Symmetry in Music

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Symmetry Operations in Music

How to get more music out of a little motif:

Translations (shifting graph vertically) $\Longleftrightarrow$ Transpositions (shifting notes up or down)
Example: Stadium sports chants (organ)

Vertical Reflection (symmetry between right and left) $\Longleftrightarrow$
Retrograde (music same forward and backward)
Example: Lean on Me

Horizontal Reflection (symmetry between top and bottom) $\Longleftrightarrow$
Inversion (what goes up, must come down)
Example: Bach, Bach and more Bach

Symmetry in Music: Transposition

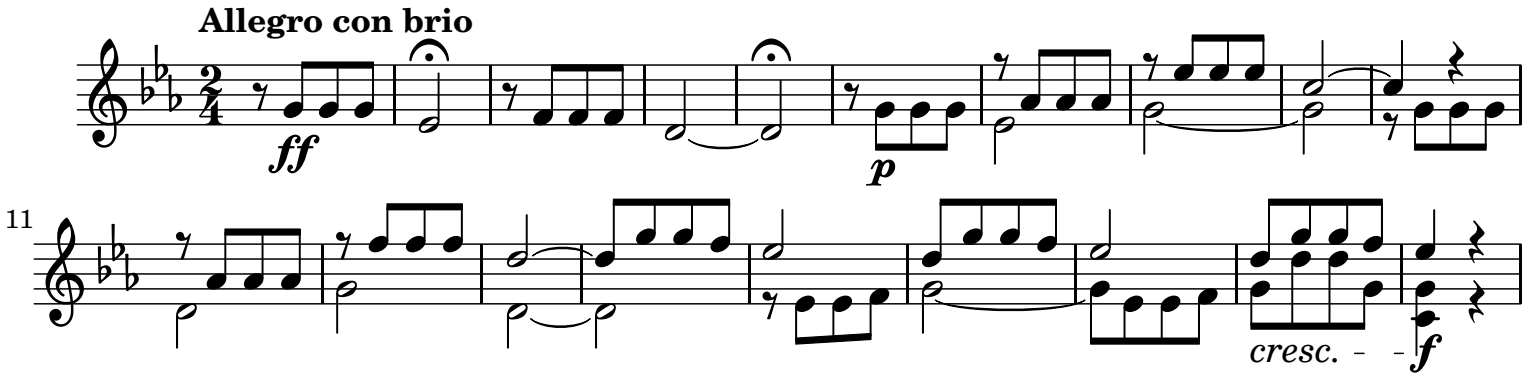


Figure: The opening measures of Beethoven's famous fifth symphony.

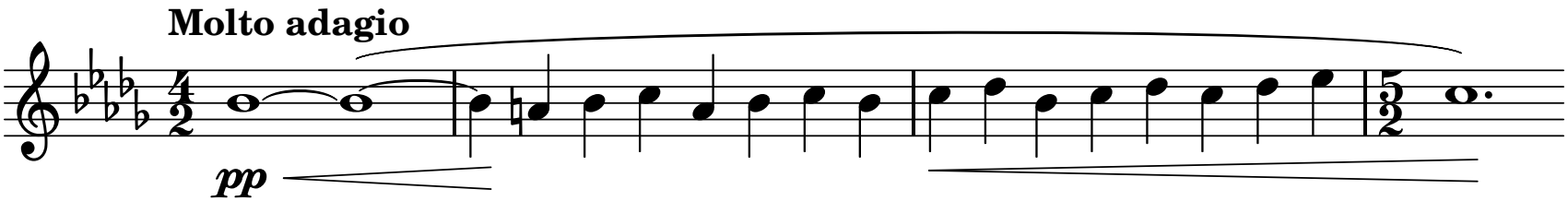
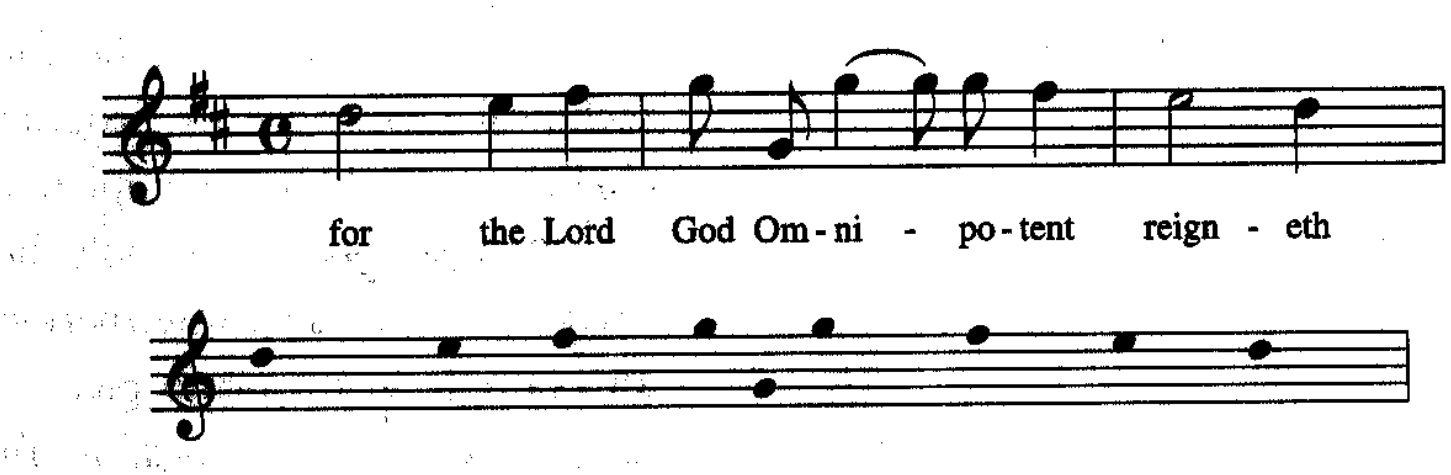


Figure: The hauntingly sublime opening melody of Samuel Barber's *Adagio for Strings*.

Symmetry in Music: Retrograde – Haydn

Figure: Joseph Haydn, *Piano Sonata in A Major*, (Hob. XVI/26; Landon 41) "Minuet in Reverse." Both the minuet and trio are exact musical palindromes.

Symmetry in Music: Retrograde and Transposition



George F. Handel, *Messiah, Hallelujah chorus* (loose retrograde – form of tone painting)

The opening minute of the piece features just two motifs, the famous "Hallelujah" motif and the excerpt above.

Johann Sebastian Bach, *Musical Offering*

- Written in 1747, three years before Bach's death, for Frederick the Great (King of Prussia).
- Upon visiting the King's palace, Bach was challenged by the King to improvise three-part and six-part fugues based on the "Royal theme."

- Bach succeeded in improvising a three-part fugue. Although he could not do a six-part fugue based on the Royal theme, he stunned the court audience by improvising a six-part fugue based on a theme of his own choosing.

