Color Visualization of Blaschke Product Mappings

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Abstract

A visualization of Blaschke product mappings can be obtained by treating them as canonical projections of covering Riemann surfaces and finding fundamental domains and covering transformations corresponding to these surfaces. A working tool is the technique of simultaneous continuation we introduced in previous papers. Here, we are refining this technique for some particular types of Blaschke products, for which coloring pre-images of annuli centered at the origin allow us to describe the mappings with a high degree of fidelity. Additional graphics and animations are provided on the web site of the project [1].

1 Blaschke Products

The building blocks of Blaschke products are Möbius transformations of the form

$$b_k(z) = e^{i\theta_k}\frac{a_k - z}{1-\overline{a}_k z}, \tag{1}$$

where $a_k \in D := \{z \in \mathbb{C}|\ |z| < 1\}$, and $\theta_k \in \mathbb{R}$. A finite (infinite) Blaschke product has the form

$$w = B(z) = \prod_{k=1}^{n} b_k(z), \tag{2}$$

where $n \in \mathbb{N}$, (respectively $n = \infty$). In the infinite case it is customary to take $e^{i\theta_k} = \dfrac{\overline{a}_k}{|a_k|}$. It is known that if $\sum\limits_{k=1}^{\infty}(1-|a_k|) < \infty$, then the infinite product converges uniformly on compact subsets of $W = \widehat{\mathbb{C}} \setminus (A \cup E)$, where $A = \{1/\overline{a}_k\}$ and E is the set of cluster points of $\{a_k\}$. For the sake of uniformity, we will always take $\dfrac{\overline{a}_k}{|a_k|}$ for the value of $e^{i\theta_k}$ regardless whether B is finite or not.

In [2] and [3] we gave a complete description of the domains Ω_k of injectivity (fundamental domains) of the mappings when the Blaschke product is of the form:

$$w = B_a(z) = \left[\frac{\overline{a}}{|a|}\frac{a-z}{1-\overline{a}z}\right]^n \tag{3}$$

and proved the following theorem.

Theorem 1.1. *The domains Ω_k bounded by consecutive arcs of circle of the form:*

$$z_k(t) = \frac{\omega_k t - r}{\omega_k t r - 1}e^{i\theta}, \qquad t \geq 0, \quad k = 0, 1, 2, ..., n-1, \tag{4}$$

where ω_k are the n^{th} roots of unity, $t \geq 0$ and $a = re^{i\theta}$, are fundamental domains of the covering Riemann surface $(\widehat{\mathbb{C}}, B_a)$.

Actually, it can be easily seen that $z_0(t)$ is always the part of the line (generalized circle) determined by a and $1/\overline{a}$ from which the segment between a and $1/\overline{a}$ has been removed. Also, if n is even, then $z_{\frac{n}{2}}(t)$ is the segment between a and $1/\overline{a}$ and the arcs $z_{\frac{n}{2}+k}(t)$ are complementary to the arcs $z_k(t)$, $k = 0, 1, ..., \frac{n}{2} - 1$. All the arcs (4) have the end points in a and $1/\overline{a}$.

If we let $a \to 0$, then $1/\overline{a} \to \infty$ and all the arcs (4) become rays starting at the origin. The covering Riemann surface $(\widehat{\mathbb{C}}, B_0)$ is the well known Riemann surface of the multivalued function $w \to w^{1/n}$. We notice that this surface has two branch points: 0 and ∞, which correspond to a and $1/\overline{a}$, the branch points of $(\widehat{\mathbb{C}}, B_a)$. This last surface is the Riemann surface of the multivalued function

$$w \to e^{i\theta} \frac{r - w^{1/n}}{1 - rw^{1/n}}$$

obtained when trying to find an inverse of the function (3).

The arcs (4) have been obtained by simultaneous continuation (see [2]) of paths in $(\widehat{\mathbb{C}}, B_a)$ over the real non negative half-axis in the w-plane. We have also found (see [3]) that the cover transformations of $(\widehat{\mathbb{C}}, B_a)$ were Möbius transformations of the form:

$$T_k(z) = \frac{a(1-\omega_k) - (|a|^2 - \omega_k)z}{1 - |a|^2\omega_k - \overline{a}(1-\omega_k)z}, \qquad k = 0, 1, 2, ..., n-1. \tag{5}$$

These transformations form a cyclic group of order n with respect to composition. In fact, the composition rule is $T_k \circ T_j = T_{k+j \ (\mathrm{mod}\ n)}$, T_0 being the identity and $T_k^{-1} = T_{n-k}$. Every domain Ω_j is mapped conformally by T_k on the domain $\Omega_{j+k \ (\mathrm{mod}\ n)}$.

The mapping (3) can be visualized by drawing the pre-image of a family of circles $w = \rho e^{i\varphi}$, $\varphi \in [0, 2\pi)$, where ρ takes different constant values. This comes to solving the equation $B(z) = \rho e^{i\varphi}$. The solutions are:

$$z_k(\rho, \varphi) = e^{i\theta}[r - \rho^{1/n} e^{i(\varphi + 2k\pi)/n}]/[1 - r\rho^{1/n} e^{i(\varphi + 2k\pi/n)}], \quad k = 0, 1, ..., n-1. \tag{6}$$

We notice that, for fixed ρ, every $z_k(\rho, \varphi)$ represents an arc of circle and that for every k, $z_k(\rho, 2\pi) = z_{k+1 \ (\mathrm{mod}\ n)}(\rho, 0)$. In particular, $z_{n-1}(\rho, 2\pi) = z_0(\rho, 0)$ and therefore the union of all these arcs is a full circle, which is mapped by B on the circle $w = \rho e^{i\varphi}$. The mapping is n to 1, every one of the previous arcs being mapped bijectively on the circle $w = \rho e^{i\varphi}$. These pre-images are orthogonal to the arcs (4) and have their centers on the line determined by a and $1/\overline{a}$. An elementary computation shows that when $\rho = 1/r^n$ the pre-image of the corresponding circle is the line (generalized circle) passing through $\frac{1}{2}(a + 1/\overline{a})$ perpendicular to the line determined by a and $1/\overline{a}$. All the pre-image circles corresponding to $\rho < 1/r^n$ are on the side of the origin and those corresponding to $\rho > 1/r^n$ are on the other side of that perpendicular line. As $\rho \to 0$ these circles accumulate to a and as $\rho \to \infty$ they accumulate to $1/\overline{a}$. We obtain an almost perfect visualization of the mapping (3) by coloring a set of annuli centered at the origin of the w-plane in different colors and with saturation increasing counterclockwise and brightness increasing outward (the saturation is determined by the argument of the point and the brightness is determined by the modulus) and imposing the same color, saturation and brightness to the pre-image of every point in these annuli. The Mathematica program allows the superposition of an orthogonal mesh on the picture in the w-plane. The mesh consists of circles centered at the origin and rays starting at the origin. The pre-image of this mesh is an orthogonal mesh in the z-plane. The colors help identify the image under B of every eye of this last mesh. Then, zooming in on the eye if necessary, we can find the corresponding points under B with high accuracy.

A variant of this ideea related to the case $n = 1$ appears in the paper *Möbius Transformations Revealed* by D.N. Arnold and J. Rogness, Notices of AMS, Vol 55, Number 10 from November 10, 2008. Our independent work has been presented at the International Conference on Complex Analysis and Related Topics, Alba Iulia, Romania, August 14-19, 2008.

Figure 1(f) shows the fundamental domains of the mapping realized by B_a for $a = 1/2 + 1/3i$ (*i.e.*, $a = re^{i\theta} \approx 0.6e^{.59i}$) and $n = 6$. We notice two unbounded fundamental domains bordered by the line $z(t) = te^{.59i}$, the two larger arcs of circle passing through a and $1/\overline{a}$, and four bounded domains bordered by the four arcs and the segment between a and $1/\overline{a}$. Each one of these six domains are mapped conformally by B on the w-plane from which the real half axis has been removed. Figures 1(a) and (b) show pre-images of annuli under B_a. Figure 1(c) zooms in on Figure 1(a) to show the behavior of B_a close to a and $1/\overline{a}$. The annuli are rendered in Figures 1 (d) and (e). Note that in all figures we show only a selection of the annuli whose pre-images are displayed. A complete collection of annuli can be viewed on the website of the project [1]. In this example we have $1/r^6 \approx 21.24$. Thus, the pre-images of circles centered at the origin and of radius less than $1/r^6$ are circles centered on the line through a and $1/\overline{a}$ situated on the side of the origin, while the pre-images of circles of radius greater than $1/r^6$ are situated on the other side of that line. Taking values of ρ between 0 and 30, we notice that indeed, in the neighborhood of $\rho = 21.24$ there is a change of convexity for the pre-images of those circles. In order to illustrate the behavior of B in the neighborhood of $1/\overline{a}$ we have drawn pre-images of annuli of very large radii. At [1] we provide an animation describing the fundamental domains of B_a depending on a and allowing a visualization of this dependence when a is moving inside the unit disc.

The next section treats a slightly different case.

2 The Case of Two Zeros of the Same Multiplicity

We study the case of

$$B(z) = b_1^n(z)b_2^n(z), \text{ where } b_j(z) = \frac{\overline{a}_j}{|a_j|}\frac{a_j - z}{1 - \overline{a}_j z}, \qquad j = 1, 2 \tag{7}$$

All solitions of the equation $B'(z) = 0$ are of order $n - 1$. The solutions inside the unit circle are a_1, a_2 and $b = re^{i\theta} = (1/a)[1 - r_1^2 r_2^2 - \sqrt{(1 - r_1^2 r_2^2)^2 - |a|^2}]$, where $a = a_1 a_2(\overline{a}_1 + \overline{a}_2) - (a_1 + a_2)$. The solution of $B'(z) = 0$ outside the unit circle is $1/\overline{b}$.

