Math 132: Calculus for Physical and Life Sciences 2
Problem Set 10 - Solutions

General Directions: You must show all work for credit on these problems.

1. The tread life (in thousands of miles) of a certain make of tire is a continuous random variable x with probability density function

$$f(x) = 0.02e^{-0.02x}$$

for $x \geq 0$.

   (a) Find the probability that a randomly selected tire of this make will have a tread life between 40,000 and 60,000 miles.
   Solution: $P(40 \leq x \leq 60) = \int_{40}^{60} 0.02e^{-0.02x}\ dx = -e^{-0.02x}|_{40}^{60} = -e^{-0.02(60)} + e^{-0.02(40)} \approx 0.15$

   (b) Find the probability that a selected tire will have a tread life of at most 30,000 miles.
   Solution: $P(0 \leq x \leq 30) = \int_{0}^{30} 0.02e^{-0.02x}\ dx = -e^{-0.02x}|_{0}^{30} = -e^{-0.02(30)} + e^{-0.02(0)} \approx 0.45$

   (c) Find the probability that a selected tire will have a tread life of at least 70,000 miles.
   Solution: $P(x \geq 70) = \int_{70}^{\infty} 0.02e^{-0.02x}\ dx = \lim_{b\to\infty} \int_{70}^{b} 0.02e^{-0.02x}\ dx = \lim_{b\to\infty} -e^{-0.02x}|_{70}^{b} = e^{-0.02(70)} \approx 0.25$

   (d) Find the mean tread life of this brand of tires.
   Solution: $E(x) = \int_0^{\infty} x(0.02e^{-0.02x})\ dx = \lim_{b\to\infty} \int_0^{b} x(0.02e^{-0.02x})\ dx = \frac{1}{0.02} = 50$
   See your class notes where we showed that the expected value of an exponential density function ke^{-kx} on $[0, \infty)$ is $E(x) = \frac{1}{k}$.

2. Suppose t is the time (in hours) it takes for calculus students to complete their final exam. Assume that all students finish within 3 hours and that the probability density function for the time t is

$$f(t) = \begin{cases} \frac{4t^3}{81} & \text{if } 0 < t < 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that $f(t)$ is a probability density function.
   Solution: $\int_0^3 \frac{4t^3}{81}\ dt = \frac{t^4}{81}|_0^3 = \frac{3^4}{81} = 1$

(b) What proportion of the students take between 1.5 and 2.5 hours to complete the exam?

Solution: $P(1.5 \leq t \leq 2.5) = \int_{1.5}^{2.5} \frac{4t^3}{81}\, dt = \frac{t^4}{81}\Big|_{1.5}^{2.5} = \frac{(2.5)^4}{81} - \frac{(1.5)^4}{81} \approx 0.42$

(c) What is the mean time for students to complete the exam?

Solution: The mean time or expected value of t is $E(t) = \int_0^3 t\frac{4t^3}{81}\, dx = \frac{4t^5}{5(81)}\Big|_0^3 = \frac{4(3)^5}{5(81)} = 2.4 = \mu$

(d) What is the median time for students to complete the exam?

Solution: The median m satisfies $P(t < m) = 0.5$. Now $P(t < m) = \int_0^m \frac{4t^3}{81}\, dt = \frac{t^4}{81}\Big|_0^m = \frac{(m)^4}{81}$. Setting this equal to 0.5 and solving for m gives the median $m = (\frac{81}{2})^{\frac{1}{4}} \approx 2.523$

(e) Find the variance and standard deviation of the random variable t associated with $f(t)$.

Solution: The variance is $Var(t) = \int_0^3 t^2\frac{4t^3}{81}\, dt - \mu^2 = \frac{4t^6}{6(81)}\Big|_0^3 - (2.4)^2 = \frac{4(3)^6}{6(81)} - (2.4)^2 \approx 0.24$.

The standard deviation is $\sigma = \sqrt{Var(t)} = \sqrt{0.24} \approx 0.49$

3. The distribution of scores on IQ exams is often modeled by a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.

(a) Give the formula for the normal probability density function that fits this description.

Solution:

$$f(x) = \frac{1}{15\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-100}{15})^2}$$

for $-\infty < x < \infty$.

(b) Estimate the fraction of the population with IQ scores between 115 and 120 by applying the Midpoint Riemann sum approximation for the appropriate integral. Use $n = 5$ subintervals in the Riemann sum.

Solution: $P(115 \leq x \leq 120) = \int_{115}^{120} f(x)\, dx$ for $f(x)$ as in part (a). This can be approximated using a midpoint Riemann sum over the interval $[115, 120]$ with $\Delta x = 1$. We get $\text{Mid}(5) = (f(115.5) + f(116.5) + f(117.5) + f(118.5) + f(119.5))(1) \approx 0.067$

(c) Estimate the fraction of the population with IQ scores between 140 and 150 by the same method as in part (b).

Solution: $P(140 \le x \le 150) = \int_{140}^{150} f(x)\,dx$ for $f(x)$ as in part (a). This can be approximated using a midpoint Riemann sum over the interval $[140, 150]$ with $\Delta x = 2$. We get $\text{Mid}(5) = (f(141)+f(143)+f(145)+f(147)+f(149))(2) \approx 0.0034$

4. Let μ and $\sigma^2 > 0$ be any two real constants.

(a) Show that the normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

has exactly one critical number at $x = \mu$, and that $f(x)$ has a local maximum at $x = \mu$.

Solution: Solving $f'(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (-2(x-\mu))/(2\sigma^2) = 0$ gives $x = \mu$. We can show that $f'(\mu - \sigma^2) > 0$ and $f'(\mu + \sigma^2) < 0$ and by the first derivative test, we see that there is a local maximum at $x = \mu$.

(b) Show that $f(x)$ has inflection points at $x = \mu + \sigma$ and $x = \mu - \sigma$.

Solution: Solving $f''(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (-\frac{1}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^4}) = 0$ gives $x = \mu \pm \sigma$. We can show that $f''(\mu \pm 2\sigma) > 0$ and $f''(\mu) < 0$. Therefore, there are points of inflection at $x = \mu \pm \sigma$.

(c) Give qualitative sketches of $y = f(x)$ with $\mu = 4$ and $\sigma^2 = 4$, and the cumulative distribution function for this normal distribution.

Solution: The graph of $f(x)$ is a bell curve which has its maximum value at $x = \mu = 4$ and inflection points at $x = \mu - \sigma = 2$ and $x = \mu + \sigma = 6$.