Bach's *Musical Offering*, cont.

- Bach returned home to compose the six-part fugue, a ricercar, as well as several other pieces, all based on the Royal theme, and sent it to the King as his *Musical Offering*.
- The work contains 13 pieces, organized symmetrically as follows:

Trio Sonata

Five Canons **Five Canons**

Ricercar **Ricercar**

- A canon is a sophisticated type of round, where a main theme is imitated in some form and played by a different part after the main theme has begun. The imitations can be direct repetition; repetition at a different interval (transposition); in inversion; or in retrograde. Sometimes the theme and its imitation begin together.

Bach's *Musical Offering* – A Musical Puzzle

- Bach used all of the different symmetry types in his canons. However, to make things interesting, Bach only wrote out the full parts for one of the 10 canons. The others were left as musical puzzles, where Bach left clues to indicate how the remaining parts were to be determined.

- Quaerendo invenietis ("Seek and ye shall find") was inscribed on certain canons, particularly those without titles.

- The "puzzle" offered by Bach was solved and first published by Bach's student Johann Philipp Kirnberger.

Bach's *Musical Offering* – Crab Canon

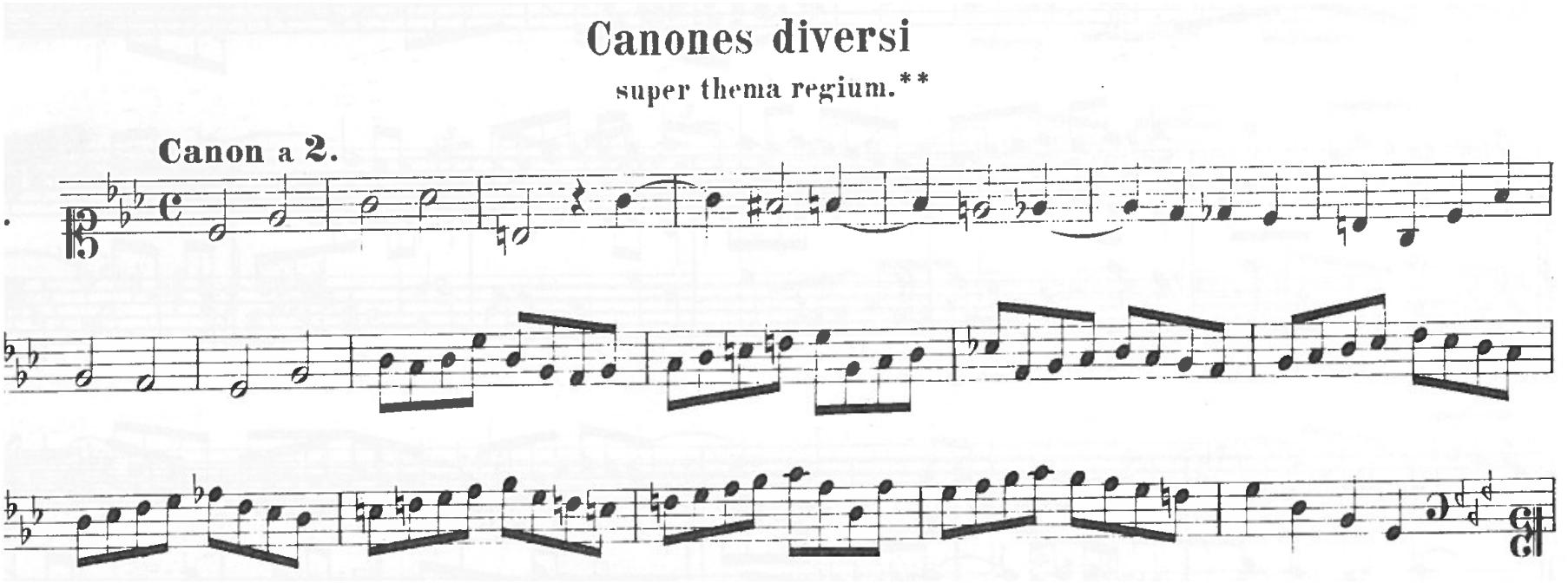


Figure: The unsolved version of one of Bach's canons from the *Musical Offering*. Notice the reflected clef and key signature at the very end of the piece.