Let β be the argument of $B(b)$ and let us perform simultaneous continuation over the ray passing through $e^{i\beta}$ from all the solutions of the equation $B(z) = e^{i\beta}$, *i.e.*, let us solve the equation $B(z) = e^{i\beta}t^n$ for every $t > 0$. If we let $\lambda = e^{i\beta/n}$, it can be easily found (see also [2]) that:

Theorem 2.1. *The domains bounded by the curves:*
$z_{1,2}^{(k)}(t) =$

$$[(r_1 - \omega_k r_2 \lambda t)e^{i\theta_1} + (r_2 - \omega_k r_1 \lambda t)e^{i\theta_2} \pm e^{i(\theta_1+\theta_2)}\sqrt{\Delta_k(t)}]/2(1 - \omega_k r_1 r_2 \lambda t), \tag{8}$$

where

$$\Delta_k(t) = [(\omega_k r_1 \lambda t - r_2)e^{-i\theta_1} - (\omega_k r_2 \lambda t - r_1)e^{-i\theta_2}]^2 + 4\omega_k(1 - r_1^2)(1 - r_2^2)\lambda t e^{-i(\theta_1+\theta_2)}$$

are fundamental domains of $(\widehat{\mathbb{C}}, B)$, *where* B *is given by* (7).

We notice that for every $k = 0, 1, ..., n-1$ we have $z_1^{(k)}(0) = a_1$, $\lim_{t\to\infty} z_1^{(k)}(t) = 1/\overline{a}_1$, $z_2^{(k)}(0) = a_2$, $\lim_{t\to\infty} z_2^{(k)}(t) = 1/\overline{a}_2$. An easy computation shows that for $t > 0$ the numbers $z_{1,2}^{(k)}(t)$ are all distinct except for the case when $z_1^{(k)}(t) = b$ and, therefore, $z_2^{(k)}(t) = 1/\overline{b}$. Thus, the curves (8) can only have in common the points a_j, respectively $1/\overline{a}_j$ and b, respectively $1/\overline{b}$ and they are mapped bijectively by B on the infinite ray passing through $e^{i\beta}$. By an obvious extension of the conformal correspondence theorem (see [4], p.154), the domains bounded by them are represented conformally by B on the complex plane from which the previously mentioned ray has been removed.

A visualization of the mapping B can be obtained as in the previous section. Namely, we plot first the curves $z_{1,2}^{(k)}(t)$ in the z-plane, $k = 0, 1, ..., n-1$. Points in concentric annuli in the w-plane and their pre-images in the z-plane are colored as described in section 1. If we denote $u_k = u_k(\rho, \varphi) = \rho^{1/n} \exp(i\frac{\varphi+2k\pi}{n})$, then the equation $B(z) = \rho e^{i\varphi}$ is equivalent to the following set of equations:

$$(1 - u_k r_1 r_2)e^{-i(\theta_1+\theta_2)}z^2 + [(r_1 u_k - r_2)e^{-i\theta_1} + (r_2 u_k - r_1)e^{-i\theta_2}]z + r_1 r_2 - u_k = 0, \qquad (9)$$

$k = 0, 1, ..., n-1$, whose solutions $z_{1,2}^{(k)}(\rho, \varphi)$, $\varphi \in [0, 2\pi)$ represent for every $\rho > 0$ the pre-image of the circle $w = \rho e^{i\varphi}$, $\varphi \in [0, 2\pi)$. For $\rho \in [\rho_1, \rho_2]$, the points $z_{1,2}^{(k)}(\rho, \varphi)$ describe the pre-image of the annulus centered at the origin of the w-plane having the radii ρ_1 and ρ_2. When $\rho_1 = 0$ and ρ_2 is a small number, the pre-image of the corresponding disc will be two disc-like shaped domains around a_1 and a_2. When ρ_1 and ρ_2 are close to 1, the pre-image of the corresponding annulus will be an annulus-like shaped domain close to the unit circle.

The intersection of these pre-images with every fundamental domain are mapped conformally by B on the previously mentioned annuli. The fundamental domains are mapped conformally one on each other by the group of covering transformations of $(\widehat{\mathbb{C}}, B)$. To find these transformations we need to solve the equation $B(\zeta) = B(z)$ in terms of ζ (see [5]). This equation is equivalent to

$$\frac{a_1 - \zeta}{1 - \overline{a}_1\zeta}\frac{a_2 - \zeta}{1 - \overline{a}_2\zeta} = \frac{a_1 - z}{1 - \overline{a}_1 z}\frac{a_2 - z}{1 - \overline{a}_2 z}\omega_k, \qquad k = 0, 1, ..., n-1.$$

Each one of these second degree equations has two solutions $\zeta = S_k(z)$ and $\zeta' = T_k(z)$ and the relation between ζ and ζ' is given by:

$$\frac{a_1 - \zeta}{1 - \overline{a}_1\zeta}\frac{a_2 - \zeta}{1 - \overline{a}_2\zeta} = \frac{a_1 - \zeta'}{1 - \overline{a}_1\zeta'}\frac{a_2 - \zeta'}{1 - \overline{a}_2\zeta'},$$

which is equivalent to

$$\zeta' = U(\zeta) = \frac{A - \zeta}{1 - \overline{A}\zeta}, \quad \text{where} \quad A = \frac{a_1 a_2(\overline{a}_1 + \overline{a}_2) - (a_1 + a_2)}{|a_1 a_2|^2 - 1}$$

We notice that U does not depend on k and it is an involution. Moreover, $U \circ S_k = T_k$ and $U \circ T_k = S_k$ for every k and $U(a_1) = a_2$, $U(1/\overline{a}_1) = 1/\overline{a}_2$, $U(a_2) = a_1$, $U(1/\overline{a}_2) = 1/\overline{a}_1$. In other words, we have proved:

Theorem 2.2. *The group of covering transformations of* $(\widehat{\mathbb{C}}, B)$ *is the group generated by* S_k, $k = 0, 1, ..., n-1$, *and* U. *The composition rule is* $U \circ S_k = T_k$, $U \circ T_k = S_k$, $S_k \circ S_j = S_{k+j \pmod n}$, *particularly* $S_k^{-1} = S_{n-k}$, $T_k^{-1} = (U \circ S_k)^{-1} = S_k^{-1} \circ U = S_{n-k} \circ U$. *The identity of the group is* S_0.

The solutions of the equations (9) are functions of $u_k = u_k(\rho, \varphi)$ of the form:

$$z_{1,2}^{(k)}(\rho, \varphi) = [(r_1 - r_2 u_k)e^{i\alpha_1} + (r_2 - r_1 u_k)e^{i\alpha_2} \pm \sqrt{\Delta_k}]/2(1 - r_1 r_2 u_k), \qquad (10)$$

where

$$\Delta_k = \Delta_k(\rho,\varphi) = [(r_1 - r_2 u_k)e^{i\alpha_1} - (r_2 - r_1 u_k)e^{i\alpha_2}]^2 + 4(1-r_1^2)(1-r_2^2)u_k e^{i(\alpha_1+\alpha_2)}.$$

We notice that when $r_2 \to 1$, then $\Delta_k(\rho,\varphi) \to [(r_1 - u_k)e^{i\alpha_1} - (1 - r_1 u_k)e^{i\alpha_2}]^2$ and therefore $z_1^{(k)}(\rho,\varphi) \to e^{i\alpha_1}[r_1 - u_k(\rho,\varphi)]/[1 - r_1 u_k(\rho,\varphi)]$, which are, for constant ρ, circles orthogonal to the boundaries of the fundamental domains of $[b_1(z)]^n$. At the same time, $z_2^{(k)}(\varphi) \to e^{i\alpha_2}$ for every $k = 0, 1, ..., n-1$.

To visualize the kind of mapping a product (7) produces, we have chosen again $n = 6$, we took a_1 to be the value a from the Figure 1 and allowed a_2 to take different values in the unit disk. An animation provided on the web site [1] shows how the fundamental domains of B from Figure 1(f) change when introducing a_2 in the picture. In particular we notice that, as a_2 approaches a point $e^{i\alpha_2}$, some of these domains shrink to $e^{i\alpha_2}$, while the others take the shape of those corresponding to the Blaschke product b_1^n. We notice that those changes are important only when a_2 is introduced in an unbounded fundamental domain of B. When a_2 belongs to a bounded fundamental domain of B, they are not so important and if a_2 is close to the unit circle in such a domain, the changes are hardly noticeable. Such a situation happens usually when, for an infinite Blaschke product B, we are switching from a partial product B_n of B to the next partial product B_{n+1}. This remark suggests that Blaschke factors generated by zeros close to the unit circle do not influence too much the picture, and therefore not only (B_n) approximate uniformly B on compact subsets of $\widehat{\mathbb{C}} \setminus (A \cup E)$, but the fundamental domains of (B_n) "approximate" in turn the fundamental domains of B.

Figure 2 illustrates the Blaschke Product (7) with two zeros of order 6. We have taken $a_1 = \dfrac{1}{2} + \dfrac{1}{3}i$, $a_2 = \frac{4}{5}e^{2.5i}$ and $n = 6$. Figure 2(f) shows the fundamental domains of B. Eleven of the fundamental are bounded and only one is unbounded. The bounded domains are bordered by arcs ending in a_1 and $1/\overline{a}_1$, a_2 and $1/\overline{a}_2$ as well as b and $1/\overline{b}$, while the unbounded domain is bordered by the arcs ending in a_1 and $1/\overline{a}_1$ and b and $1/\overline{b}$. The Blaschke product B mapps conformally each one of these domains on the w-plane from which the ray passing through $B(b)$ has been removed. Figures 2(a) and (b) show the pre-images of concentric annuli. Some of the annuli are shown in Figure 2(d) and (e). A complete collection of the annuli is displayed at the web site [1]. Figure 2(c) zooms in on the pre-image of B close to a_2 and $1/\overline{a}_2$. Figures 2(g) and (h) show the fundamental domains of B above with a_2 replaced by $.99e^{2.5i}$ and $.999e^{2.5i}$ respectively.

