8 FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

8.1 Arc Length and Surface Area

Preliminary Questions

1. Which integral represents the length of the curve \( y = \cos x \) between 0 and \( \frac{\pi}{4} \)?

\[
\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos^2 x} \, dx, \quad \int_0^{\frac{\pi}{4}} \sqrt{1 + \sin^2 x} \, dx
\]

**Solution** Let \( y = \cos x \). Then \( y' = -\sin x \), and \( 1 + (y')^2 = 1 + \sin^2 x \). Thus, the length of the curve \( y = \cos x \) between 0 and \( \pi \) is

\[
\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin^2 x} \, dx.
\]

2. By rotating the line \( y = r \) about the \( x \)-axis, for \( x \) in the interval \([0, h]\), and applying the surface area formula, obtain the well-known fact that the surface area of a cylinder of radius \( r \) and length \( h \) is given by \( 2\pi rh \).

**Solution** Let \( y = r \). Then \( y' = 0 \), \( 1 + (y')^2 = 1 \), and the surface area is

\[
2\pi \int_0^h r \sqrt{1 + h^2} \, dx = 2\pi rh.
\]

3. If \( 0 \leq f(x) \leq g(x) \) for \( x \) in the interval \([a, b]\), can the surface obtained by rotating the graph of \( y = g(x) \) around the \( x \)-axis over the interval have less surface area than the surface obtained by rotating the graph of \( y = f(x) \) around the \( x \)-axis over the same interval?

**Solution** Yes, the surface obtained by rotating the graph of \( y = g(x) \) around the \( x \)-axis over the interval can have less surface area than the surface obtained by rotating the graph of \( y = f(x) \) around the \( x \)-axis over the same interval. Although \( f(x) \leq g(x) \), it is possible that \( \sqrt{1 + [f'(x)]^2} \) could be larger than \( \sqrt{1 + [g'(x)]^2} \). For example, consider \( f(x) = 0.9 + 0.1 \sin(10\pi x) \) and \( g(x) = 1 \) for \( x \) in the interval \([0, 1]\). Then, \( 0 \leq f(x) \leq g(x) \), but the surface obtained by rotating the graph of \( y = g(x) \) around the \( x \)-axis has an area of

\[
2\pi \int_0^1 f(x) \sqrt{1 + [f'(x)]^2} \, dx \approx 6.3,
\]

while the surface obtained by rotating the graph of \( y = f(x) \) around the \( x \)-axis has an area of

\[
2\pi \int_0^1 f(x) \sqrt{1 + [f'(x)]^2} \, dx \approx 13.0,
\]

where the value of the integral was obtained using **Mathematica**.

4. Use the formula for arc length to show that for any constant \( C \), the graphs \( y = f(x) \) and \( y = f(x) + C \) have the same length over every interval \([a, b]\). Explain geometrically.

**Solution** The graph of \( y = f(x) + C \) is a vertical translation of the graph of \( y = f(x) \); hence, the two graphs should have the same arc length. We can explicitly establish this as follows:

\[
\text{length of } y = f(x) + C = \int_a^b \sqrt{1 + \left( \frac{d}{dx}(f(x) + C) \right)^2} \, dx = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \text{length of } y = f(x).
\]

5. Use the formula for arc length to show that the length of a graph over \([1, 4]\) cannot be less than 3.

**Solution** Note that \( f'(x)^2 \geq 0 \), so that \( \sqrt{1 + [f'(x)]^2} \geq \sqrt{1} = 1 \). Then the arc length of the graph of \( f(x) \) on \([1, 4]\) is

\[
\int_1^4 \sqrt{1 + [f'(x)]^2} \, dx \geq \int_1^4 1 \, dx = 3
\]
Exercises

1. Express the arc length of the curve \( y = x^4 \) between \( x = 2 \) and \( x = 6 \) as an integral (but do not evaluate).

SOLUTION Let \( y = x^4 \). Then \( y' = 4x^3 \) and

\[
s = \int_2^6 \sqrt{1 + (4x^3)^2} \, dx = \int_2^6 \sqrt{1 + 16x^6} \, dx.
\]

2. Express the arc length of the curve \( y = \tan x \) for \( 0 \leq x \leq \frac{\pi}{4} \) as an integral (but do not evaluate).

SOLUTION Let \( y = \tan x \). Then \( y' = \sec^2 x \), and

\[
s = \int_0^{\pi/4} \sqrt{1 + (\sec^2 x)^2} \, dx = \int_0^{\pi/4} \sqrt{1 + \sec^4 x} \, dx.
\]

3. Find the arc length of \( y = \frac{1}{12} x^3 + x^{-1} \) for \( 1 \leq x \leq 2 \). Hint: Show that \( 1 + (y')^2 = \left( \frac{1}{4}x^2 + x^{-2} \right)^2 \).