Solution for the Crab Canon

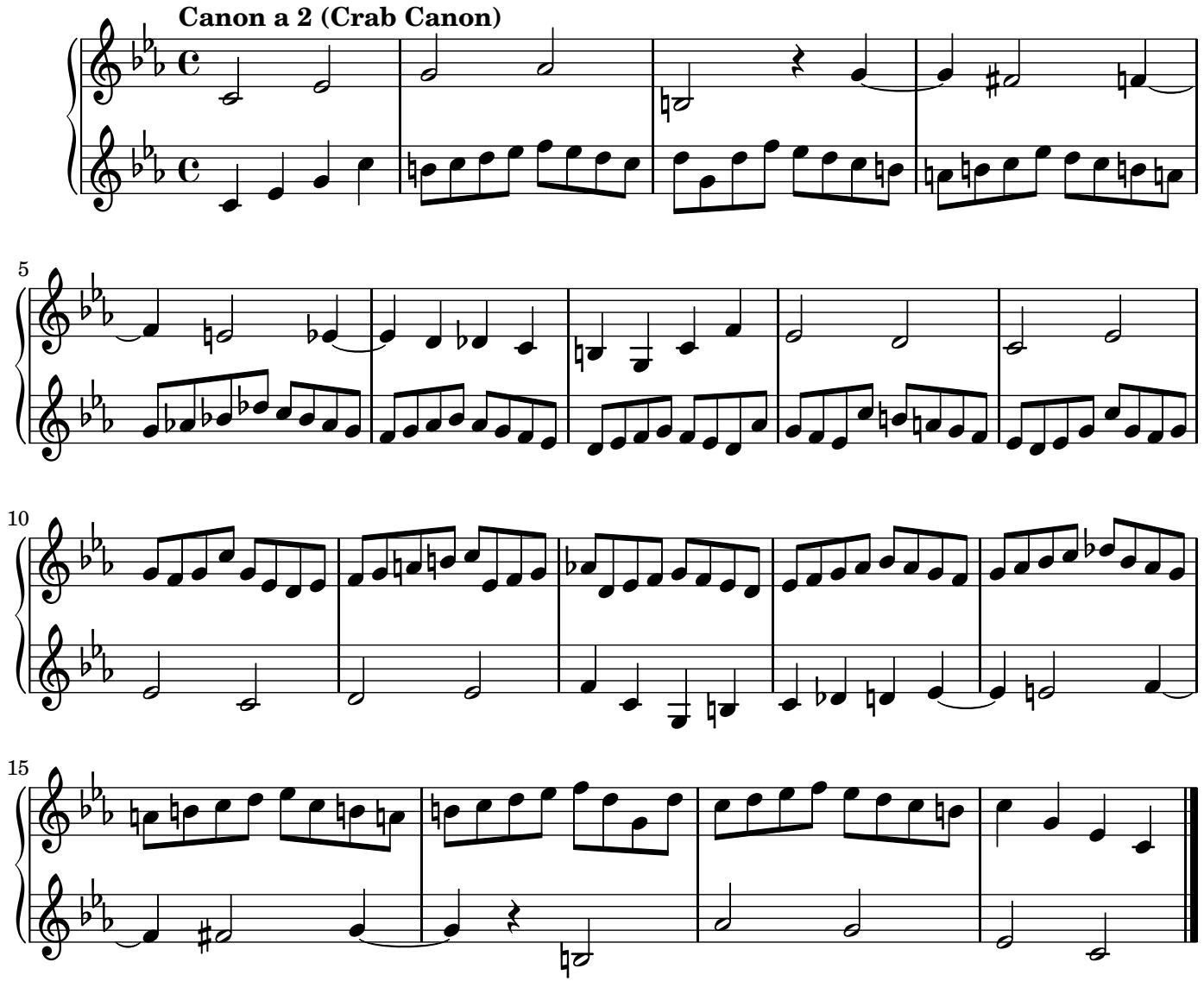


Figure: The *Crab Canon* from Bach's *Musical Offering*

Bach's Crab Canon – Analysis

- The primary theme (first part) consists of the Royal theme followed by an eighth-note countermelody. The entire part sounds perfectly fine in retrograde (played backwards). Thus, the second part plays the primary theme backwards simultaneously as the first part plays it forwards.

- Alternatively, a vertical reflection (retrograde) occurs at the end of measure nine. Each part moves in retrograde, but the parts are interchanged; the first parts plays the second part backwards and vice-versa.

- Mathematically, this last interpretation can be visualized on a Möbius strip! Take the primary theme and cut it in half. Glue the two parts together, but make a twist before gluing. Each player now travels in opposite directions around the strip, with the vertical reflection taking place when the two parts pass each other after one "loop." The twist represents the interchanging of the parts.

▸ Bach on a Möbius strip

Inversions

- An inversion occurs when the main theme is reflected horizontally about some note. If the melody goes up by a fourth, then the inversion goes down by a fourth, etc.

- Two types of inversions: tonal and exact. A tonal inversion is one where the inversion remains in the given key; an exact inversion requires all intervals to be reflected precisely.

- For example, in the key of C major, a melody that begins on a C and goes up a major third to E, would be reflected in a tonal inversion about C to the notes C and A (down a minor third), in order to avoid any accidentals. If the inversion were exact, then it would be C to A♭ (down a major third).

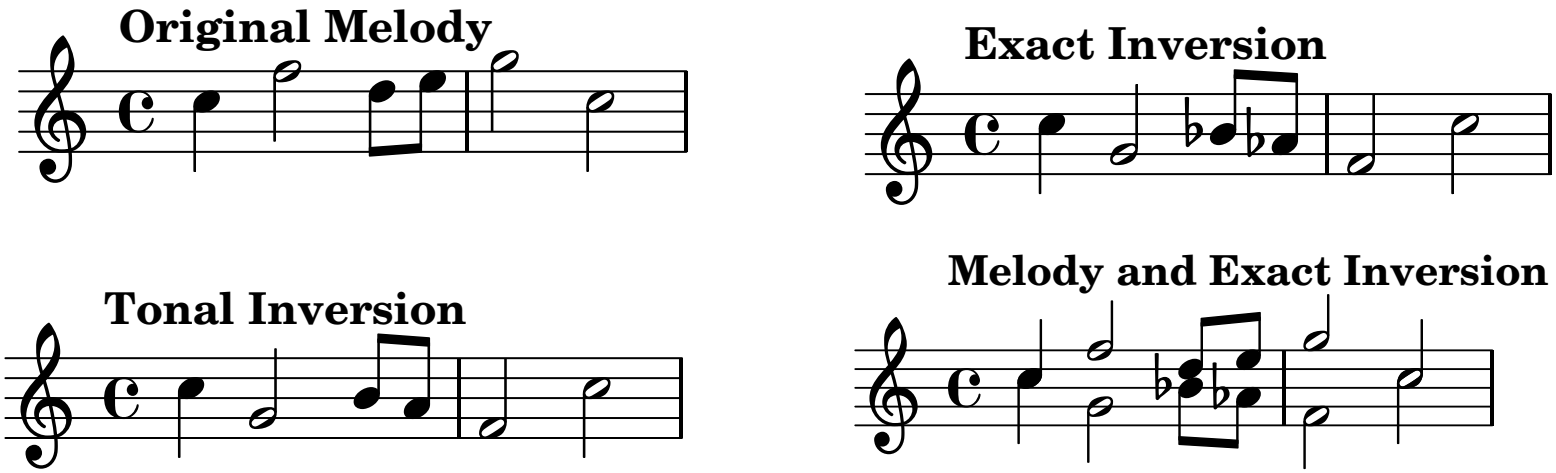
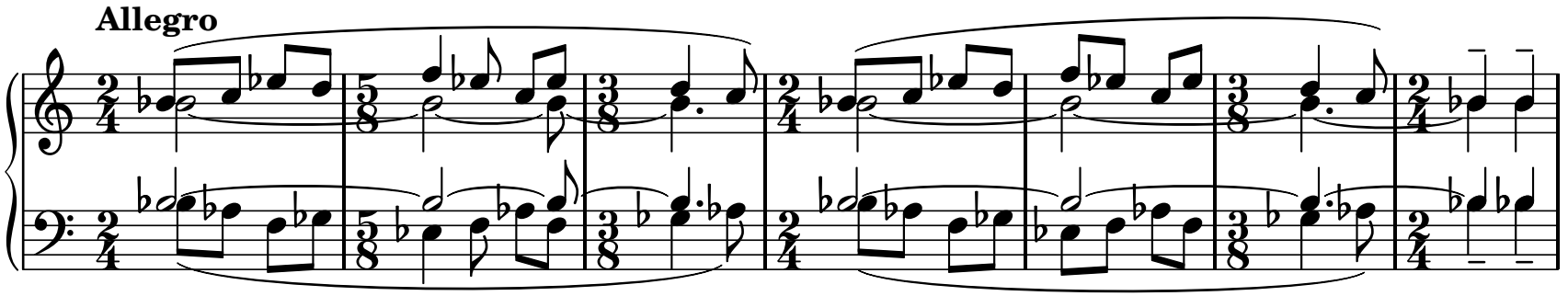


Figure: A simple melody along with its tonal and exact inversions. Here the horizontal reflection is about C, as can be viewed clearly in the lower-right excerpt.