3 The Case of Zeros of the Same Module and of Arguments $\alpha + 2k\pi/n$

Suppose that $a_k = re^{i\alpha}\omega_k$, $k = 0, 1, 2, ..., n-1$, where ω_k are the roots of order n of unity. Then $\dfrac{\overline{a}_k}{|a_k|} = e^{-i\alpha}\omega_{n-k}$. Therefore $\displaystyle\prod_{k=0}^{n-1} \frac{\overline{a}_k}{|a_k|} = (-1)^{n-1}e^{-n\alpha i}$ and

$$w = B(z) = e^{-in\alpha}\frac{z^n - r^n e^{in\alpha}}{e^{-in\alpha}r^n z^n - 1} \tag{11}$$

Theorem 3.1. *The domains bounded by consecutive rays $z_k(t) = e^{i[\alpha+(2k+1)\pi/n]}t$, $t \geq 0$ are fundamental domains of $(\widehat{\mathbb{C}}, B)$, where B is given by (11). The image by B of every one of these domains is the w-plane from which the interval $(r^n, 1/r^n)$ of the real axis has been removed. The* *covering transformations of $(\widehat{\mathbb{C}}, B)$ are rotations $T_j(z) = \omega_j z$, where ω_j are the roots of order n of unity.*

Proof: The equation $B(z) = t$, $t \geq 0$ (simultaneous continuation over the real non negative half-axis) has the solutions

$$z_k(t) = \begin{cases} [(r^n - t)/(1 - r^n t)]^{1/n} e^{i\alpha}\omega_k & \text{if } t \in [0, r^n] \cup (1/r^n, \infty) \\ [(t - r^n)\ /\ (1 - r^n t)]^{1/n} e^{i(\alpha+\pi/n)}\omega_k & \text{if } t \in [r^n, 1/r^n), \end{cases} \tag{12}$$

where $k = 0, 1, ..., n-1$

We notice that $z_k(r^n) = 0$ and $\lim\limits_{t\to 1/r^n} z_k(t) = \infty$ for every k, and as t varies between r^n and $1/r^n$ the argument of $z_k(t)$ remains the same, namely $\alpha + (2k+1)\dfrac{\pi}{n}$. Therefore, $z_k(t)$ describes the ray issued from the origin and forming the angle $\alpha + (2k+1)\frac{\pi}{n}$ with the positive real half-axis. The intersection of this ray with the unit circle is the point $\zeta_k = z_k(1) = e^{[\alpha+(2k+1)\pi/n]i}$ and it obviously does not depend on r. This means that the respective ray will remain the same as r varies between 0 and 1.

The above mentioned rays are the borders of fundamental domains Ω_k, which are mapped conformally by B on the domain obtained when removing from the w-plane the interval $[r^n, 1/r^n]$. The expressions of $z_k(t)$, for $t \in [0, r^n] \cup (1/r^n, \infty)$ show simply that the image by B of every segment between 0 and a_k is the segment $[0, r^n]$, with $B(0) = r^n$, $B(a_k) = 0$, and that of every interval $(1/a_k, \infty)$, on the ray $z(t) = e^{i\alpha}\omega_k t$, is the interval $(1/r^n, \infty)$, with $B(\infty) = 1/r^n$ and $B(1/a_k) = \infty$ for every $k = 0, 1, ..., n-1$. A small circle around the origin of the w-plane has as pre-image by B the union of n curves, each one around an a_k and situated in Ω_k. Every one of those curves is mapped by B bijectively on the respective circle. A very big circle around the origin of the w-plane has as pre-image by B the union of n small curves each one around a point $1/\overline{a}_k$. Every one of these curves is mapped by B bijectively on the respective circle. A circle $w(\tau) = \rho e^{i\tau}$, $\tau \in [0, 2\pi)$, with ρ close to 1 has as pre-image by B a unique curve close to the unit circle. Its intersection with every Ω_k, is mapped by B bijectively on the respective circle. That curve is orthogonal to every ray $z(t) = e^{i[\alpha+(2k+1)\pi/n]}t$, $t > 0$.

The domains Ω_k are mapped one on each other by the covering transformations $T_j(z) = \omega_j z$. It can be easily checked that, for every j, we have $B \circ T_j = B$, *i.e.*, T_j are indeed covering transformations. Obviously, the set of transformations $\{T_j\}$ is a cyclic group of order n for which the composition law is $T_j \circ T_k = T_{j+k \pmod n}$. Every Ω_k is mapped conformally by T_j on $\Omega_{k+j \pmod n}$.

In Figure 3 we consider the Blaschke product of oder 6 defined in (11) with zeroes $a_k = re^{i\alpha}\omega_k$, where $r = 2/3$, $\alpha = \pi/5$ and $k = 0, 1, \ldots, 5$. Figure 3(g) shows the fundamental domains of B and represents six infinite sectors bounded by the rays $t \to e^{i[\alpha+(2k+1)\pi/6]}, k = 0, 1, \ldots, 5$. Every such sector is mapped by B conformally on the w-plane from which the interval $(r^6, 1/r^6)$ has been removed. Figures 3(a) and (b) show the pre-images under B of concentric annuli. Several of these annuli have been displayed in figures 3(e) and (f). The complete collection of annuli is available at [1]. Every small annulus around the origin in the w-plane has as pre-image six annular domains, each one around an a_k. The spectrum of colors in every one of these domains coincide with that of the annulus. Every annulus of radii close to 1 has as pre-image a unique annular domain close to the unit circle. The spectrum of colors of the intersection of this annular domain with every Ω_k coincide with that of the annulus. Figure 3(c) zooms in on the pre-image of B close to one of the zeros. Figure 3(d) zooms in on the pre-image of B close to the origin.

4 Two Sets of Zeros of Arguments $\alpha_1 + 2k\pi/n$ and $\alpha_2 + 2k\pi/n$

Following the pattern of the section 2, for $0 \le r_1 \le r_2 < 1$, we consider Blaschke products having the zeros $a_k = r_1 e^{i\alpha_1}\omega_k$, and $b_k = r_2 e^{i\alpha_2}\omega_k$, namely

$$B(z) = \frac{(z^n e^{-in\alpha_1} - r_1^n)(z^n e^{-in\alpha_2} - r_2^n)}{(r_1^n e^{-in\alpha_1} z^n - 1)(r_2^n e^{-in\alpha_2} z^n - 1)} \tag{13}$$

In the following we will study in detail the case $\alpha_1 = \alpha_2 = \alpha$ and at the end of the section we will make some remarks about the general case.

Theorem 4.1. *The covering Riemann surface* $(\widehat{\mathbb{C}}, B)$, *where* B *is given by* (13), *has as fundamental domains the sectors bounded by consecutive infinite rays* $z(\tau) = e^{i(\alpha + k\pi/n)}\tau$, $\tau \ge 0$, $k = 0, 1, ..., 2n-1$. *The group of covering transformations is generated by the uniform branches of the multivalued function*

$$z \to \{[(1 + r_1^n r_2^n)z^n - (r_1^n + r_2^n)e^{in\alpha}]/[(r_1^n + r_2^n)e^{-in\alpha} z^n - (1 + r_1^n r_2^n)]\}^{1/n}$$

and the transformation $z \to e^{2\pi i/n} z$. *The respective sectors are mapped conformally by* B *on the* w*-plane from which a part of real axis has been removed.*

Proof: The equation $B(z) = t$, where B is given by (13) has $2n$ solutions, which can be written in the form:

$$z_{1,2}^{(k)}(t) = \begin{cases} e^{i\alpha}\omega_k K_{1,2}^{1/n}(t) & \text{if } K_{1,2}(t) > 0 \\ e^{i(\alpha + \pi/n)}\omega_k [-K_{1,2}(t)]^{1/n} & \text{if } K_{1,2}(t) < 0 \end{cases} \tag{14}$$

where

$$K_{1,2}(t) := \frac{(r_1^n + r_2^n)(1 - t) \pm \sqrt{\Delta(t)}}{2(1 - r_1^n r_2^n t)} \tag{15}$$

and

$$\Delta(t) = (r_1^n + r_2^n)^2 (1 - t)^2 - 4(r_1^n r_2^n - t)(1 - r_1^n r_2^n t). \tag{16}$$

We notice that $\sqrt{\Delta(1)} = 2(1 - r_1^n r_2^n)$ and hence $K_1(1) = 1$ and $K_2(1) = -1$, which implies that $z_1^{(k)}(1) = e^{i\alpha}\omega_k$ and $z_2^{(k)}(1) = e^{i(\alpha + \pi/n)}\omega_k$. Indeed, it is easy to check that $B(e^{i\alpha}\omega_k) = B(e^{i(\alpha+\pi/n)}\omega_k) = 1$. On the other hand, we have

$$z_2^{(k)}(r_1^n r_2^n) = 0 \quad \text{and} \quad z_1^{(k)}(r_1^n r_2^n) = \frac{e^{i\alpha}\omega_k (r_1^n + r_2^n)^{1/n}}{(1 + r_1^n r_2^n)^{1/n}}.$$

This shows that, as t varies from $r_1^n r_2^n$ to 1 every point $z_1^{(k)}(t)$ moves on the ray $e^{i\alpha}\omega_k\tau$, $\tau \ge 0$, from $e^{i\alpha}\omega_k (r_1^n + r_2^n)^{1/n}/(1 + r_1^n r_2^n)^{1/n}$ to the unit circle, while every point $z_2^{(k)}(t)$ moves from the origin to the unit circle on the ray $e^{i(\alpha+\pi/n)}\omega_k\tau$, $\tau \ge 0$. This time the simultaneous continuation over $w(t) = t$, $0 \le t \le 1$, does not bring us to a satisfactory situation, since not all corresponding paths meet in branch points of $(\widehat{\mathbb{C}}, B)$. These branch points can be found by solving the equation $B'(z) = 0$. The localization problem of the zeros of B'(z) will reoccur in the following sections (see [6] for more details). The solutions of the equation $B'(z) = 0$ are $c_{1,2}^{(k)} = e^{i\alpha}\omega_k \rho_{1,2}$, $k = 0, 1, ..., n-1$, where

$$\rho_{1,2} = \frac{(r_1^n r_2^n + 1 \pm \sqrt{(r_1^n r_2^n + 1)^2 - (r_1^n + r_2^n)^2})^{1/n}}{(r_1^n + r_2^n)^{1/n}}. \tag{17}$$

For every k, the points $c_1^{(k)}$ and $c_2^{(k)}$ are symmetric with respect to the unit circle, since $\rho_1\rho_2 = 1$. We notice also that $r_1 < \rho_2 < r_2$, hence $1/r_2 < \rho_1 < 1/r_1$ and that $-1 < B(c_2^{(k)}) < 0$, hence $B(c_1^{(k)}) < -1$. We conclude from here that as τ varies between $r_1$and r_2, the point $B(e^{i\alpha}\omega_k\tau)$ varies on the real negative half-axis between $B(c_2^{(k)})$ and 0. Similarly, as τ varies between $1/r_2$ and $1/r_1$, the point $B(e^{i\alpha}\omega_k\tau)$ varies on the real negative half-axis between $B(c_1^{(k)})$ and $-\infty$. Therefore the continuation should also be performed over the intervals from $B(c_2^{(k)})$ to 0, respectively from $B(c_1^{(k)})$ to $-\infty$. In other words, the image of any ray passing through $e^{i\alpha}\omega_k$ is the set obtained by removing the interval from $B(c_1^{(k)})$ to $B(c_2^{(k)})$ from the real axis.

