

# Topics in Mathematics: Math and Music

## CD #3: Symmetry in Music

This CD contains some of the examples of symmetrical music we have been discussing in class. Each piece uses some type of mathematical symmetry: translation, horizontal or vertical reflection, or rotation by  $180^\circ$ , or some combination of these symmetries. Taken as a set, they form a subgroup of a well-known mathematical group called the dihedral group of degree 4 (symmetries of the square; see Section 5.3 of the course textbook). Some composers use these ideas in music quite intentionally to generate new thematic material out of an original idea. Bach was certainly one of the great masters of this, although it is important to note that he did so while never losing site of the musical art in his creations. Not every inversion or retrograde is exactly accurate for want of a musical flavor or style.

**What to listen for:** The music for many of the pieces on this CD can be found in the textbook or in the class lecture notes on symmetry in music. It is important to follow the score as you listen, as this helps identify the composer's intent and reveal the symmetry transformation. It is also a good way to engage your brain while you listen. Try to hear where a retrograde or an inversion takes place in the music. As with CD #2, this collection contains a wide variety of musical styles and various tonalities. Try to discern the differences between earlier and later forms of composition as well as between different forms of music (fugue, oratorio, symphony, sonata, song, etc.) The music is organized by symmetry types, which are underlined and highlighted in boldface below.

1. Ludwig Van Beethoven, *Symphony No. 5 in C minor*, 1st movement, Op. 67, 1807–1808. Track 1, Disc 3 from the complete collection of Beethoven's Nine Symphonies (he actually wrote 10, but the 10th was unfinished) performed by the Chicago Symphony Orchestra under the direction of Sir Georg Solti. This famous symphony took the idea of elaborating upon a simple motif to new heights. The opening “da da da dum” motif, once called “fate knocking at the door” by Beethoven himself, is **transposed**, inverted, elongated, shortened, and continually repeated throughout the work to create the bulk of the music (see Figure 5.10 in the textbook). The entire first minute of this symphony consists solely of this famous motif. Although space restrictions allow for only the first movement, you are encouraged to listen to the entire symphony to see how Beethoven brings the motif back unexpectedly in the final movement.
2. Samuel Barber, *Adagio for Strings*, 1936. Track 8 on a Sony Classical CD featuring the New York Philharmonic, conducted by Leonard Bernstein (no. 39 of 100 in the Royal Edition series). This is the orchestral version of the piece “Agnus Dei” from CD #2. Barber, one of the most well-known 20th century composers in the U.S., took the slow movement from his String Quartet in B minor, Op. 11, 1936, and reworked it into this hauntingly beautiful work for string orchestra. This music was used for Oliver Stone's great Vietnam War film *Platoon*. Much later in his career Barber set the music to the traditional Latin “Agnus Dei” text. The piece is in a minor key and drips with luscious harmonies and a tantalizingly slow, somber melody that appears to be continuously climbing up the  $B^b$  minor scale (see Figure 5.11 of the course textbook). The melody features a **transposition** of a three-note motif that remains within the given key. Barber's work has often been described as neo-romantic, harking back to the great Romantic composers (e.g., Beethoven and Brahms.)