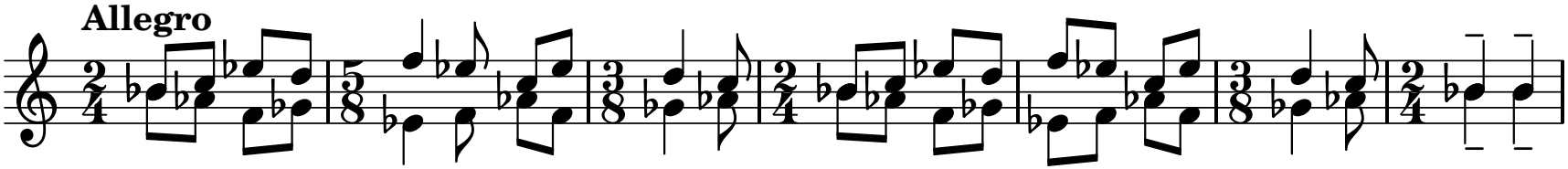
Excerpt	Sequence of Intervals
Original Melody	↑ P4, ↓ m3, ↑ whole step, ↑ m3, ↓ P5
Exact Inversion	↓ P4, ↑ m3, ↓ whole step, ↓ m3, ↑ P5
Tonal Inversion	↓ P4, ↑ M3, ↓ whole step, ↓ M3, ↑ P5

Table: The interval sequences for the excerpts in the above figure.

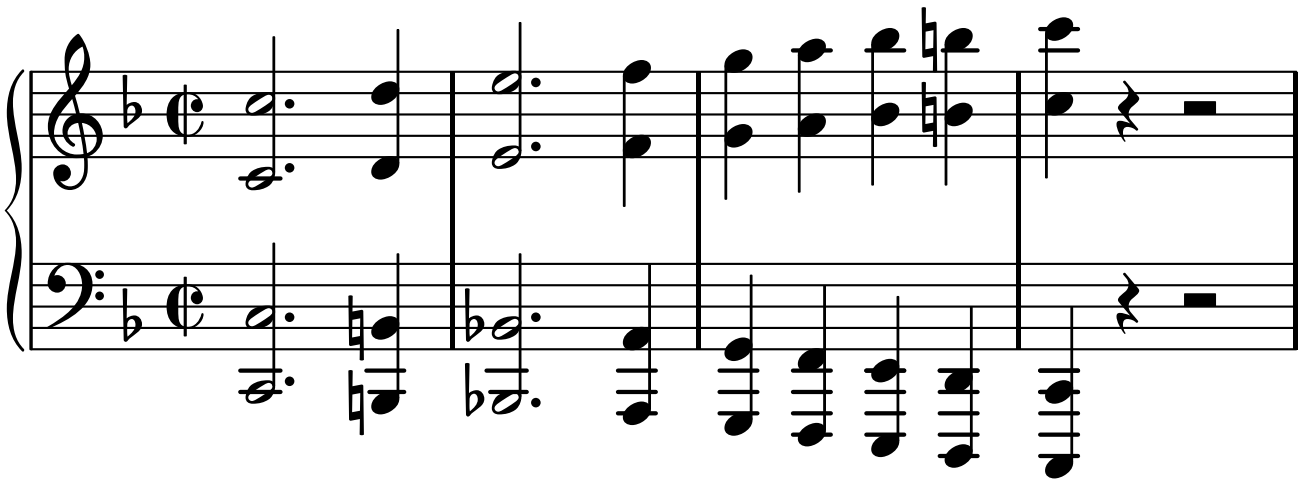
Symmetry in Music: Inversion – Bartók



Béla Bartók, *Mikrokosmos, No. 141, Subject and Reflection* – exact inversion about B♭.



Symmetry in Music: Inversion – Sousa



John Philip Sousa, opening of the march *The Thunderer*. (Analyze for HW.)

