Theorems of the Propositional Calculus

Equivalence and true

(3.1) **Axiom, Associativity of** $\equiv$**:** $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of** $\equiv$**:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of** $\equiv$**:** $\mathit{true} \equiv q \equiv q$

(3.4) true

(3.5) **Reflexivity of** $\equiv$**:** $p \equiv p$

Negation, inequivalence, and false

(3.8) **Axiom, Definition of** false**:** $\mathit{false} \equiv \neg \mathit{true}$

(3.9) **Axiom, Distributivity of** $\neg$ **over** $\equiv$**:** $\neg(p \equiv q) \equiv \neg p \equiv q$

(3.10) **Axiom, Definition of** $\not\equiv$**:** $(p \not\equiv q) \equiv \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of** false**:** $\neg \mathit{false} \equiv \mathit{true}$

(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv \mathit{false}$

(3.16) **Symmetry of** $\not\equiv$**:** $(p \not\equiv q) \equiv (q \not\equiv p)$

(3.17) **Associativity of** $\not\equiv$**:** $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$

Disjunction

(3.24) **Axiom, Symmetry of** $\vee$**:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of** $\vee$**:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of** $\vee$**:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of** $\vee$ **over** $\equiv$**:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

(3.29) **Zero of** $\vee$**:** $p \vee \mathit{true} \equiv \mathit{true}$

(3.30) **Identity of** $\vee$**:** $p \vee \mathit{false} \equiv p$

(3.31) **Distributivity of** $\vee$ **over** $\vee$**:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction

(3.35) **Axiom, Golden rule:** $p \wedge q \equiv p \equiv q \equiv p \vee q$

(3.36) **Symmetry of** $\wedge$**:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of** $\wedge$**:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of** $\wedge$**:** $p \wedge p \equiv p$

(3.39) **Identity of** $\wedge$**:** $p \wedge \mathit{true} \equiv p$

(3.40) **Zero of** $\wedge$**:** $p \wedge \mathit{false} \equiv \mathit{false}$

(3.41) **Distributivity of** $\wedge$ **over** $\wedge$**:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv \mathit{false}$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of** $\vee$ **over** $\wedge$**:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of** $\wedge$ **over** $\vee$**:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) **Definition of** $\equiv$**:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

Implication

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) **Definition of implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Definition of implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of** $\Rightarrow$ **over** $\equiv$**:** $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv \mathit{true}$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

(3.71) **Reflexivity of** $\Rightarrow$**:** $p \Rightarrow p \equiv true$

(3.72) **Right zero of** $\Rightarrow$**:** $p \Rightarrow true \equiv true$

(3.73) **Left identity of** $\Rightarrow$**:** $true \Rightarrow p \equiv p$

(3.74) $p \Rightarrow false \equiv \neg p$

(3.75) $false \Rightarrow p \equiv true$

(3.76) **Weakening/strengthening:** (a) $p \Rightarrow p \vee q$

(b) $p \wedge q \Rightarrow p$

(c) $p \wedge q \Rightarrow p \vee q$

(d) $p \vee (q \wedge r) \Rightarrow p \vee q$

(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:** (a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Leibniz as an axiom

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E_e^z = E_f^z$

(3.84) **Substitution:** (a) $(e = f) \wedge E_e^z \equiv (e = f) \wedge E_f^z$

(b) $(e = f) \Rightarrow E_e^z \equiv (e = f) \Rightarrow E_f^z$

(c) $q \wedge (e = f) \Rightarrow E_e^z \equiv q \wedge (e = f) \Rightarrow E_f^z$

(3.85) **Replace by** $true$**:** (a) $p \Rightarrow E_p^z \equiv p \Rightarrow E_{true}^z$

(b) $q \wedge p \Rightarrow E_p^z \equiv q \wedge p \Rightarrow E_{true}^z$

(3.86) **Replace by** $false$**:** (a) $E_p^z \Rightarrow p \equiv E_{false}^z \Rightarrow p$

(b) $E_p^z \Rightarrow p \vee q \equiv E_{false}^z \Rightarrow p \vee q$

(3.87) **Replace by** $true$**:** $p \wedge E_p^z \equiv p \wedge E_{true}^z$

(3.88) **Replace by** $false$**:** $p \vee E_p^z \equiv p \vee E_{false}^z$

(3.89) **Shannon:** $E_p^z \equiv (p \wedge E_{true}^z) \vee (\neg p \wedge E_{false}^z)$

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of** $\vee$**:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of** $\wedge$**:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof techniques

(4.4) **Deduction:** To prove $P \Rightarrow Q$, assume P and prove Q.

(4.5) **Case analysis:** If E_{true}^z, E_{false}^z are theorems, then so is E_P^z.

(4.6) **Case analysis:** $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$

8.24: b≤c≤d ⇒ (b≤i<d ≡ b≤i<c ∨ c≤i<d)

General Laws of Quantification

either with b,c,d:=0,n,n+1
or with b,c,d:=0,1,n+1

For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$

(8.14) **Axiom, One-point rule:** Provided $\neg occurs('x', 'E')$,
$(\star x \mid x = E : P) = P[x := E]$

(8.15) **Axiom, Distributivity:** Provided $P, Q{:}\mathbb{B}$ or R is finite,
$(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

(8.16) **Axiom, Range split:** Provided $R \wedge S \equiv false$ and $P{:}\mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.17) **Axiom, Range split:** Provided $P{:}\mathbb{B}$ or R and S finite,
$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.18) **Axiom, Range split for idempotent** $\star$**:**
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.19) **Axiom, Interchange of dummies:** Provided $\star$ is idempotent or R, Q are finite, $\neg occurs('y', 'R')$, and $\neg occurs('x', 'Q')$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