3. Guillaume de Machaut (c. 1300 - 1377), *Ma Fin Est Mon Commencement* (“My End is My Beginning”). Track 19 from a Hyperion CD titled *Messe De Notre Dame* performed by the outstanding Hilliard Ensemble. As the text indicates, this work is a **retrograde** canon in music and almost in words as well. This is one of the earliest known examples of a retrograde in music that was actually written down. Apparently the piece is sung using Pythagorean tuning. This is particularly recognizable near the cadences or ends of each phrase. The singers are keeping as many perfect fifths as possible in a 3:2 ratio to produce the Pythagorean tuning using their voices.
4. George Frideric Handel, *Hallelujah Chorus* from the *Messiah*, 1741. Track 18, Disc 2 from a BMG Entertainment CD titled *Messiah* performed by Musica Sacra, directed by Richard Westenburg. Arguably Handel’s most famous work, the *Messiah* is a grand oratorio telling the story of Jesus that Handel obsessively composed in only 24 days. The text is a compilation of verses from the Bible. Upon finishing the now famous Hallelujah Chorus, Handel exclaimed, “I did think I did see all Heaven before me, and the Great God Himself!” Early on in the chorus (about 22 seconds in) we hear all the voices in unison singing “for the Lord God Omnipotent reigneth,” a phrase which if one neglects duration, is in **retrograde** (see the excerpt on page 5 of the lecture notes). By using vertical symmetry, Handel is announcing the brilliance and balance of The Lord God (perhaps an early form of tone painting.) To emphasize this significance, the motif appears throughout this movement. Traditionally the audience stands when the Hallelujah Chorus is sung during a performance of the *Messiah*, often a powerfully spiritual moment for performers and audience alike.
5. Johann Sebastian Bach, *Crab Canon* from the *Musical Offering*, BWV 1079, 1747. This is track 9 off of the accompanying audio CD to the book *The Math Behind the Music* by Leon Harkleroad, Cambridge University Press, 2006. This is a fabulous example of a perfect **retrograde** over 18 measures of music (see Figure 5.14 of the course text). The top part plays the Royal Theme (see Figure 5.13) followed by an eighth-note counter melody while the bottom part plays the exact same thing, only *backwards*! Alternatively, a vertical reflection (retrograde) occurs at the end of measure nine, except that the parts flip, with the first part playing the second part backwards and vice versa. Mathematically, the interchange and reflection can best be visualized on a Möbius strip. Be sure to watch the wonderful YouTube video of this construction (click here).
6. Joseph Haydn, *Piano Sonata in A Major, “Minuet and Trio”*, Landon 41, Hob. XVI:26, 1774–1775. Track 13, Disc 6 from the complete collection of Haydn’s piano sonatas performed by John McCabe (Decca Record Company Limited, London). The opening minuet is titled “Menuet Al Rovescio” (Minuet in Reverse) and is an exact musical palindrome (**retrograde**) with the vertical reflection occurring at the end of measure ten (see Figure 5.12 in the course text). Note that McCabe even reverses the trill (a quick flourish of extraneous notes) in bars eight and nine. He plays it forwards the second time through the first section and backwards the first time through the second section, following the palindrome precisely. The  $\frac{3}{4}$  meter seems crucial to making the retrograde work harmonically as Haydn often emphasizes two of the three notes of the underlying chord in beats one and three of each measure. Thus, on playing such a measure backwards, the chord heard remains unchanged. Haydn was so proud of his clever musical palindrome that he used it in both his *Sonata No. 4 for Piano and Violin* as well as his *Symphony No. 47 in G major*, “The Palindrome.”

