Introduction to Persistent Homology

Persistent homology applies the homology construction to the subcomplexes of a filtration of a simplicial complex.

The starting point is the induced map on homology for the inclusion of one complex in another. The next two slides review this concept.
Intuition: If $X$ is a simplicial complex and $Y$ is a subcomplex, then every cycle in $Y$ (of any dimension) is also a cycle in $X$.

This is because the property of being a cycle is about how the simplices in the cycle fit together.

Example: $z$ is a cycle no matter where $A$ is located.

$dz = 0$ in each case.
Suppose $Y$ is a subcomplex of $X$. Denote the simplicial map that maps each simplex of $Y$ to itself in $X$ by $\text{In}: Y \to X$.

$\exists$:

\[ Y \xrightarrow{\text{In}} X \]

$H_1(Y) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Cycles in $Y$ can become boundaries.

Induced Map:

$[\mathbb{Z}_2] \to [\mathbb{Z}_2]$  

$[\mathbb{Z}_2] \to [0]$ (a boundary) 

$X$ can contain cycles not in $Y$. 

\[ H_1(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \]
If $Y$ is a subcomplex of $X$, $\text{In}: Y \to X$, we have induced maps $\text{In}_*: \quad$

$\implies \quad \text{In}_*: H_0(Y) \to H_0(X)$

$\implies \quad \text{In}_*: H_1(Y) \to H_1(X)$

$\implies \quad \text{In}_*: H_2(Y) \to H_2(X)$

$\vdots$
If $X$ is a simplicial complex with filtration

$$X_1 \subset X_2 \subset X_3 \subset \ldots \subset X_n = X$$

We have inclusion maps

$$X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_i \hookrightarrow X_{i+1} \hookrightarrow \ldots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

And induced maps for each dimension $j$

$$\tilde{H}_j(X_1) \to \tilde{H}_j(X_2) \to \ldots \to \tilde{H}_j(X_i) \to \tilde{H}_j(X_{i+1}) \to \ldots \to \tilde{H}_j(X_{n-1}) \to \tilde{H}_j(X)$$
From cycles to bars in a bar code

\[ \cdots \rightarrow H_j(X_{i-1}) \rightarrow H_j(X_i) \rightarrow \cdots \rightarrow H_j(X_{k-1}) \rightarrow H_j(X_k) \rightarrow \cdots \]

\[ \begin{array}{c}
\Rightarrow \quad \text{In}_{k-1}^i \\
\Rightarrow \quad [z] \\
\Rightarrow \quad [z] \text{ is created}
\end{array} \]

\[ \begin{array}{c}
\Rightarrow \quad [z] \not\in \text{Im} \left( \text{In}_{i-1}^i \right) \\
\Rightarrow \quad [z] \in \ker \left( \text{In}_{k-1}^i \right)
\end{array} \]

\[ [z] \text{ is born at time } i \quad [z] \text{ dies at time } k \]