Let us denote by Ω_k' the domains bounded by the rays

$$z = \tau e^{i(\alpha-\pi/n)}\omega_k, \tau \geq 0 \quad \text{and} \quad z = \tau e^{i\alpha}\omega_k, \tau \geq 0, \quad k = 0, 1, 2, ..., n-1$$

and by Ω_k'' the domains bounded by the rays

$$z = \tau e^{i\alpha}\omega_k, \tau \geq 0 \quad \text{and} \quad z = \tau e^{i(\alpha+\pi/n)}\omega_k, \tau \geq 0, \quad k = 0, 1, 2, ..., n-1.$$

By the conformal correspondence theorem (see [4], page 154), the domains Ω_k' and Ω_k'' are mapped conformally by B on the complex plane from which the part of the real axis between $B(c_2^{(k)})$ and $+\infty$ and between $-\infty$ and $B(c_1^{(k)})$ has been removed.

The website [1] provides several plots showing how these fundamental domains change when varying r_1 and/or r_2. We notice that the intersections $e^{i\alpha}\omega_k$ and $e^{i(\alpha+\pi/n)}\omega_k$ of the above mentioned rays with the unit circle remain the same, *i.e.*, they do not depend on the particular values of r_1 and r_2, but only on α and n. On the other hand, if we let just one point a_k move on the ray $e^{i\alpha}\omega_k\tau$, then all these intersections will suffer perturbations.

Notice also that $B(\omega_k z) = B(z)$ and therefore $T_k(z) = \omega_k z$ are covering transformations of the covering Riemann surface $(\widehat{\mathbb{C}}, B)$. These are not the only such transformations. In order to find all the covering transformations of $(\widehat{\mathbb{C}}, B)$ we need to find, as in the previous section, ζ as a function of z for which $B(\zeta) = B(z)$. It can be easily checked that this happens if and only if either $\zeta^n = z^n$ (hence $\zeta = \omega_k z$), or

$$\zeta^n = \frac{(1 + r_1^n r_2^n)z^n - (r_1^n + r_2^n)e^{in\alpha}}{(r_1^n + r_2^n)e^{-in\alpha}z^n - (1 + r_1^n r_2^n)} \tag{18}$$

For $z = \rho e^{i(\alpha\pm\pi/n)}\omega_k$, $\rho \geq 0$, we have $z^n = -\rho^n e^{in\alpha}$ and the expression (18) becomes

$$\zeta^n = \frac{([(1 + r_1^n r_2^n)\rho^n + (r_1^n + r_2^n)]e^{in\alpha}}{(r_1^n + r_2^n)\rho^n + (1 + r_1^n r_2^n)}$$

with the solutions

$$\zeta_j(\rho) = \left(\frac{(1 + r_1^n r_2^n)\rho^n + (r_1^n + r_2^n)}{(r_1^n + r_2^n)\rho^n + (1 + r_1^n r_2^n)}\right)^{1/n} e^{i\alpha}\omega_j, \tag{19}$$

where $j = 0, 1, 2, ..., n-1$.

For $z = \rho e^{i\alpha}\omega_k$, $\rho \geq 0$, we have $z^n = \rho^n e^{in\alpha}$ and the expression (18) becomes

$$\zeta^n = \frac{[(1 + r_1^n r_2^n)\rho^n - (r_1^n + r_2^n)]e^{in\alpha}}{(r_1^n + r_2^n) - (1 + r_1^n r_2^n)}$$

with the solutions

$$\zeta_j'(\rho) = \left(\frac{(1 + r_1^n r_2^n)\rho^n - (r_1^n + r_2^n)}{(r_1^n + r_2^n)\rho^n - (1 + r_1^n r_2^n)}\right)^{1/n} e^{i\alpha}\omega_j, \quad j = 0, 1, 2, ..., n-1, \tag{20}$$

if $\rho \in (0,\ [r_1^n + r_2^n]^{1/n}\ /\ [1 + r_1^n r_2^n]^{1/n}) \cup ([1 + r_1^n r_2^n]^{1/n} /\ [r_1^n + r_2^n]^{1/n}, +\infty)$, and

$$\zeta'_j(\rho) = \frac{-(1 + r_1^n r_2^n)\rho^n + (r_1^n + r_2^n)}{[(r_1^n + r_2^n)\rho^n - (1 + r_1^n r_2^n)]^{1/n}} e^{i(\alpha+\pi/n)}\omega_j \quad j = 0, 1, 2, ..., n-1, \tag{21}$$

if $\rho \in ([r_1^n + r_2^n]^{1/n}\ /\ [1 + r_1^n r_2^n]^{1/n}, [1 + r_1^n r_2^n]^{1/n} /\ [r_1^n + r_2^n]^{1/n})$.

If we choose the uniform branches $S_j(z)$ of the multivalued function

$$\left(\frac{(1 + r_1^n r_2^n)z^n - (r_1^n + r_2^n)e^{in\alpha}}{(r_1^n + r_2^n)e^{-in\alpha}z^n - (1 + r_1^n r_2^n)} \right)^{1/n} \tag{22}$$

such that $S_j(\rho e^{i(\alpha-\pi/n)}\omega_k) = \zeta_{j+k}(\rho)$, then S_j maps conformally the interior of every Ω'_k on the interior of Ω''_{k+j} and the interior of every Ω''_k on the interior of $\Omega'_{k+j+1.}$ Indeed, we notice that

$$S_j(0) = \zeta_{j+k}(0) = \frac{[r_1^n + r_2^n]^{1/n} e^{i\alpha}\omega_{j+k}}{[1 + r_1^n r_2^n]^{1/n}} = \zeta'_{j+k}(0) \tag{23}$$

and

$$S_j(\infty) = \zeta_{j+k}(\infty) = \frac{[1 + r_1^n r_2^n]^{1/n} e^{i\alpha}\omega_{j+k}}{r_1^n + r_2^n} = \zeta'_{j+k}(\infty) \tag{24}$$

As z moves on the ray $z(\rho) = \rho e^{i(\alpha-\pi/n)}\omega_k$, $\rho \geq 0$ from 0 to ∞, $S_j(z(\rho))$ moves on the ray $z(\tau) = \tau e^{i\alpha}\omega_{j+k}$ between the two values $\zeta_{j+k}(0)$ and $\zeta_{j+k}(\infty)$. Also, as z varies on the ray $z(\rho) = \rho e^{i\alpha}\omega_k$ from 0 to $\zeta_k(0)$ and from $\zeta_k(\infty)$ to ∞, $S_j(z(\rho))$ varies from $\zeta_{j+k}(0)$ to 0, respectively from ∞ to $\zeta_{j+k}(\infty)$. In other words, the ray $z(\tau) = \tau e^{i\alpha}\omega_{j+k}$, $\tau \geq 0$ is the bijective image by S_j of the path formed with the ray $z(\rho) = \rho e^{i(\alpha-\pi/n)}\omega_k$ and the two intervals from 0 to $\zeta_k(0)$ and from $\zeta_k(\infty)$ to ∞ of the ray $z(\rho) = \rho e^{i\alpha}\omega_k$. On the other hand, if we let z vary on the ray $z(\rho) = \rho e^{i\alpha}\omega_k$ between $\zeta_k(0)$ and $\zeta_k(\infty)$, $S_j(z(\rho))$ describes the ray $z(\tau) = \tau e^{i(\alpha+\pi/n)}\omega_{j+k}$. The fact that the mappings we mentioned above are conformal is again a corollary of the conformal correspondence theorem. Moreover, it is obvious that every mapping S_j is conformal throughout, except for the points

$$0, \quad \left(\frac{r_1^n + r_2^n}{1 + r_1^n r_2^n} \right)^{1/n} e^{i\alpha}\omega_k, \quad \left(\frac{1 + r_1^n r_2^n}{r_1^n + r_2^n} \right)^{1/n} e^{i\alpha}\omega_k, \quad \text{and} \quad \infty.$$

It can be easily checked that $B(S_j(z)) = B(z)$ for every j and hence S_j are covering transformations of $(\widehat{\mathbb{C}}, B)$.