7. Johann Sebastian Bach (1685–1750), *Fugue, “The Little”*, BWV 578. Track 8 from a CBS Records CD titled Bach: Great Organ Favorites performed by E. Power Biggs. This is one of Bach’s most popular fugues written for the organ. Recall that a fugue is a piece in which a main subject is presented successively in different voices (in this case different registers of the organ), sometimes coming back in inversion or retrograde or even retrograde-inversion. Meanwhile, a countersubject is often presented which complements the main subject heard at the outset. Often this countersubject is more active than the subject and it too may return in inversion or retrograde. Bach was arguably the greatest composers of fugues. Try and hear each of the different instances when the main subject reappears, noting that sometimes it shows up in a major key and sometimes in a minor key. The piece is written in G minor although it ends triumphantly on a major chord.
8. Johann Sebastian Bach, *Fugue No. 8 in D<sup>♯</sup> minor* from *The Well-Tempered Clavier*, BWV 853, 1722–1744. Track 16, Disc 1 from a Radio-Canada CD titled The Well-Tempered Clavier performed on harpsichord by Scott Ross. Bach wrote 48 preludes and fugues, one each for the 12 major keys and 12 minor keys, over a period of 22 years. The two volume work has been referred to as the “pianist’s Old Testament.” As the title indicates, Bach’s *Well-Tempered Clavier* was intended to be performed on a keyboard that was well-tempered, despite the daunting challenge of playing preludes and fugues in all 24 major and minor keys. Many have mistakenly assumed that Bach wrote the massive work to champion equal-temperament as a superior tuning system. In fact, there is ample evidence to suggest that this was not the case. The music for the full fugue, including the subject and inverted subjects, is on pp. 16–19 of the lecture notes. As you follow along with the music, notice the ways in which Bach brings the opening subject back, sometimes rhythmically altered (syncopated or augmented), sometimes in **inversion** and sometimes both. Occasionally the subject appears in all three voices, but staggered, as in a round.
9. John Philip Sousa, *The Thunderer* (excerpt), 1889. Track 11 off the Harkleroad CD, the opening to this march is an easily discernible **inversion**. Note that this inversion is different than those typically used by Bach in that it happens *simultaneously* rather than delayed (see Figure 5.17 of the course text). You should analyze this excerpt for homework (Exercise 5 in Section 5.1).
10. Béla Bartók, *Mikrokosmos*, Book 6, BB 105: No. 141 *Subject and Reflection*, 1926–1939. Track 45 from a complete recording of this famous set of piano pieces, performed by Jenő Jandó (Naxos). The entire *Mikrokosmos* consists of 153 pieces for piano ranging in difficulty from beginner to advanced and featuring some interesting musical traits including the use of whole-tone scales, Hungarian folk tunes, syncopation, and modal harmonies. In Bartók’s own words, *Mikrokosmos* “appears as a synthesis of all the musical and technical problems which were treated and in some case only partially solved in the previous piano works.” This particular piece is a fine example of an **inversion** and as with the previous musical example, this inversion happens simultaneously (at least for most of the work) so that the musical symmetry is readily apparent in the score as well as upon listening (see Figures 5.7 and 5.8 of the course text). The inversion is exact in the sense that the musical intervals are mirrored precisely (right-hand up a minor third means left-hand down a minor third), a compositional technique not harmonically accepted during Bach’s time, but one that Bartók clearly intended as evidenced by the title (even the number is symmetric). The inversion is slightly offset in the second half of the piece, which can be described mathematically as a *glide reflection*.

11. Franz Liszt, *Hungarian Rhapsody #2* (excerpt), 1847. Track 13 off the Harkleroad CD, this is a fine example of combining many of our symmetry transformations. Here we can find transpositions, inversions, and retrogrades in just a few measures. As with the Sousa excerpt, the symmetry relationships are fairly easy to hear and see (see page 21 of the lecture notes). The full piece is one of the most popular piano solos around and was used in many animated cartoons including the Bugs Bunny short *Rhapsody Rabbit* (Warner Brothers) and Tom and Jerry's *The Cat Concerto* (MGM) which won the 1946 Academy Award for best cartoon.
12. George Gershwin, *I Got Rhythm*, 1930. Track 6 off of a Concord Records, Inc. CD titled Someone to Watch Over Me, sung by Susannah McCorkle. This is a quaint intro track discussing the lyrics of the tune written by the composer's brother, lyricist Ira Gershwin.
13. George Gershwin, *I Got Rhythm*, 1930. Track 7 off the previous CD. Gershwin, who grew up in Brooklyn, NY, the son of Russian-Jewish parents, wrote this popular tune in his musical *Girl Crazy*. The music for this song is on page 22 of the lecture notes. The main opening melody is four measures long and features a simple retrograde at the end of bar two, but with an important shift in the rhythm to give it a jazzy feel. The form of the song is an AABA structure with the B section containing an inversion (more in the recording than in the score) as well as a transposition. Thus, this simple 32-measure song contains three of our four mathematical symmetries.