MATH 136-04, Fall 2010

Chapter 1 Review Sheet

This chapter focuses on the major functions we will be studying throughout the semester. Many of these functions are excellent **models** for real-world phenomenon and are essential in the natural and social sciences. Much of this material is standard in pre-calculus courses. Please complete all the examples.

1.1 Four Ways to Represent a Function

A **function** is a rule that assigns to each input element in the **domain** a unique output element in the **range**. The set of inputs to a function is called the **domain**, while the set of outputs is called the **range**. If a function $f : A \mapsto B$ maps from the set A to the set B (but not necessarily all of B), then we often call B the **co-domain**.

The four ways to represent a function referred to in the text are analytically (an explicit formula), graphically, numerically (table) and verbally (described in words). Typically, we will use x and t as the independent variables (inputs) and letters such as y, N, s (for position), v (for velocity) or a (for acceleration) as the dependent variables (outputs). When graphing a function, we will always assume the independent variable is plotted on the horizontal axis while the dependent variable is plotted on the vertical axis. In order to represent a function, a graph must pass the **vertical line test**, that is, any vertical line through the graph can only pass through at most one point. Otherwise, one input in the domain would have more than output in the range, violating the definition of a function.

Example 0.1 *The number of toads in Australia seems to double every year. At first there were only 200, but now there are millions. Let $N(t)$ equal the number of toads in t years. Initially we have 200 so $N(0) = 200$. One year later the number doubles so we will have $N(1) = 400$ and $N(2) = 800$. Find an analytic formula for $N(t)$ and sketch the graph of the function. What type of function is N?*

We say that a function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

Likewise, f is **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

This is much easier to see visually. Increasing functions move upwards from left to right while decreasing functions move downwards.

A function that satisfies $f(-x) = f(x)$ for all x in its domain is called an **even** function. The graph of an even function is symmetric about the vertical axis. If a function satisfies $f(-x) = -f(x)$ for all x in its domain, then it is an **odd** function. The graph of an odd function is symmetric about the origin (after reflecting about both the horizontal and vertical axes, the same graph is obtained.)

Examples of even functions include $7, x^2, x^4, |x|, x^{-2}, \cos x$. Examples of odd functions include $x, x^3, x^5, 1/x, \sin x, \tan x$.

Example 0.2 *Is the sum of two even functions even, odd or neither? Prove it. What about the sum of two odd functions? What about the sum of an even and an odd function?*

1.2 Mathematical Models: A Catalog of Essential Functions

Linear $f(x) = mx + b$

The constant m represents the slope, while b is the y-intercept (where the line crosses the vertical axis). If x increases by one unit, then $f(x)$ increases (or decreases) by m units. If $m > 0$, then the line is increasing while if $m < 0$, the line is decreasing. When $m = 0$, the line has zero slope and is horizontal (constant function).

Exponential $f(x) = ca^x$

Note the variable x is an **exponent**, hence the name of the function. The constant a is called the **base** and should always be positive. If $a > 1$, then we have **exponential growth** while if $0 < a < 1$, then we have **exponential decay**. The constant c is arbitrary and represents the **initial population** in an exponential population model. The key difference between linear and exponential functions is that a constant change in x yields a constant change in y for a linear function, but a constant **ratio** in y for an exponential function. In other words, for an exponential function, if x increases by one unit, then $f(x)$ increases (or decreases) by a **factor** of a.

Piecewise Function

A function may be defined in different pieces by specifying which formula is to be used over a particular subset of the domain.

Example 0.3 *Sketch the graph of the piecewise function*

$$g(x) = \begin{cases} x^2 & \textit{if } x < 0 \\ 2 & \textit{if } 0 \le x \le 3 \\ 4 - x & \textit{if } x > 3. \end{cases}$$

Absolute Value $f(x) = |x|$

One particular piecewise function is critical in mathematics, the **absolute value** function. The graph of this function is a V with vertex at the origin. Although you may have learned that the absolute value is always positive, this hardly captures the meaning of this function. The absolute value is used to measure distance. For example, $|4| = 4$ and $|-4| = 4$ both indicate that the points 4 and -4 are each four units from 0 on the number line. The expression $|a - b|$ gives the distance between the numbers a and b on the number line. Thus, $|2 - 5| = 3$ since 2 and 5 are 3 units apart on the number line. Similarly, $|3 + 4| = |3 - -4| = 7$ since 3 and -4 are 7 units apart.

The piecewise definition for $|x|$ is

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Polynomial $p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$

A **polynomial** is a sum of terms of the form ax^n where a is any constant and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The degree of p is the largest power n in the expression. For example, $p(x) = 3x^2 + x - 4$ is a degree 2 polynomial or a **quadratic** function. Any quadratic has the form $p(x) = ax^2 + bx + c$ for some constants a, b, c. The function $p(x) = 5x^3 - x + \pi$ is a **cubic** polynomial (degree 3) and the function $p(x) = 17x + \sqrt{2}\,x^3 - 12x^4$ is a **quartic** polynomial (degree 4).