(8.20) **Axiom, Nesting:** Provided $\neg occurs('y', 'R')$,
$(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$

(8.21) **Axiom, Dummy renaming:** Provided $\neg occurs('y', 'R, P')$,
$(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$

(8.22) **Change of dummy:** Provided $\neg occurs('y', 'R, P')$, and f has an inverse, $(\star x \mid R : P) = (\star y \mid R[x := f.y] : P[x := f.y])$

(8.23) **Split off term:** $(\star i \mid 0 \le i < n + 1 : P) = (\star i \mid 0 \le i < n : P) \star P_n^i$ or
(⋆i|0 ≤ i < n+1:P) = P[i:=0] ⋆ (⋆i|0 < i < n+1:P)

Theorems of the Predicate Calculus

Universal quantification

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.3) **Trading:** (a) $(\forall x \mid R : P) \equiv (\forall x \mid : \neg R \vee P)$
(b) $(\forall x \mid R : P) \equiv (\forall x \mid : R \wedge P \equiv R)$
(c) $(\forall x \mid R : P) \equiv (\forall x \mid : R \vee P \equiv P)$

(9.4) **Trading:** (a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$
(b) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$
(c) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$
(d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$

(9.5) **Axiom, Distributivity of $\vee$ over $\forall$:** Prov. $\neg occurs(`x', `P')$,
$P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

(9.6) Provided $\neg occurs(`x', `P')$, $(\forall x \mid R : P) \equiv P \vee (\forall x \mid : \neg R)$

(9.7) **Distributivity of $\wedge$ over $\forall$:** Provided $\neg occurs(`x', `P')$,
$\neg(\forall x \mid : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$

(9.8) $(\forall x \mid R : true) \equiv true$

(9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$

(9.10) **Range weakening/strengthening:** $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$

(9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$

(9.12) **Monotonicity of $\forall$:**
$(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$

(9.13) **Instantiation:** $(\forall x \mid : P) \Rightarrow P[x := e]$

(9.16) P is a theorem iff $(\forall x \mid : P)$ is a theorem.

EXISTENTIAL QUANTIFICATION

(9.17) **Axiom, Generalized De Morgan:**
$(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.18) **Generalized De Morgan:** (a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$
(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$
(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.20) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$

(9.21) **Distributivity of $\wedge$ over $\exists$:** Provided $\neg occurs(`x', `P')$,
$P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$

(9.22) Provided $\neg occurs(`x', `P')$, $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid : R)$

(9.23) **Distributivity of $\vee$ over $\exists$:** Provided $\neg occurs(`x', `P')$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$

(9.24) $(\exists x \mid R : false) \equiv false$

(9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$

(9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$

(9.27) **Monotonicity of $\exists$:**
$(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$

(9.28) **$\exists$-Introduction:** $P[x := E] \Rightarrow (\exists x \mid : P)$

(9.29) **Interchange of quantifications:**
Provided $\neg occurs(`y', `R')$ and $\neg occurs(`x', `Q')$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.30) Provided $\neg occurs(`\hat{x}', `Q')$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem

Table of Precedences

(a)	$[x := e]$ (textual substitution)	(highest precedence)
(b)	. (function application)	
(c)	unary prefix operators: $+ \; - \; \neg \; \# \; \sim \; \mathcal{P}$	
(d)	$**$	
(e)	$\cdot \; / \; \div$ **mod gcd**	
(f)	$+ \; - \; \cup \; \cap \; \times \; \circ \; \bullet$	
(g)	$\downarrow \; \uparrow$	
(h)	$\#$	
(i)	$\triangleleft \; \triangleright \; \hat{}$	
(j)	$= \; < \; > \; \in \; \subset \; \subseteq \; \supset \; \supseteq \; \mid$	(conjunctional, see page 29)
(k)	$\vee \; \wedge$	
(l)	$\Rightarrow \; \Leftarrow$	
(m)	$\equiv$	(lowest precedence)

All nonassociative binary infix operators associate to the left, except $**$, $\triangleleft$, and $\Rightarrow$, which associate to the right.

The operators on lines (j), (l), and (m) may have a slash / through them to denote negation —e.g. $b \not\equiv c$ is an abbreviation for $\neg(b \equiv c)$.

Greek letters and their Transliterations

Name	Sign	Tr.	Name	Sign	Tr.	Name	Sign	Tr.
Alpha	α	a	Iota	ι	i	Rho	ρ	r
Beta	β	b	Kappa	κ	k	Sigma	σ	s
Gamma	γ	g	Lambda	λ	l	Tau	τ	t
Delta	δ	d	Mu	μ	m	Upsilon	υ	y, u
Epsilon	ϵ	e	Nu	ν	n	Phi	ϕ	ph
Zeta	ζ	z	Xi	ξ	x	Chi	χ	ch
Eta	η	e	Omicron	o	o	Psi	ψ	ps
Theta	θ	th	Pi	π	p	Omega	ω	o

Types Used in this Text

integer	$\mathbb{Z}$	integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
nat	$\mathbb{N}$	natural numbers: $0, 1, 2, \ldots$
positive	$\mathbb{Z}^+$	positive integers: $1, 2, 3, \ldots$
negative	$\mathbb{Z}^-$	negative integers: ...,-3,-2,-1
rational	$\mathbb{Q}$	rationals i/j for i, j integers, $j \neq 0$
reals	$\mathbb{R}$	real numbers
positive reals	$\mathbb{R}^+$	positive real numbers
bool	$\mathbb{B}$	booleans: *true*, *false*
sets	$set(t)$	set of elements of type t
bags	$bag(t)$	bag of elements of type t
sequences	$seq(t)$	sequence of elements of type t