Symmetry in Music: Inversion – Bach

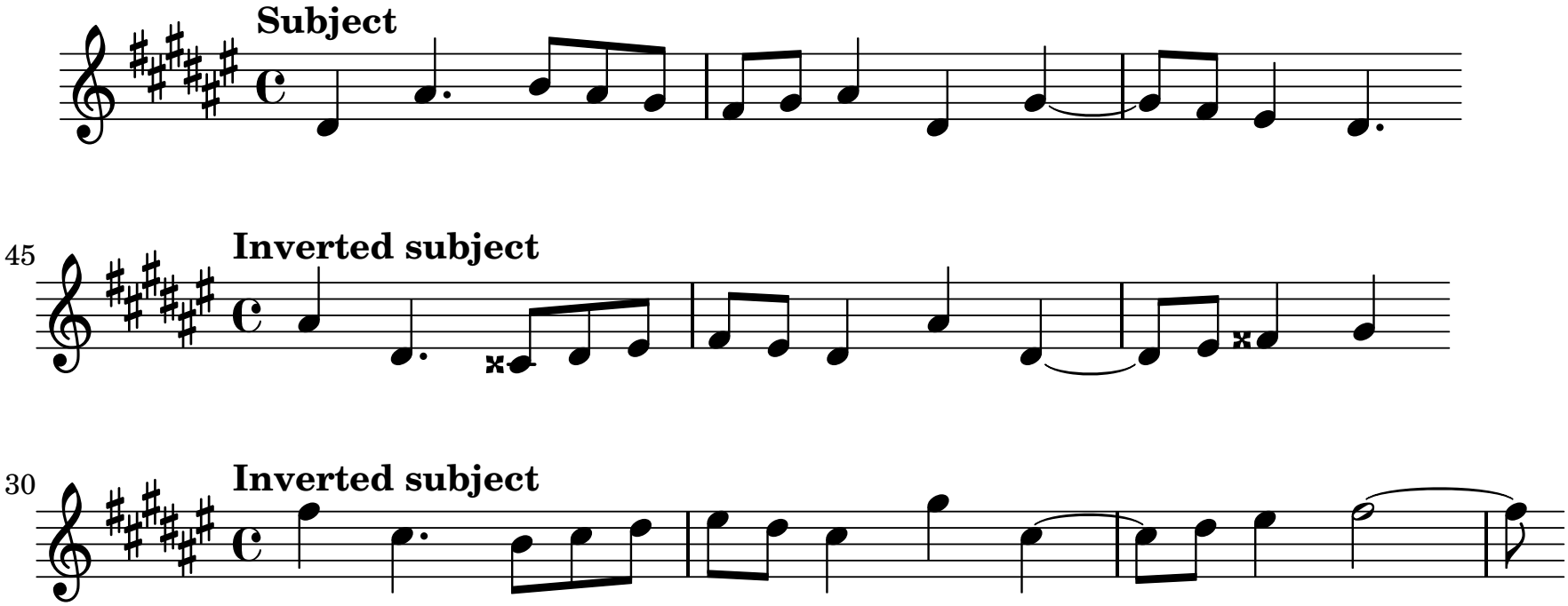
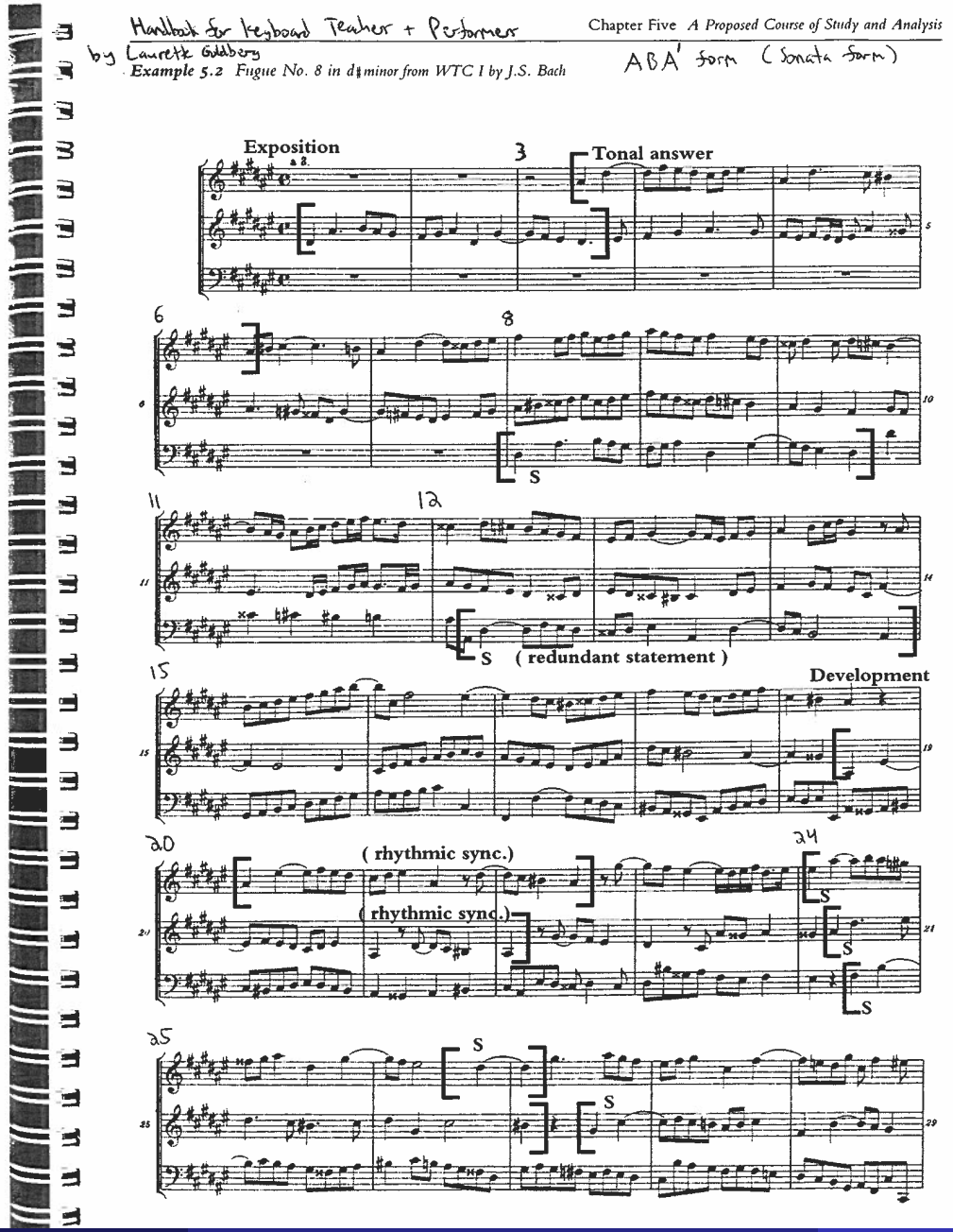