In order to find the composition law of the mappings S_j, we notice that the ray $z(\rho) = \rho e^{i(\alpha-\pi/n)}\omega_k$ is mapped bijectively by $S_{j'}$ on the interval between $S_{j'}(0)$ and $S_{j'}(\infty)$ on the ray $z(\tau) = \tau e^{i\alpha}\omega_{k+j'}$, which is mapped in turn bijectively by S_j on the ray $z(t) = te^{i\alpha}\omega_{k+j'+j}$. Therefore, $S_j \circ S_{j'}$ maps bijectively the ray $z(\rho) = \rho e^{i(\alpha-\pi/n)}\omega_k$, $\rho \geq 0$ on the ray $z(t) = te^{i\alpha}\omega_{k+j'+j}$ with $S_j \circ S_{j'}(0) = 0$ and $S_j \circ S_{j'}(\infty) = \infty$. In a similar way we find that the ray $z(\rho) = \rho e^{i\alpha}\omega_k$, $\rho \geq 0$ is mapped bijectively by $S_j \circ S_{j'}$ on the ray $z(t) = te^{i(\alpha+\pi/n)}\omega_{k+j'+j}$, $t \geq 0$, which implies that the domain Ω'_k is mapped conformally by $S_j \circ S_{j'}$ on the domain $\Omega'_{k+j'+j}$ with $S_j \circ S_{j'}(0) = 0$, $S_j \circ S_{j'}(\infty) = \infty$ and the mapping is conformal on the closure of Ω'_k, except perhaps in 0 and ∞. In a similar way we find that $S_j \circ S_{j'}$ maps conformally Ω''_k on $\Omega''_{k+j'+k}$ and this happens for every k. It is a simple fact that such a transformation must coincide with $z \to \omega_{j'+j}z$. In other words $S_j \circ S_{j'} = T_{j+j'}$. It is also obvious that $S_j \circ T_k = T_k \circ S_j = S_{j+k}$.

Figure 4 illustrates the Blaschke product defined by (13) with $n = 4$, $\alpha = \pi/3$, $r_1 = 3/5$ and $r_2 = 4/5$. Thus, $a_k = 3/5e^{\pi/3}\omega_k$ and $b_k = 4/5e^{\pi/3}\omega_k$, $k = 0, 1, 2, 3$. Figure 4(g) shows the fundamental domains of B. It represent the unit circle and 8 infinite rays passing through

$e^{i(\alpha+k\pi/4)}, k = 0, 1, ..., 7$. The points a_j, b_j and $c_{1,2}^{(j)}$, are on the ray $z \to e^{i(\alpha+j\pi/2)}z$, $j = 0, 1, 2, 3$. Figures 4(a) and (b) show the pre-images under B of concentric annuli. Several of these annuli have been displayed in figures 4(c) and (d). The complete collection of annuli is available at [1]. Figures 4(c) and (d) zoom in on the pre-image plot close to the zeros in the second quadrant inside and outside the unit circle respectively.

The study of the general case when α_1 and α_2 are arbitrary is more laborious, yet still feasible. The images by B of the branch points are no longer real if $\alpha_1 \neq \alpha_2$ and the simultaneous continuation should be performed over a path passing through those images. Moreover, the equation for the branch points is more complicated. We present below the final results of our computation. With the notations $a = r_1 e^{i\alpha_1}$ and $b = r_2 e^{i\alpha_2}$ the equation $B'(z) = 0$ becomes

$$z^{n-1}[\overline{A}z^{2n} - 2z^n + A] = 0, \tag{25}$$

where

$$A = \frac{a^n + b^n - a^n b^n (\overline{a}^n + \overline{b}^n)}{1 - |a|^{2n}|b|^{2n}}. \tag{26}$$

The origin is a branch point of order n (and so is $z = \infty$). There are $2n$ more distinct branch points. If we denote by b_k the non zero solutions of the above equation situated in the unit disc, then $1/\overline{b}_k,\ k = 0, 1, ..., n-1$ are also solutions and they are situated in the exterior of the unit disk. Let β be the argument of $B(b_k)$, which does not depend on k, and it is the same as the argument of $B(1/\overline{b}_k)$. We need to perform simultaneous continuation over the ray $w = e^{i\beta}t$, for t between 1 and $|B(b_k)|$ and then over the segment between $B(b_k)$ and $B(0)$. After reflection in the unit circle, the curves obtained will give us the boundaries of the fundamental domains of B. It is obvious that $B(\omega_k z) = B(z)$ for every n-th root of unity ω_k, hence $T_k(z) = \omega_k z$ are cover transformations of the covering Riemann surface $(\widehat{\mathbb{C}}, B)$. On the other hand, solving the equation $B(\zeta) = B(z)$ in terms of ζ we obtain $\zeta = z\omega_k,\ k = 0, 1, ..., n-1$, or

$$\zeta^n = \frac{z^n - A}{\overline{A}z^n - 1}, \tag{27}$$

where A is given by (26).

Let us denote by $\zeta = S_j(z)$ the uniform branches of the above multivalued function. By direct computation, we find as expected that $B \circ S_j = B$, $j = 0, 1, ..., n-1$. If we conveniently choose the principal branch of (27) as in (22), then we have again $S_j \circ S_k = S_k \circ S_j = T_j \circ S_k = S_j \circ T_k = S_{j+k}$ for every j and k, and we can easily infer that the group generated by the transformations S_j and $z \to e^{2\pi i/n}z$ is the group of covering transformations of $(\widehat{\mathbb{C}}, B)$.

5 Infinite Blaschke Products

It is known (see [2] and [3]) that if the Blaschke product $w = B(z)$ has the (generalized) Cantor set E as the set of cluster points of its zeros, then the equation $B(z) = 1$ has infinitely many distinct solutions in every "removed" open arc $I_n \subset \partial D \setminus E$. The only cluster points of these solutions are the end points of I_n. Therefore, $I_n = \cup_{k=-\infty}^{+\infty} \Gamma_{n,k}$, where $\Gamma_{n,k}$ are half open arcs of the unit circle, which are mapped by B continuously and bijectively on the unit circle in the w-plane and they accumulate to the ends of I_n. Every fundamental domain $\Omega_{n,k}$ of B contains the interior of a unique arc $\Gamma_{n,k}$. These domains also accumulate to every point of E. Moreover, if $K \subset \mathbb{C} \setminus (A \cap E)$ is a compact set, then there is only a finite number of domains $\Omega_{n,k}$ such that $\Omega_{n,k} \cap K \neq \emptyset$. Here $A = \{1/\overline{a}_{n,k}, n \in N, k \in Z\}$ is the set of the poles of B.

We have seen in Section 2 that, for the type of Blaschke product studied there, the fundamental domains corresponding to a_2 could be made as small as we wanted if a_2 was moved close enough to the unit circle. We conjecture here that this property is true for any Blaschke product. Moreover, for an infinite Blaschke product B with E a Cantor set, if we choose the sequence of partial products $\{B_m\}$ adding always a_k with non decreasing module, then there are sequences of fundamental domains $\{\Delta_{m,j}\}$ of B_m convergent to every fundamental domain $\Omega_{n,k}$ of B. In other words, not only $\{B_m\}$ approximates B, but the fundamental domains of $\{B_m\}$ "approximate" every fundamental domain of B. Consequently, for m large enough, the picture of fundamental domains of B_m, as well as the mapping of those domains by B_m give a fairly accurate description of the first m fundamental domains of B, respectively of the mapping of those domains by B. The remaining fundamental domains of B can be enclosed in the complement of a compact set $K \subset \mathbb{C} \setminus (A \cup E)$. More exactly, if K is given, then there exists $m_0 \in \mathbb{N}$ depending only on K, such that for any $m \geq m_0$ only the first m_0 fundamental domains intersect K. Then, to characterize B, we can simply deal with a B_m for a large m. Solving, as usual, the equations $B_m(z) = 1$, and $B_m'(z) = 0$ is in principle a feasible task, and so is the corresponding simultaneous continuation.

Let us consider the example of the infinite Blaschke product B defined by the Blaschke sequence $\{a_n^{(k)}\}$, where

$$a_n^{(k)} = \left[1 - \frac{1}{(n+1)^2}\right] \omega_k, \quad k = 0, 1, 2; \quad n \in \mathbb{N}, \tag{28}$$

and ω_k are the roots of order three of unity. If we let $a_n^{(0)} = a_n$, then

$$B(z) = \prod_{n-1}^{\infty} \frac{z^3 - a_n^3}{a_n^3 z^3 - 1}.$$

We notice that for this Blaschke product we have $E = \{\omega_k,\ k = 0, 1, 2\}$. We illustrate by a picture the fact that B is fairly well approximated by the partial product having just nine factors, namely the product obtained letting $n \in \{1, 2, 3\}$, $k \in \{0, 1, 2\}$. An easy computation shows that:

$$B_9(z) = \frac{z^9 - \sigma_1 z^6 + \sigma_2 z^3 - p}{p z^9 - \sigma_2 z^6 + \sigma_1 z^3 - 1}, \tag{29}$$

where

$$\sigma_1 = a_1^3 + a_2^3 + a_3^3, \qquad \sigma_2 = a_1^3 a_2^3 + a_1^3 a_3^3 + a_2^3 a_3^3, \qquad p = a_1^3 a_2^3 a_3^3.$$

The following equations are equivalent.

$$B_9(z) = 1$$

$$(1-p)z^9 + (\sigma_2 - s_1)z^6 + (\sigma_2 - s_1)z^3 + (1-p) = 0$$

$$(z^3 + 1)[(1-p)z^6 - (1 - p + \sigma_1 - \sigma_2)z^3 + (1-p)] = 0$$

The last equation can be easily solved, and we obtain the solutions:

$$z_k^{(1)} = e^{i\pi/3}\omega_k, \qquad z_k^{(2)} = e^{-i\theta/3}\omega_k, \qquad z_k^{(3)} = e^{i\pi/3}\omega_k,$$

where ω_k are again the roots of order three of unity and $\theta = 11.5°$. It is obvious that these nine solutions are all on the unit circle, as we expected. Now we need the solutions of the equation $B_9'(z) = 0$. We notice that $b_0 = 0$ is a double solution of this equation. After the substitution $u = z^3$, the remaining solutions are those of the equation

$$\frac{d}{du}\left(\frac{u^3 - \sigma_1 u^2 + \sigma_2 u - p}{p u^3 - \sigma_2 u^2 + \sigma_1 u - 1}\right) = 0.$$

An elementary computation shows that this equation is equivalent to:

$$(s_2 - p\sigma_1)(u + 1/u)^2 + 2(p\sigma_2 - \sigma_1)(u + 1/u) + (\sigma_1 + p)^2 + 4(1 - p^2) - (1 + \sigma_2)^2 = 0$$

We notice that the images by B_9 of all these branch points are real, hence we expect to obtain fundamental domains for B_9 by performing simultaneous continuation over the real axis of the w-plane starting from the solutions of $B_9(z) = 1$. However, since the curves obtained in this way pass through $a_n^{(k)}$, they cannot border fundamental domains, given the fact that $a_n^{(k)}$ are simple zeros and therefore they should remain inside of such domains. In order to avoid this contradiction, we turn around the origin of the w-plane on a small half-circle when performing simultaneous continuation. Figure 6(a) below shows the fundamental domains of B_9 obtained in this way.