Example 0.4 *Find the quadratic function that is even and passes through the points* $(-1, 1)$ *and* $(2, 13)$.

Rational $R(x) = p(x)/q(x)$

A rational function is simply the ratio of two polynomials. For example,

$$R(x) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{and} \qquad S(x) = \frac{x^5 - 12x^3 + 6}{x^3 - 1}$$

are each rational functions.

Power Function $f(x) = x^p, p \in \mathbb{R}$

The power function generalizes the monomials in a polynomial by allowing for any kind of exponent, not just natural numbers. Some examples of power functions include:

$$x^2, \sqrt{x} = x^{1/2}, x^{-1} = \frac{1}{x}, x^{\pi}, x^{\sqrt{2}}$$

You should know the graphs of $1/x$ and $\sqrt{x}$.

Trig Functions $\sin x, \cos x, \tan x$

Trigonometric functions are extremely important. When writing $\sin x$ or $\cos x$, it is always assumed that x is measured in **radians** not degrees. An angle of 1 radian is equivalent to the angle which cuts off 1 unit of arc length of the unit circle. This is approximately $57°$. The key formula to remember is that π radians equals $180°$. This follows from the fact that the circumference of the unit circle is 2π.

For a given angle θ, let l_θ represent the ray emanating from the origin that makes an angle of θ with the positive x-axis. It is important to remember that $\cos\theta$ equals the x-coordinate of the point of intersection between l_θ and the unit circle and that $\sin\theta$ equals the y-coordinate of this same point of intersection. Since the unit circle has the equation $x^2 + y^2 = 1$, we quickly have the important identity

$$\cos^2\theta + \sin^2\theta \;=\; 1.$$

Trig functions are called **periodic** functions because they repeat themselves after some time (called the period).

The other trig functions are defined as follows:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

Example 0.5 *What are the domains and ranges of all six trig functions? It may help to draw graphs. Try doing this without your calculator.*

1.3 New Functions from Old Functions

This section focuses on how to create new functions by shifting, stretching, reflecting and composing familiar functions. One key point to remember is that changes in the output of the function are changes in the range and therefore effect the vertical direction of the graph. Changes in the input (replacing x by $x + 3$ for example), result in changes in the horizontal direction of the graph.

Suppose that c is a positive constant. The graph of $y = f(x) + c$ is the graph of f shifted up vertically by c units. Likewise, the graph of $y = f(x) - c$ is the graph of f shifted down vertically by c units. These are both output changes, thus vertical shifts. The graph of $y = f(x + c)$ is the graph of f shifted left horizontally by c units while the graph of $y = f(x - c)$ is the graph of f shifted right horizontally by c units. Since these are domain changes, they effect the graph horizontally.

This last item may seem counterintuitive because adding on the number line means moving right, so why does $f(x + c)$ shift *left*? This is because we are doing a domain change, thus actually shifting the x *axis* to the right, which has the net effect of shifting the graph to the left. That's pretty cool! All domain changes tend to do the *opposite* of what we would expect.

Suppose that $c > 1$. The graph of $y = cf(x)$ stretches the graph of f vertically by a factor of c units (range change) while $y = \frac{1}{c}f(x)$ compresses the graph of f vertically by a factor of c units. The graph of $y = f(cx)$ compresses the graph of f horizontally by a factor of c units (domain change so we get opposite effect) while $y = f(\frac{1}{c}x)$ stretches the graph of f horizontally by a factor of c units.

Example 0.6 *Plot the graphs of the functions* $\sin x, \sin(2x)$ *and* $\sin(\frac{1}{2}x)$ *on the same axis over the interval* $0 \le x \le 2\pi$. *What is the* **period** *of each function? Compare your findings with the above facts on stretching and compressing graphs.*

The graph of $y = -f(x)$ is found by reflecting the graph of f about the x-axis (output flips signs). The graph of $y = f(-x)$ is found by reflecting the graph of f about the y-axis (input flips signs).

Another important way to obtain new functions from old ones is to compose two functions together. The notation

$$(f \circ g)(x) \;=\; f(g(x))$$

is read "f of g of x". The new function created is referred to as $f \circ g$. The input x first gets plugged into g then the output $g(x)$ gets plugged into f, yielding the output of the function $f \circ g$.

Example 0.7 *If* $f(x) = x^2 - 3$ *and* $g(x) = \frac{1}{x^2}$, *find* $(f \circ g)(1)$, $(g \circ f)(1)$, $(f \circ f)(1)$, $(g \circ g)(1)$ *and* $(g \circ g \circ \cdots (2 \text{ million times}) \circ g)(1)$.

1.5 Exponential Functions

As described in Section 1.2, an exponential function is one of the form $f(x) = ca^x$, where the variable is an **exponent**. If the **base** $a > 1$, then we have **exponential growth** (graph is increasing and concave up) while if $0 < a < 1$, then we have **exponential decay** (graph is decreasing and concave up). The constant c is arbitrary and represents the **initial population** in an exponential population model. In other words, one might use the model

$$P(t) = P_0 a^t$$

to model a population that starts at P_0 and then grows by a factor of a units for every unit of time t, $P_0, P_0 a, P_0 a^2, \ldots$.