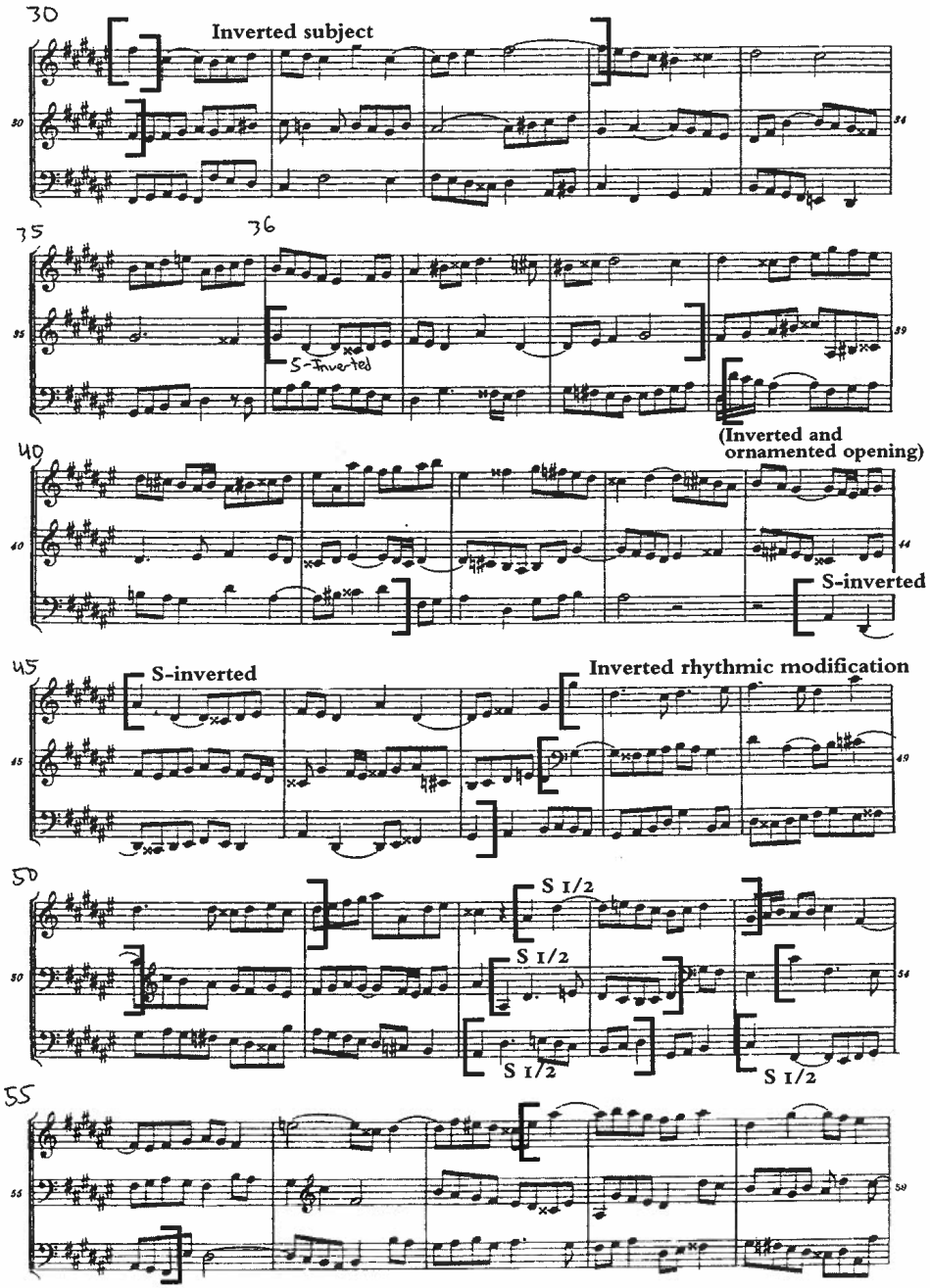
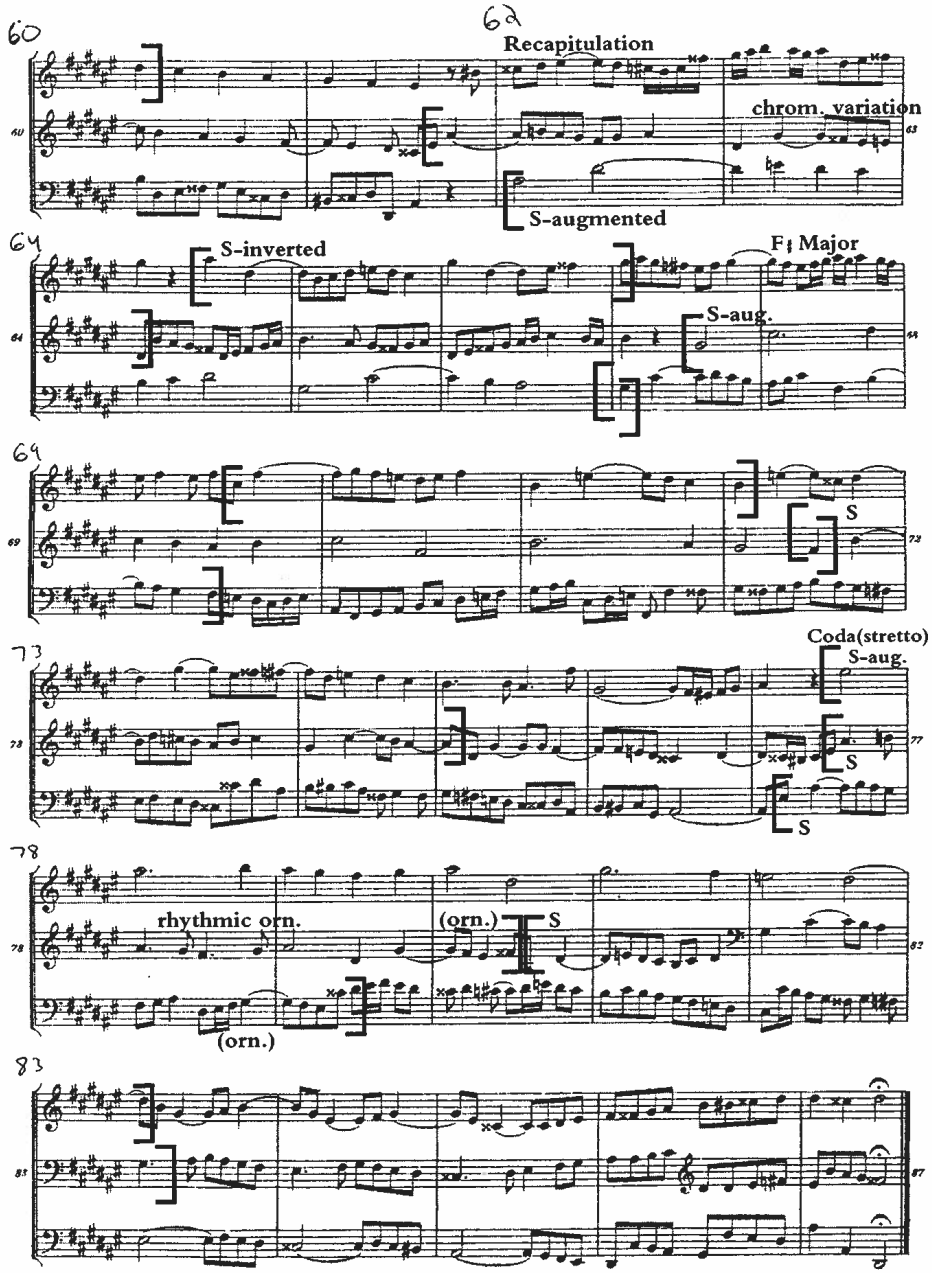