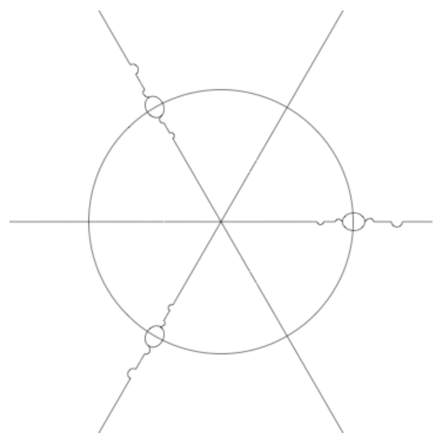


Figure 6(a)

We perform the same computation for B_{12} and we obtain Figure 6(b).

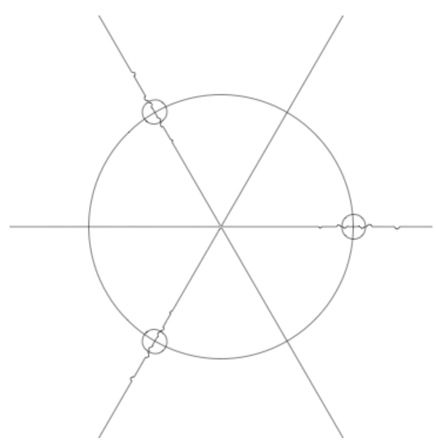
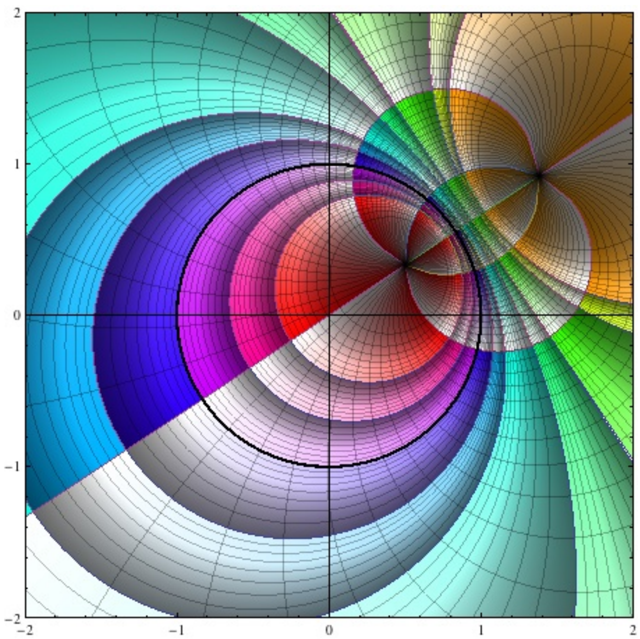


Figure 6(b)

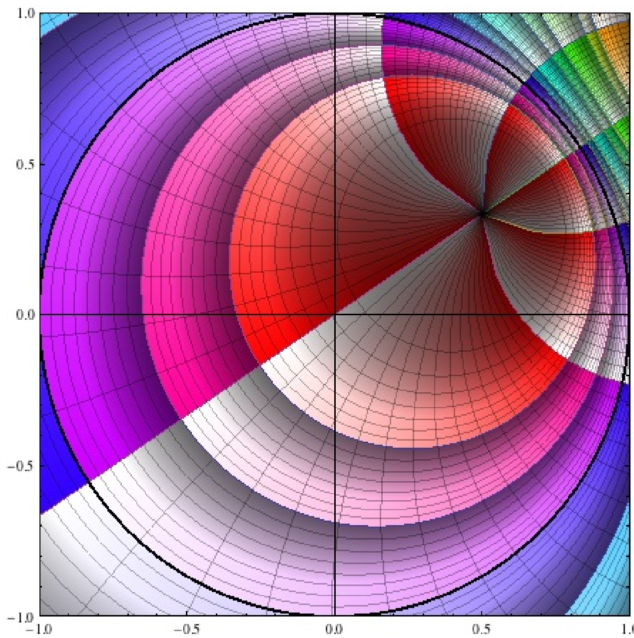
The unbounded fundamental domains remained practically the same, while every bounded domain simply split into two. If we focus just on a compact set $K \subset \mathbb{C} \setminus \cup_{k=0}^{3} D_k$, where D_k are discs of radius 0.05 centered at ω_k respectively, then, starting with a given $m = 9$, all these domains will belong to $\mathbb{C} \setminus K$ and will be out of sight. In other words, as long as we deal only with K, the finite Blaschke product B_m is the same as the infinite Blaschke product B.

Figure 5 illustrates the approximation of the infinite Blaschke product defined by the sequence (28) by the partial products B_9 and B_{12}. Figures 5(g) and (h) zoom in close to $z = 1$ on the fundamental domains of B_9 and B_{12} respectively. Figures 5(a) and (b) show the pre-images under B_9 and B_{12} respectively of concentric annuli. Several of these annuli have been displayed in figures 5(e) and (f). The complete collection of annuli is available at [1]. Figures 5(c) and (d) zoom in close to $z = 1$ on the respective pre-images given in figures 5(a) and (b). The pre-images under B_9 and B_{12} are hardly distinguishable, the difference appearing only in the zoom, where in Figure 5(d) we can notice a very small additional fundamental domain.

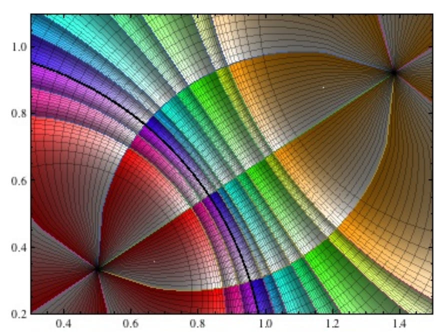
Figure 1



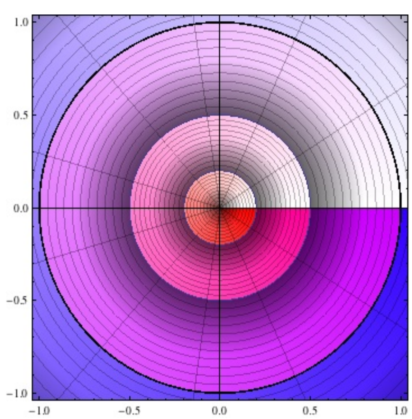
(a)



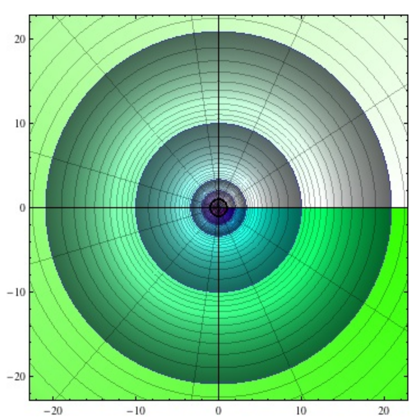
(b)



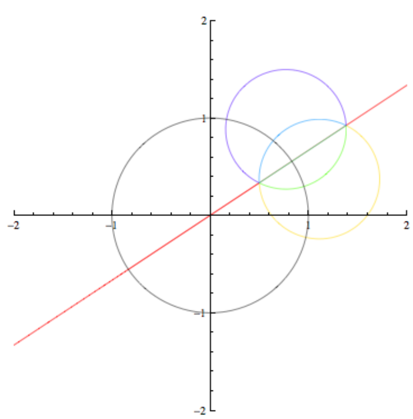
(c)



(d)

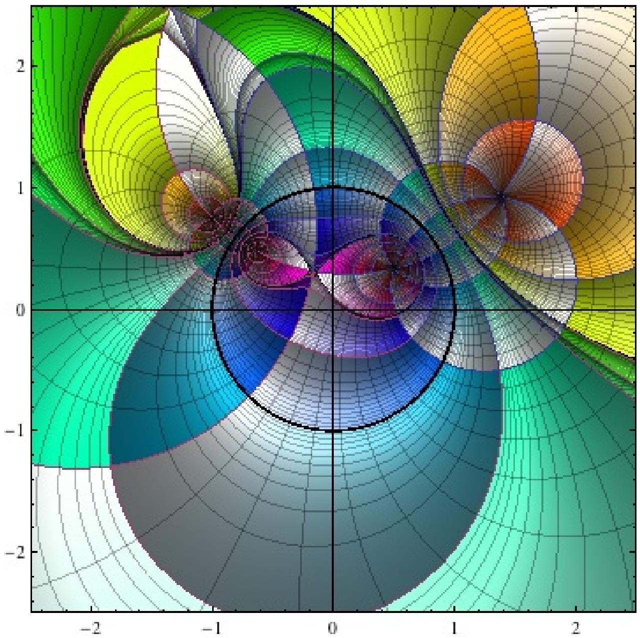


(e)

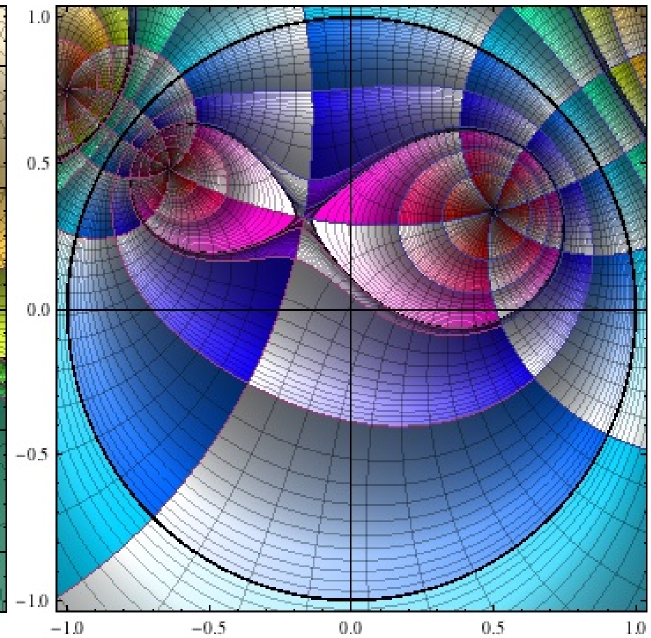


(f)

