Math and Music: The Science of Sound

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Sound (Section 3.1)

- Sound is formed by changes in air pressure.
- Sound waves are caused by a vibrating object in some medium (e.g., air).
- Air molecules (sensitive to pressure changes) bounce against each other to create a sound wave.
- Note: The actual molecules move transverse to the wave (up and down at 500 m/s). Their combined movement creates the actual wave — think of the wave in a sports stadium. Sound travels through air at 343 m/s or 767 miles/hr.
- Sound needs a medium to travel through (e.g., air, water). In a vacuum (e.g., space), there is no sound!

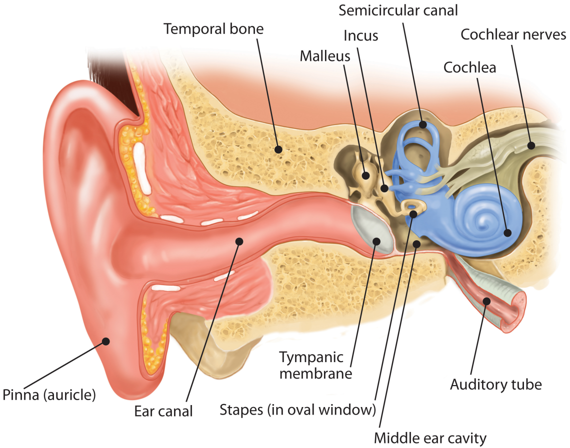

Figure: The human ear. Pressure variations reaching the pinna are converted into frequency data that immediately travel to the brain via the cochlear nerves. © Kocakayaali; Dreamstime.com—Human Ear Photo.

The Human Ear

- Sound vibrations collected in outer ear (the pinna).
- Changes in pressure are passed down the ear canal to vibrate our ear drum.
- The hammer, anvil, and stirrup work to propagate the vibrations into the fluid in the snail-shaped cochlea.
- Cochlea filters the sound by frequency (number of cycles per second) using tiny rows of hairs on the amazing basilar membrane (16,000 hair cells joined by tip links of width equal to 10 atoms).
- A physical-to-electrical transformation occurs as ion channels transmit frequency information from the basilar membrane to nerve impulses heading to the brain.
- Feedback Effect: The brain can request information from the basilar membrane to enhance certain frequencies.

The Key Role of the Brain

- Can decompose different sounds and identify the source of each one (e.g., can distinguish different instruments in an orchestra even if they are all playing the same note). The basilar membrane feedback mechanism is crucial here.
- Can store different sounds and instantly recall them.
- Can adapt and adjust to new sounds (e.g., becoming comfortable with a new accent).
- Can focus on certain sounds while ignoring others (e.g., the ability to have a conversation with your neighbor in a loud and crowded room). Think of professional athletes ignoring hecklers.

Section 3.2: The Attributes of Sound

Perceptual	Physical	Units
Loudness	Intensity	dB (decibels)
Pitch	Frequency	Hz (cycles per second)
Duration	Length of time	s (seconds)
Timbre	Spectrum	

Timbre refers to the unique characteristics of a sound. Different instruments (e.g., string versus horn) produce sound with different timbres and this helps us distinguish one instrument from another.

Loudness and Decibels

The human ear can perceive sound in a very wide range of loudness.

Loudness (sound intensity, denoted by I) is measured on a logarithmic scale using decibels (dB) according to the formula

$$\text{number of decibels} \;=\; 10\log_{10}\left(\frac{I}{I_0}\right),$$

where I_0 is the threshold of human hearing ($I_0 = 1 \times 10^{-12}$ watts/m^2).

Key Fact: Multiplying the sound intensity by a factor of d means adding $10\log_{10}(d)$ decibels.

Example: Increasing the sound by a factor of 100 means adding only 20 dB, because

$$10\log_{10}(100) \;=\; 10 \cdot 2 \;=\; 20.$$

Sound	Decibels (dB)
Threshold of human hearing	0
Whisper	15
Mosquito buzz	40
Regular conversation	60
Jackhammer	100
Rock concert	120
Threshold of pain	130
Jet engine at 30 meters	150

Table: Some sounds and their approximate intensity measured in decibels. A logarithmic scale condenses the gaps between different degrees of loudness.

Logarithms

Key Idea: The output of a logarithm is an exponent.

Example 1: $\log_2(16) = 4$, since $2^4 = 16$.

Example 2: $\log_{10}(1000) = 3$, since $10^3 = 1000$.

In general,

$$\log_b(d) \;=\; x \quad \text{means} \quad b^x \;=\; d.$$

b is the base and x is the exponent.

To compute $\log_b(d)$, we find the exponent for which b raised to that value gives the number d. In other words, we solve the equation

$$b^x \;=\; d$$

for x.