Figure: The subject and two different inversions of the subject in Bach's *Fugue No. 8 in D♯ minor* from the *Well-Tempered Clavier*, vol. I

Bach: The Well-Tempered Clavier, Fugue No. 8 in D♯ minor




30
Inverted subject
35
36
S-inverted
(Inverted and ornamented opening)
40
S-inverted
45
S-inverted
Inverted rhythmic modification
50
S 1/2
S 1/2
S 1/2
S 1/2
55


60
62
Recapitulation
chrom. variation
S-augmented
64
S-inverted
F♮ Major
S-aug.
69
S
73
Coda(stretto)
S-aug.
S
S
78
rhythmic orn.
(orn.)
S
(orn.)
83

Symmetry in Music: Retrograde-inversion

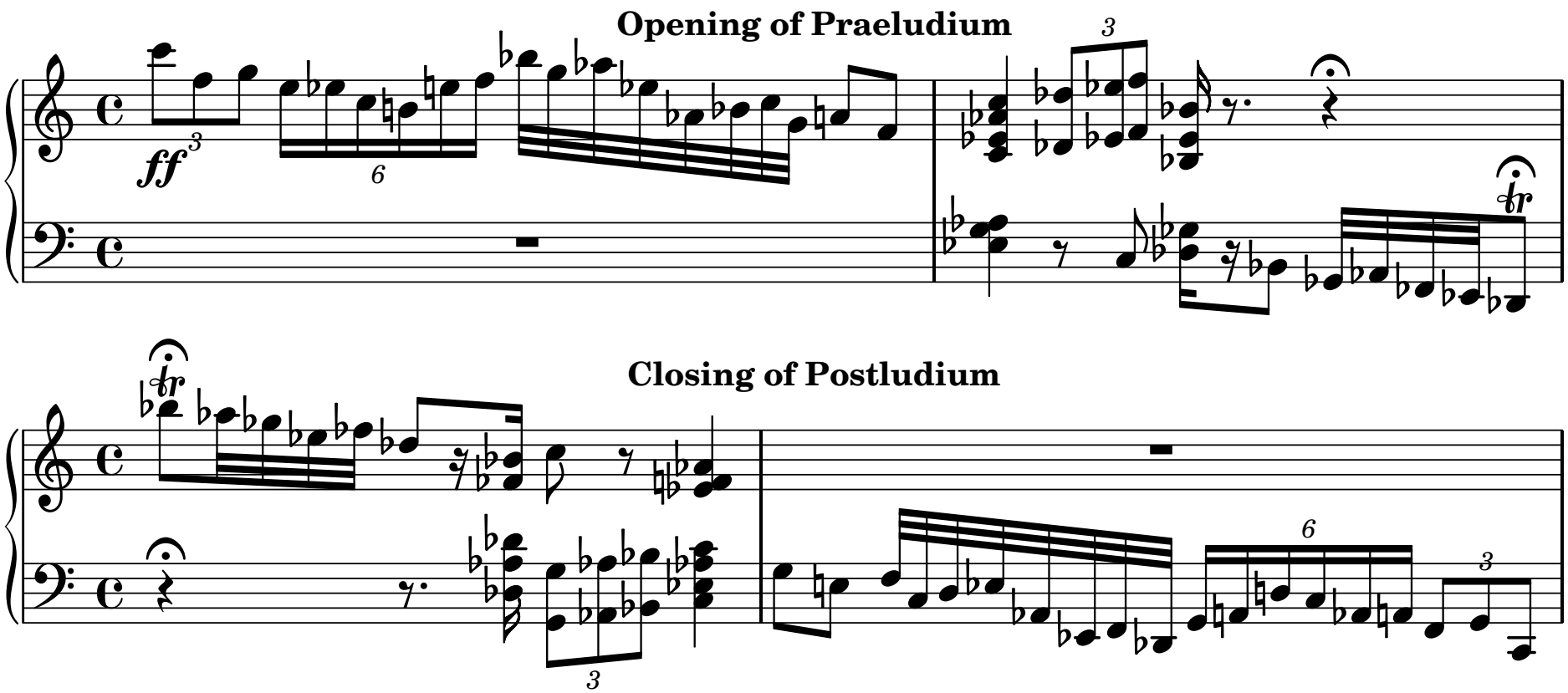


Figure: Paul Hindemith, *Ludus Tonalis* ("Game of Tones"), beginning and end. The ending Postludium is an exact retrograde-inversion (180° rotation) of the opening Praeludium.