Example 0.8 *Which function grows faster,* $f(x) = x^2$ *or* $g(x) = 2^x$? *Explain.*

Example 0.9 *Find the exponential function passing through the points* $(1, 4)$ *and* $(2, 12)$.

The Number e

There is one particular base that is more important than any other. It is the special number $e \approx 2.718281828\cdots$. This is an irrational number so the decimal pattern does *not* repeat nor ever terminate. The number e can be defined in many different ways. For us, it is the precise base such that when the graph of the exponential function $y = e^x$ is plotted, the slope of the tangent line through the point $(0, 1)$ is precisely one. This may seem like the strangest definition you've ever heard of, but from this, it follows that

$$\frac{d}{dx}\left(e^x\right) \;=\; e^x.$$

This is the *only* function with the property that it is equal to its own derivative. This reminds me of the old novelty song "I'm My Own Grandpa" by Dwight Latham and Moe Jaffe.

1.6 Inverse Functions and Logarithms

One of the simplest ways to obtain a new function from an old one is to simply flip the domain and range. So if $f(2) = 7$, then the new function f^{-1}, called the **inverse of** f, has $f^{-1}(7) = 2$. The function f^{-1} simply maps each element in the range of f back to the element in the domain it came from. It "inverts" f. However, what if there were two elements in the domain whence it came from? Then we'd have a problem, and f^{-1} would not be a function. To rectify this, we must assume that f is **one-to-one**, that is, each element in the range of f has one and only one pre-image that was sent to it in the domain.

For example, the function $g(x) = x^2$ is not one-to-one because both -2 and 2 are each sent to the same element in the range, $g(-2) = g(2) = 4$. This function fails the **horizontal line test** and is really two-to-one. However, if we restrict the domain (yes, it's ok to do this) to $x \geq 0$, then g suddenly becomes one-to-one and now the inverse is actually a function. You know it already as $g^{-1}(x) = \sqrt{x}$. By definition (ie. restricting the domain of x^2), $\sqrt{x}$ only spits out non-negative values.

Key point: The only functions with well-defined inverses are those that are one-to-one.

Note that the notation here is not the usual exponent notation. In other words,

$$f^{-1} \;\neq\; \frac{1}{f}$$

The choice of -1 as the exponent is mathematical shorthand for **inverse**. Don't confuse this!

Based on the definition of the inverse of a function, the following formulas should make sense:

$$f^{-1}(f(x)) \;=\; x \qquad \text{and} \qquad f(f^{-1}(x)) \;=\; x.$$

Simply put, the inverse of f reverses what f does to x. Likewise, f reverses the action of f^{-1}, that is, the inverse of f^{-1} is just f.

Graphically, if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}. Thus, to obtain the graph of f^{-1} from the graph of f (or vice versa), just reflect the graph of f about the line $y = x$. This is also how one obtains an analytic formula for the inverse of a function: interchange the variables x and y and solve for y.

Example 0.10 *Find the inverse of the function $f(x) = 1 + \sqrt{2 + 3x}$. What should the domain of f^{-1} be in order to have f as its inverse?*

One of the most important classes of inverses are the **logarithms**. In general, the inverse of the exponential function a^x (which is one-to-one), is called the **logarithm to the base** a, denoted as $\log_a x$. For example, since $2^5 = 32$, then $\log_2 32 = 5$. Or, since $10^{-3} = 0.001$, we have that $\log_{10} 0.001 = -3$. It helps to remember that the logarithm function outputs an exponent since the exponential function (its inverse) *inputs* an exponent. We have that

$$\log_a a^x \;=\; x \qquad \text{and} \qquad a^{\log_a x} \;=\; x \tag{1}$$

by the definition of an inverse function.

Example 0.11 *Without a calculator, compute* $\log_4 64, \log_2 \frac{1}{32}$ *and* $\log_{10} 1,000,000$. *What happens if you type* $\log 0$ *on a calculator? Explain why.*

Just as e is the most important base of the exponential functions, the inverse of e^x is the most important logarithm, called the **natural logarithm** and denoted as $\ln x$. Thus, $\ln e^4 = 4$ since the function e^x sends 4 to e^4. We have that $\ln 1 = 0$ since $e^0 = 1$. Using equation (1), we have that

$$\ln e^x \;= x \qquad \text{and} \qquad e^{\ln x} \;= x.$$

Example 0.12 *Sketch the graph of the function* e^x *and its inverse* $\ln x$ *without a calculator. What are the domain and range of* $\ln x$*?*

By inverting the rules of exponents, we obtain some important properties of logarithms. These properties are true no matter which base is used.

$$\log(cd) \;=\; \log(c) + \log(d), \quad \log(\frac{c}{d}) \;=\; \log(c) - \log(d) \quad \text{and} \quad \log(c^d) \;=\; d \log c$$

Example 0.13 *Derive the second formula on the previous line from the first and third.*