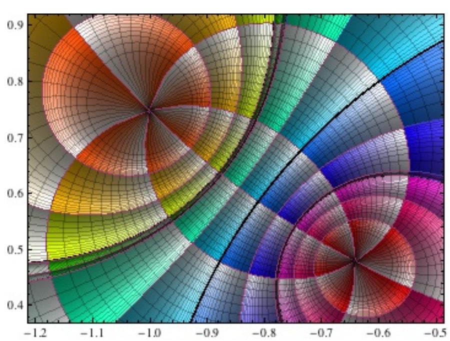
Figure 2



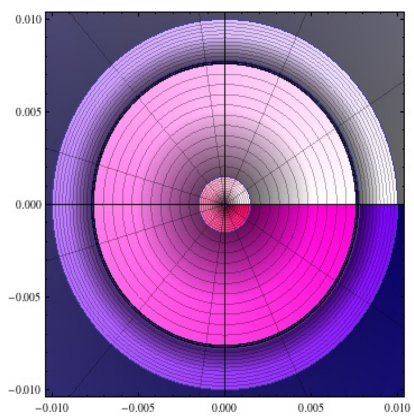
(a)



(b)



(c)



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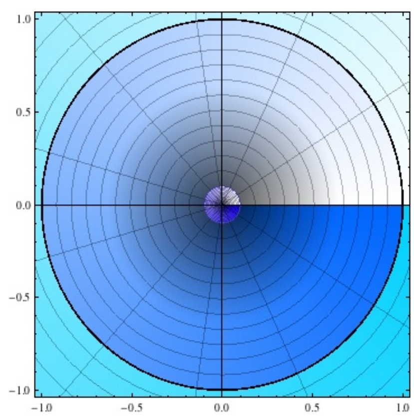

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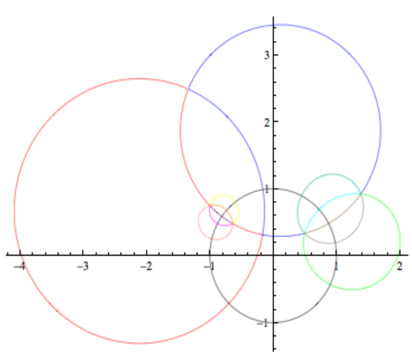

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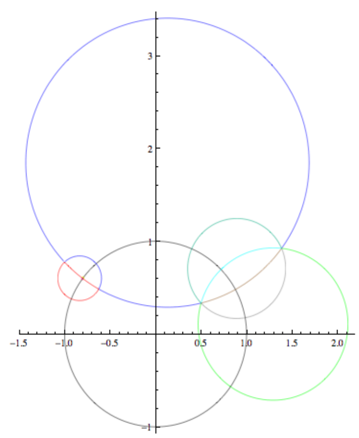

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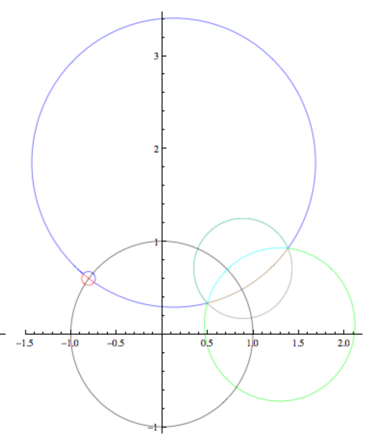

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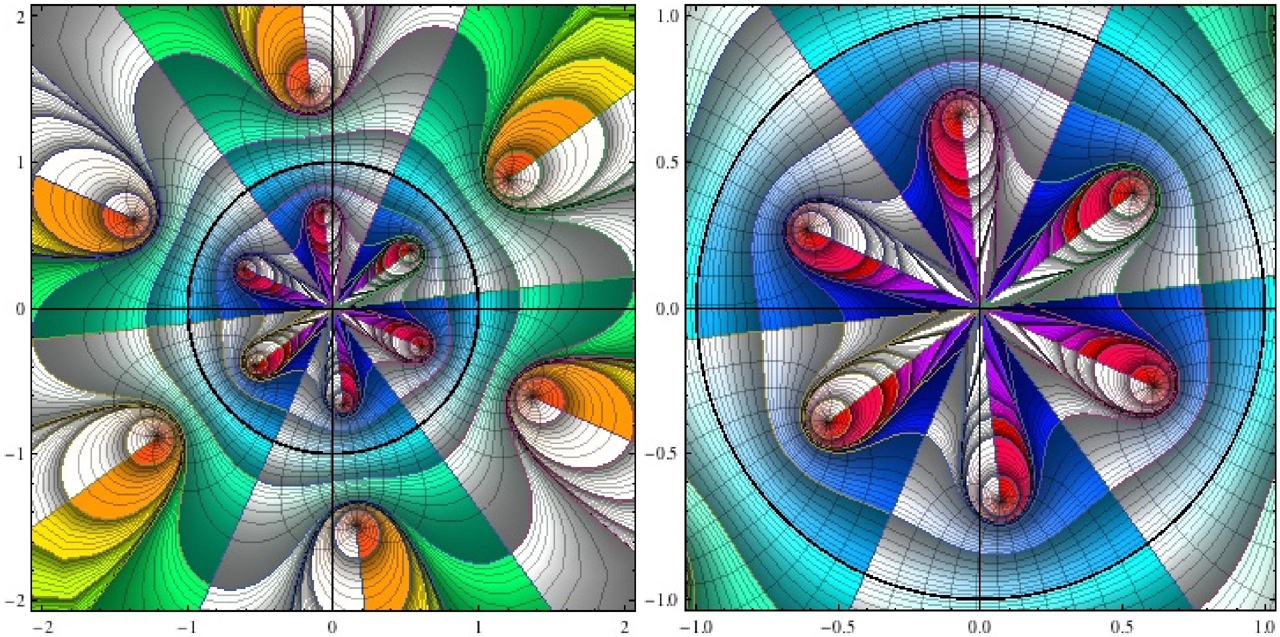
Figure 3



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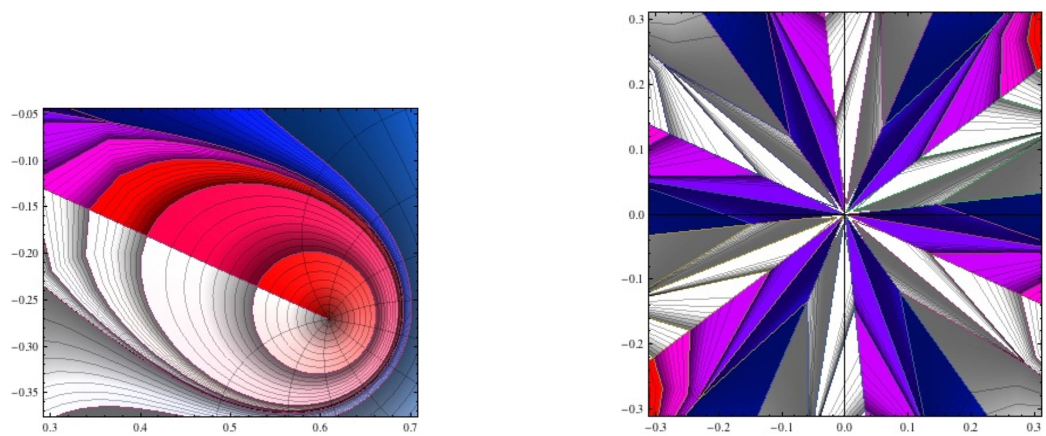

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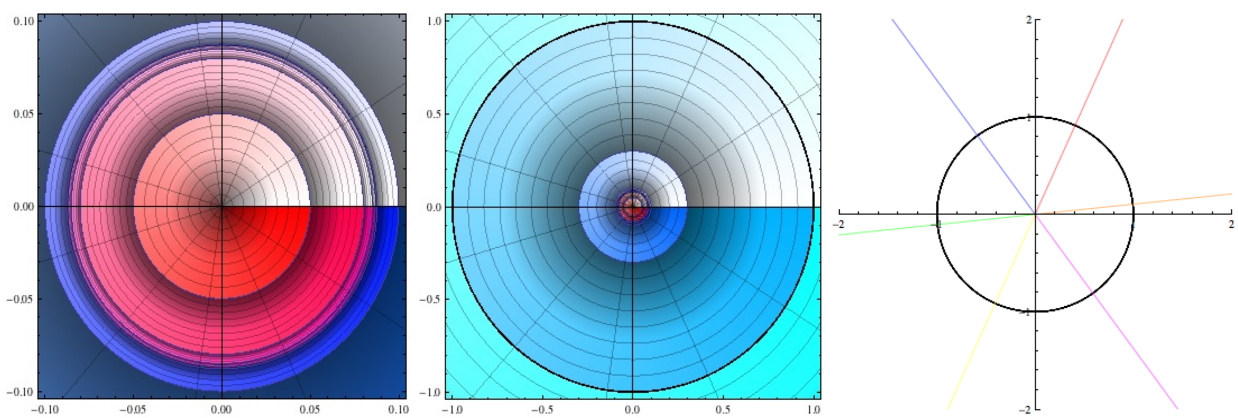