Combining Symmetries – Liszt

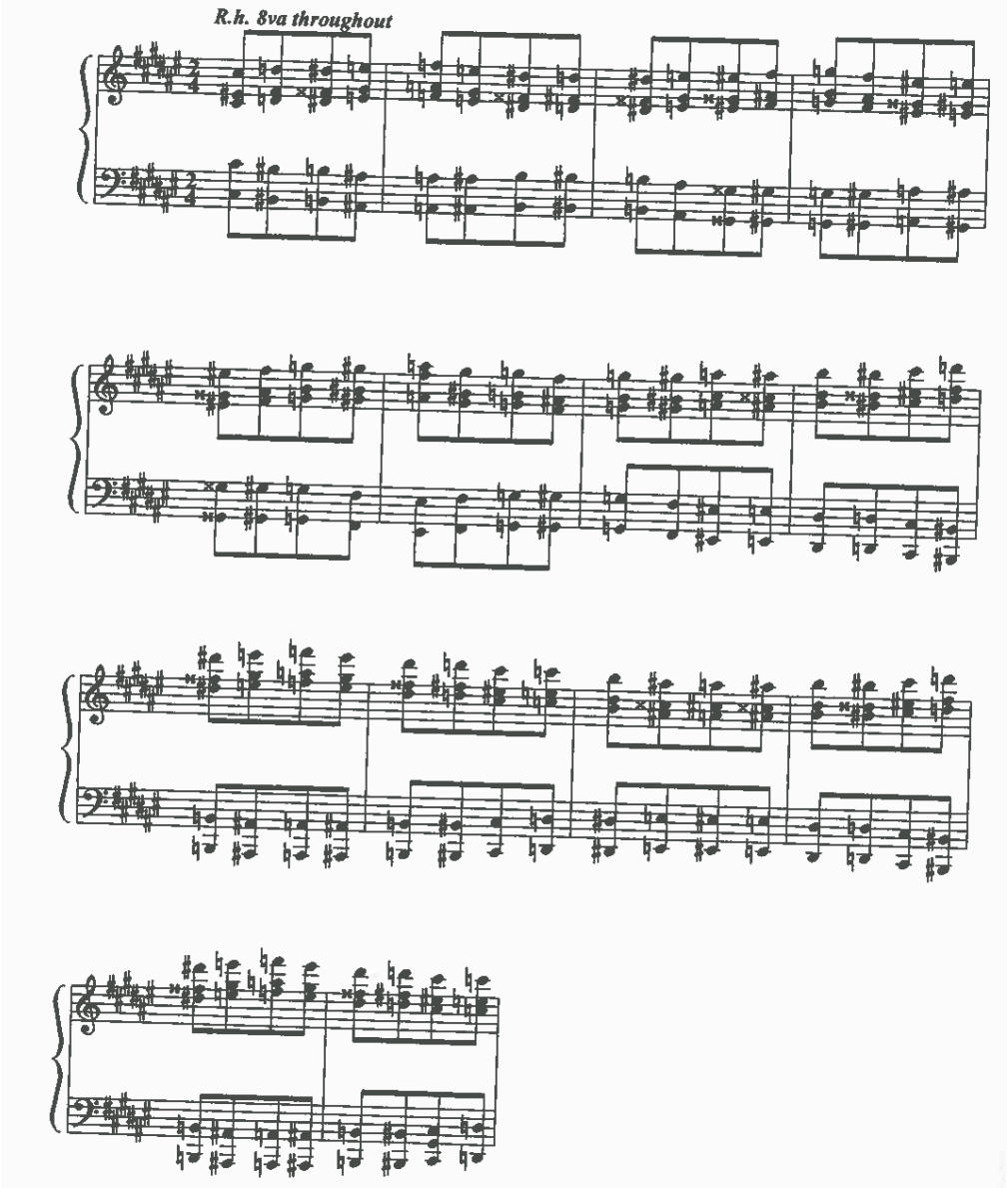


Figure: Franz Liszt, excerpt from *Hungarian Rhapsody #2*

Combining Symmetries – Gershwin

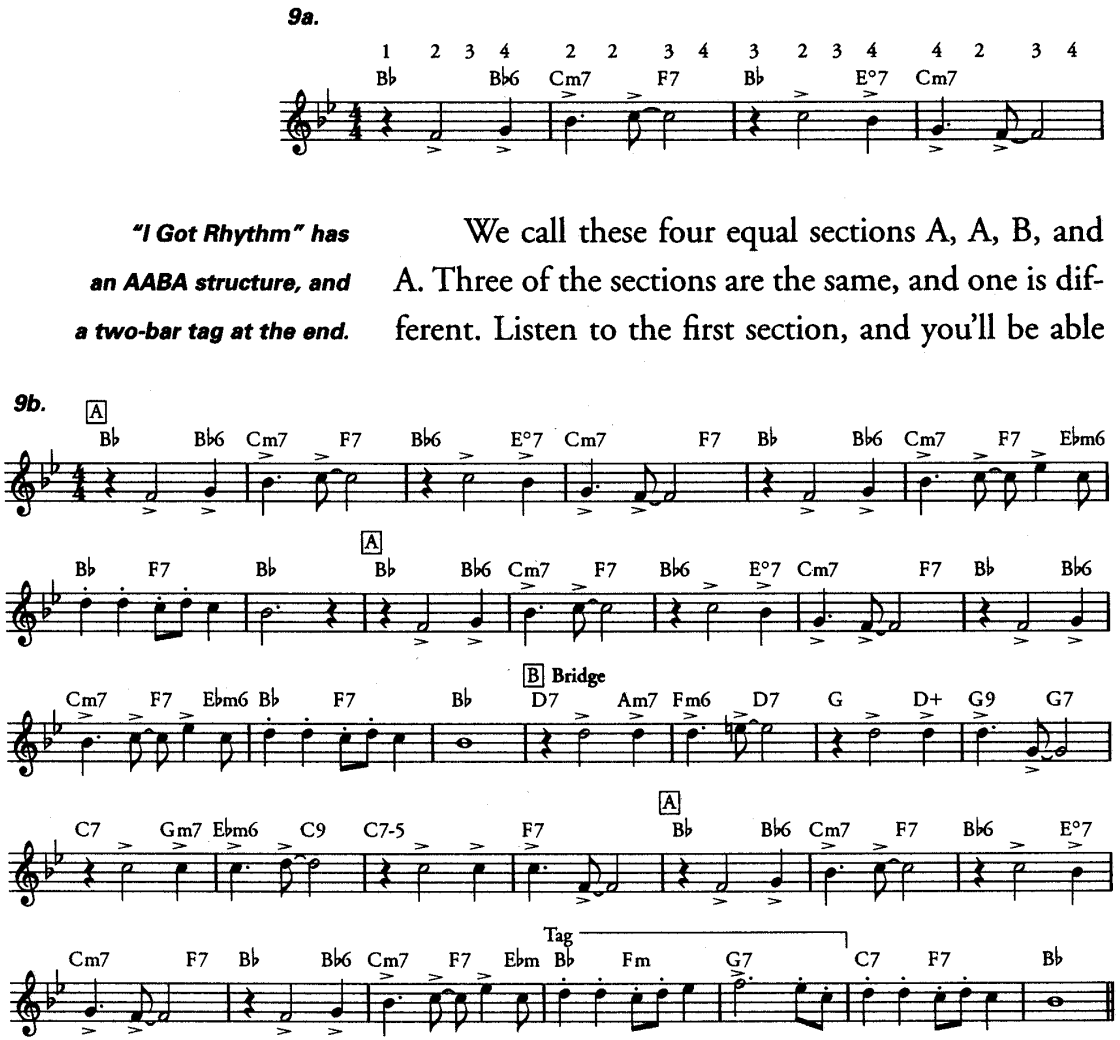
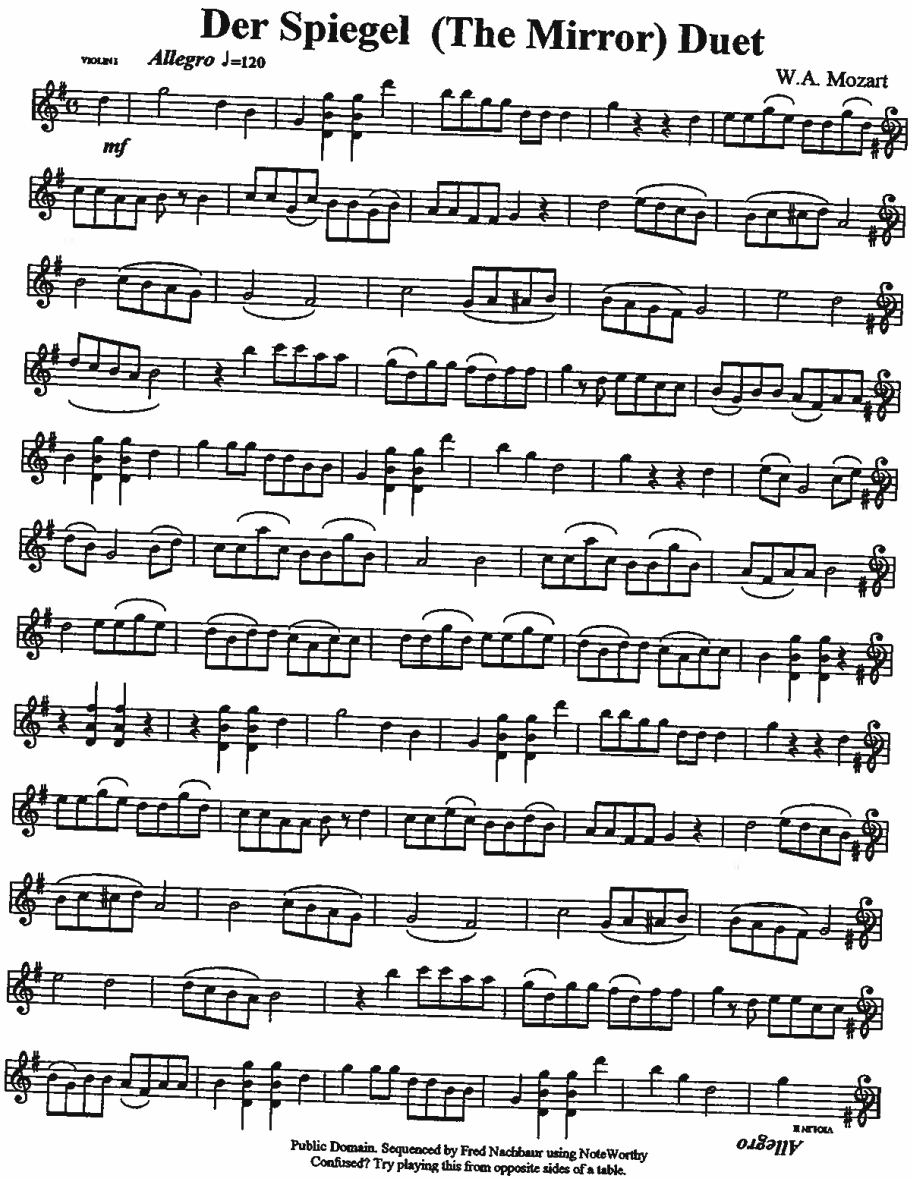


Figure: George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)



(Note: the attribution to Mozart is dubious)

Figure 9.6.