.)

83

Symmetry in Music: Retrograde-inversion

Opening of Praeludium

Closing of Postludium

The image displays two musical staves from Paul Hindemith's *Ludus Tonalis*. The top staff, titled "Opening of Praeludium", shows a treble clef with a common time signature. It begins with a fortissimo (*ff*) dynamic and a triplet of eighth notes. This is followed by a sixteenth-note scale in the right hand, while the left hand remains silent. The piece concludes with a triplet of eighth notes and a fermata. The bottom staff, titled "Closing of Postludium", shows a bass clef with a common time signature. It begins with a fermata, followed by a sixteenth-note scale in the left hand, while the right hand remains silent. The piece concludes with a triplet of eighth notes and a fermata. The two sections are exact retrograde-inversions of each other.

Figure: Paul Hindemith, *Ludus Tonalis* ("Game of Tones"), beginning and end. The ending Postludium is an **exact** retrograde-inversion (180° rotation) of the opening Praeludium.

Combining Symmetries – Liszt

R.h. 8va throughout

The image displays a musical score for the right-hand part of Franz Liszt's Hungarian Rhapsody #2, marked "R.h. 8va throughout". The score is written in G major and 2/4 time. It consists of four systems of music, each with a treble and bass clef staff. The first system shows a series of chords in the right hand and a rhythmic accompaniment in the left hand. The second system continues this pattern with more complex chordal textures. The third system features a more active right hand with sixteenth-note patterns. The fourth system concludes with a final chordal cadence. The notation includes various accidentals, including naturals and sharps, and rests.

Figure: Franz Liszt, excerpt from *Hungarian Rhapsody #2*

Combining Symmetries – Gershwin

9a.

1 2 3 4 2 2 3 4 3 2 3 4 4 2 3 4

Bb Bb6 Cm7 F7 Bb E^o7 Cm7

"I Got Rhythm" has an AABA structure, and a two-bar tag at the end. We call these four equal sections A, A, B, and A. Three of the sections are the same, and one is different. Listen to the first section, and you'll be able

9b.

A Bb Bb6 Cm7 F7 Bb6 E^o7 Cm7 F7 Bb Bb6 Cm7 F7 Ebm6

A Bb F7 Bb Bb Bb6 Cm7 F7 Bb6 E^o7 Cm7 F7 Bb Bb6

B Bridge D7 Am7 Fm6 D7 G D+ G9 G7

A C7 Gm7 Ebm6 C9 C7-5 F7 Bb Bb6 Cm7 F7 Bb6 E^o7

Tag Cm7 F7 Bb Bb6 Cm7 F7 Ebm6 Bb Fm G7 C7 F7 Bb

Figure: George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)

Der Spiegel (The Mirror) Duet

VOLTA 1 *Allegro* $J=120$

W.A. Mozart

mf

Allegro

Public Domain. Synchronized by Fred Nachbauer using NoteWorthy
Codemore? Try playing this from opposite sides of a table.

(Note: the attribution to Mozart is dubious)

Figure 9.6.