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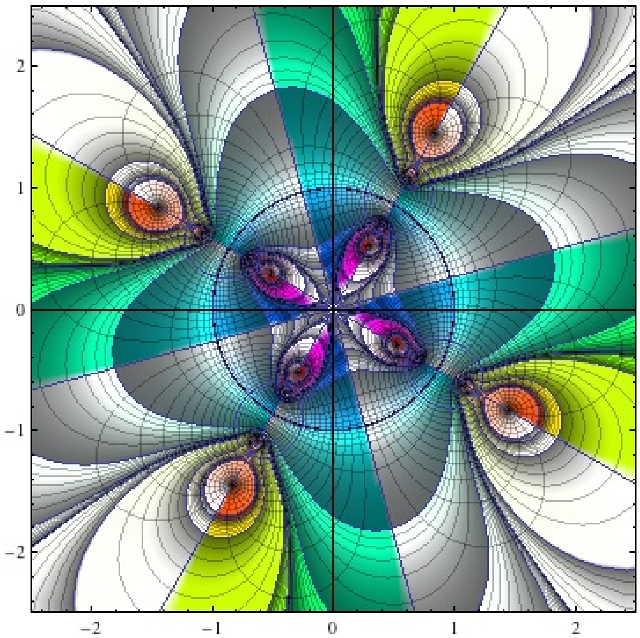
Figure 4



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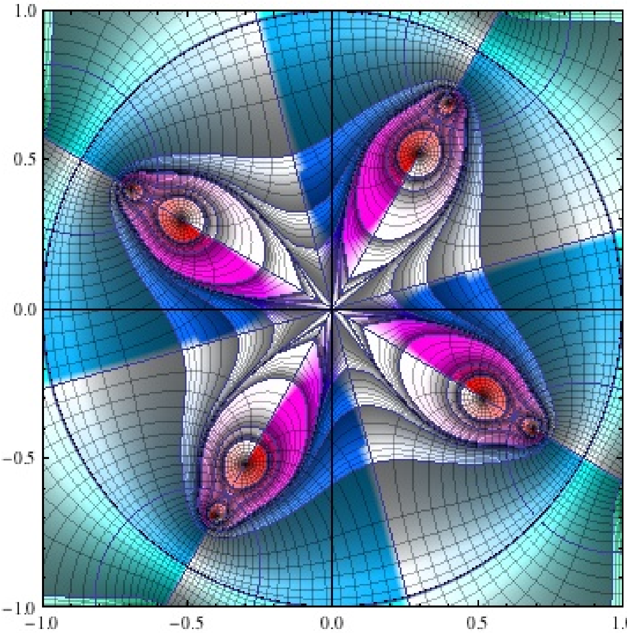

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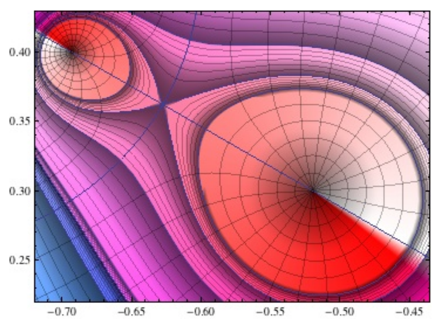

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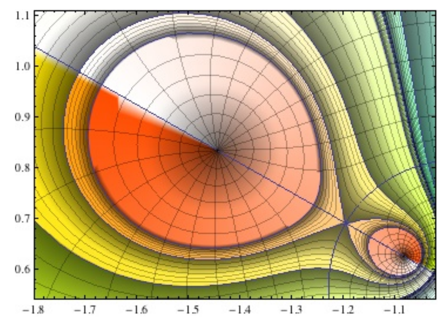

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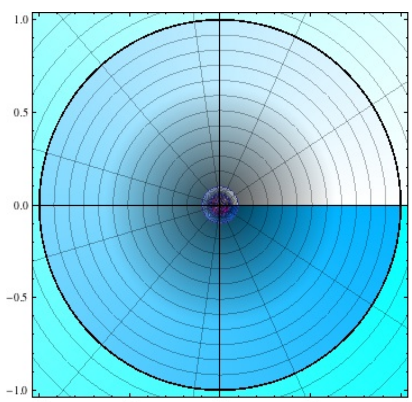

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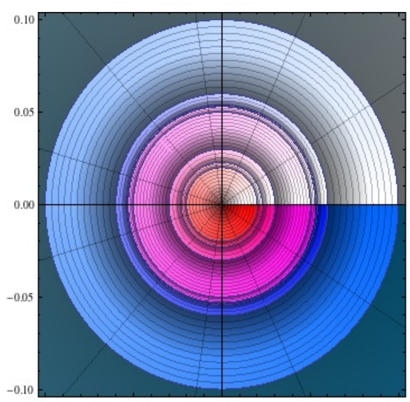

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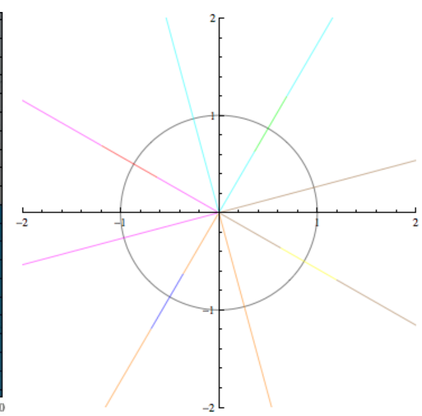

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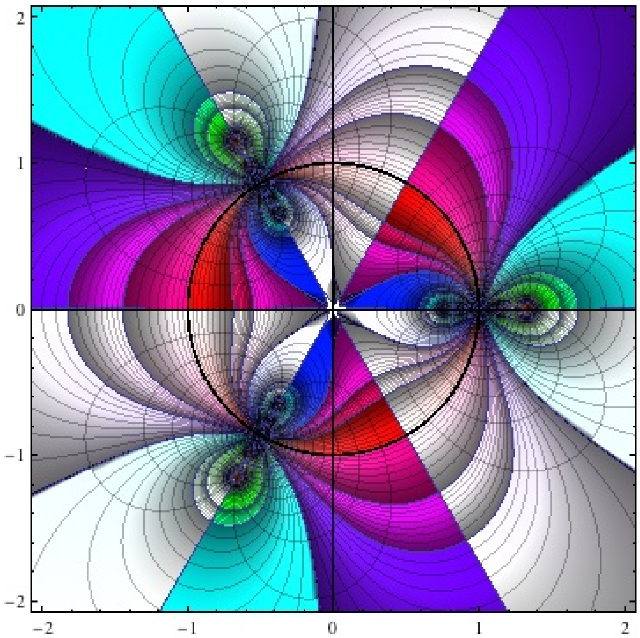
Figure 5



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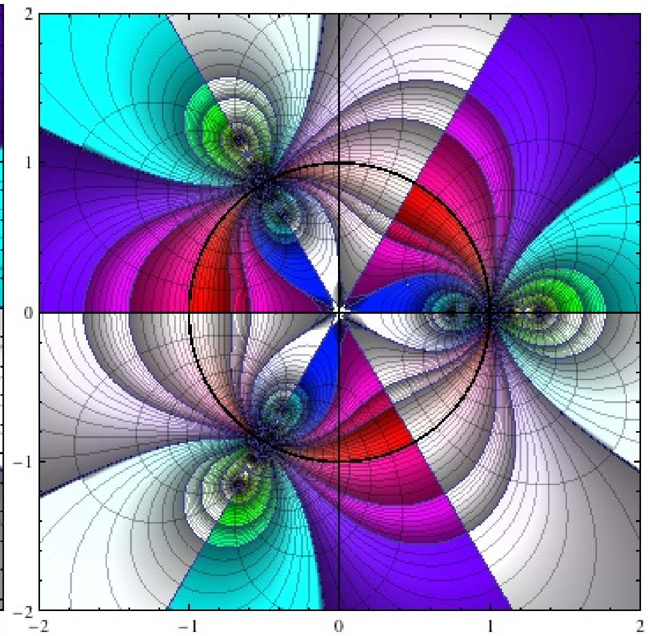

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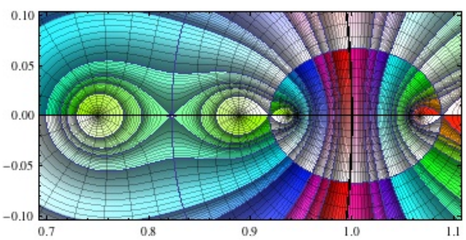

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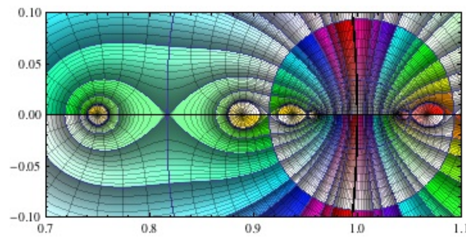

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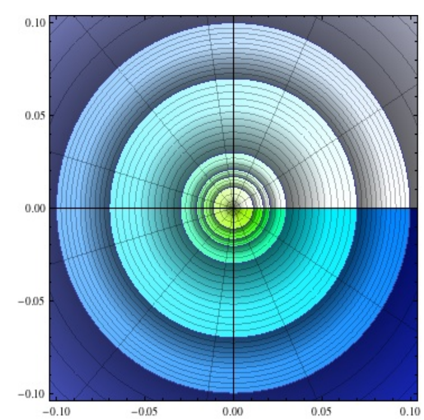

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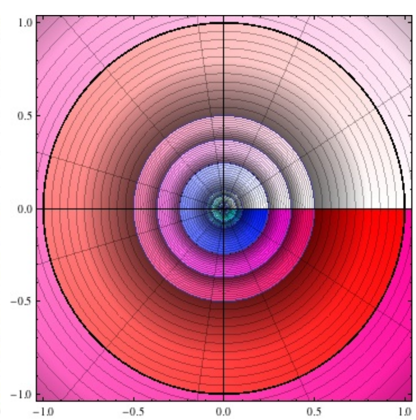

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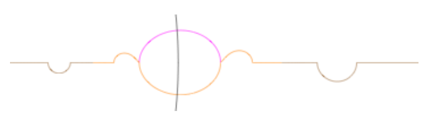

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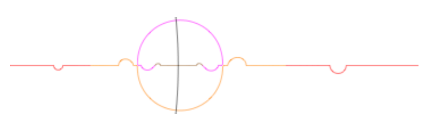

(h)

6 Technical details

All images for this article have been created using the software *Mathematica 6* on a MacBookPro with a 2.33 GHz Intel Core 2 Duo processor. Sample code is available on the website of the project [1].

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