Exercises with Logarithms

Find the value of each logarithm:

1. $\log_3(27) =$
2. $\log_2(1/32) =$
3. $\log_{10}(1{,}000{,}000{,}000) =$
4. $\log_9(3) =$
5. $\log_5(1) =$
6. $\log_5(0) =$

Properties of Logarithms

The following properties hold for any base b (must have the same base on each side of the equation).

1. $\log(xy) = \log(x) + \log(y)$
2. $\log\left(\frac{x}{y}\right) = \log(x) - \log_b(y)$
3. $\log(x^m) = m\log(x)$
4. $\log(1) = 0$

Exercise: Suppose that the volume of sound coming from a speaker is increased by 40 decibels. By what factor has the sound's intensity increased?

Frequency

Frequency is determined by how fast a sound wave is traveling.

Frequency is measured by the number of cycles a wave makes in a second. Unit of measurement is a Hertz (Hz).

100 Hz means 100 cycles in one second.

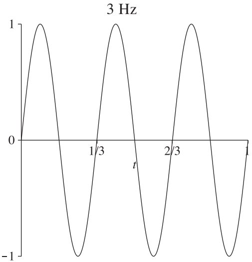

Figure: A wave with a frequency of 3 Hz (three cycles completed in one second).

Frequency versus Pitch

Faster waves (higher frequency) correspond to higher notes (higher pitch).

Moving up the piano means increasing the frequency of each note. Strings at the top of the keyboard vibrate faster than those at the bottom.

Key Reference Note: The A above middle C on the piano has a frequency of 440 Hz. This note, commonly referred to as A440, is the one that orchestras tune to at the start of a concert. It is a universal standard.

Key Idea: The ratio between two frequencies determines the musical interval between the pitches (not the difference). The most important ratio is $2:1$, which gives the octave. Thus, the first A below middle C has a frequency of 220 Hz since $440/2 = 220$.

Mammal or Instrument	Frequency Range (Hz)
Human Ear	20–20,000
Dog	50–46,000
Dolphin	1000–130,000
Bat	2000–110,000
Gerbil	100–60,000
Piano	27–4186
Violin	196–3520
Tuba	40–440
Soprano	262–1047
Bass (voice)	80–330

Table: The approximate frequency range heard by some mammals contrasted with the range of some instruments and voices.

Sine Waves

When a tuning fork is struck, its vibrations produce a nearly perfect sine wave.

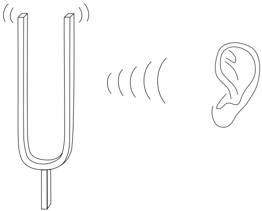

Figure: A vibrating tuning fork emits a simple sinusoidal sound wave toward our ear.

▸ Click Here for Video Oscilloscope measurements of a tuning fork, flute, and violin all playing the same note (The Open University).

Definition

The sine of t, denoted by $\sin(t)$ or just $\sin t$, is the y-coordinate of the point of intersection between the unit circle and a ray emanating from the origin at an angle of t radians.

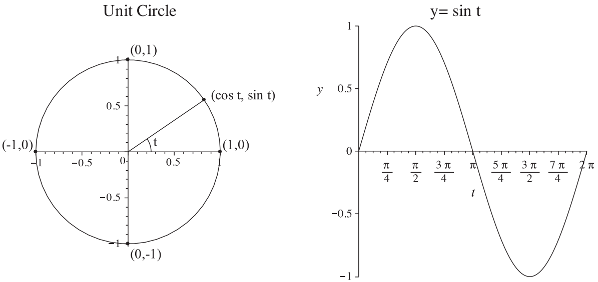

Figure: On the left is the unit circle. The values of $\cos t$ and $\sin t$ are defined as the x- and y-coordinates, respectively, of the point of intersection at angle t. On the right is a graph of $y = \sin t$ over one cycle.

Radians

Sine function: $y = \sin t$ or $f(t) = \sin t$

The input into the sine function is an angle (measured in radians) and the output is a number between -1 and 1.

Domain: $\mathbb{R}$ (all real numbers) Range: $-1 \leq y \leq 1$

Definition

One radian is the angle formed by traveling one unit of length along the unit circle.

Recall: Circumference of a circle is $2\pi r$. For the unit circle, $r = 1$. Thus

$$\boxed{360^\circ \;=\; 2\pi \text{ rad} \qquad \text{or} \qquad 180^\circ \;=\; \pi \text{ rad.}}$$

Trig Exercises

Find the values of each expression:

1. $\sin(0) =$
2. $\sin(\pi) =$
3. $\sin(12\pi) =$
4. $\cos(13\pi) =$
5. $\cos(9\pi/2) =$
6. $\cos^2(\pi/75) + \sin^2(\pi/75) =$