SOLUTION Let \( y = \frac{1}{12} x^3 + x^{-1} \). Then \( y' = \frac{x^2}{4} - x^{-2} \), and

\[
(y')^2 + 1 = \left( \frac{x^2}{4} - x^{-2} \right)^2 + 1 = \frac{x^4}{16} - \frac{1}{2} + x^{-4} + 1 = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left( \frac{x^2}{4} + x^{-2} \right)^2.
\]

Thus,

\[
s = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \sqrt{\left( \frac{x^2}{4} + \frac{1}{x^2} \right)^2} \, dx = \int_1^2 \frac{x^2}{4} + \frac{1}{x^2} \, dx
\]

\[
= \int_1^2 \left( \frac{x^2}{4} + \frac{1}{x^2} \right) \, dx \quad \text{since} \quad \frac{x^2}{4} + \frac{1}{x^2} > 0
\]

\[
= \left( \frac{x^3}{12} - \frac{1}{x} \right) \bigg|_1^{\frac{\pi}{4}} = \frac{13}{12}.
\]

4. Find the arc length of \( y = \left( \frac{x}{2} \right)^4 + \frac{1}{2x^2} \) over \([1, 4]\). Hint: Show that \( 1 + (y')^2 \) is a perfect square.

SOLUTION Let \( y = \left( \frac{x}{2} \right)^4 + \frac{1}{2x^2} \). Then

\[
y' = 4 \left( \frac{x}{2} \right)^3 \left( \frac{1}{2} \right) - \frac{1}{x^3} = \frac{x^3}{4} - \frac{1}{x^3}
\]

and

\[
(y')^2 + 1 = \left( \frac{x^3}{4} - \frac{1}{x^3} \right)^2 + 1 = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} + 1 = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left( \frac{x^3}{4} + \frac{1}{x^3} \right)^2.
\]

Hence,

\[
s = \int_1^4 \sqrt{1 + y'^2} \, dx = \int_1^4 \sqrt{\left( \frac{x^3}{4} + \frac{1}{x^3} \right)^2} \, dx = \int_1^4 \frac{x^3}{4} + \frac{1}{x^3} \, dx
\]

\[
= \int_1^4 \left( \frac{x^3}{4} + \frac{1}{x^3} \right) \, dx \quad \text{since} \quad \frac{x^3}{4} + \frac{1}{x^3} > 0 \text{ on } [1, 4]
\]

\[
= \left( \frac{x^4}{16} + \frac{x^{-2}}{-2} \right) \bigg|_1^4 = \frac{525}{32}.
\]

In Exercises 5–10, calculate the arc length over the given interval.

5. \( y = 3x + 1 \), \([0, 3]\)

SOLUTION Let \( y = 3x + 1 \). Then \( y' = 3 \), and

\[
s = \int_0^3 \sqrt{1 + 9} \, dx = 3\sqrt{10}.
\]
6. \( y = 9 - 3x \), \([1, 3]\)

**SOLUTION**  Let \( y = 9 - 3x \). Then \( y' = -3 \), and \( s = \int_{1}^{3} \sqrt{1 + 9} \, dx = 3\sqrt{10} - \sqrt{10} = 2\sqrt{10} \).

7. \( y = x^{3/2} \), \([1, 2]\)

**SOLUTION**  Let \( y = x^{3/2} \). Then \( y' = \frac{3}{2}x^{1/2} \), and

\[
s = \int_{1}^{2} \sqrt{1 + \left( \frac{3}{2}x^{1/2} \right)^2} \, dx = \int_{1}^{2} \sqrt{\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}} \, dx
\]

Since \( \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} > 0 \),

\[
= \left( \frac{1}{3}x^{3/2} + x^{1/2} \right) \bigg|_{1}^{2} = \frac{17\sqrt{2}}{3}.
\]

8. \( y = \frac{1}{4}x^{3/2} - x^{1/2} \), \([2, 8]\)

**SOLUTION**  Let \( y = \frac{1}{4}x^{3/2} - x^{1/2} \). Then

\[
y' = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2},
\]

and

\[
1 + (y')^2 = 1 + \left( \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2} \right)^2
\]

Hence,

\[
s = \int_{2}^{8} \sqrt{1 + (y')^2} \, dx = \int_{2}^{8} \sqrt{\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}} \, dx
\]

9. \( y = \frac{1}{4}x^2 - \frac{1}{2} \ln x \), \([1, 2e]\)

**SOLUTION**  Let \( y = \frac{1}{4}x^2 - \frac{1}{2} \ln x \). Then

\[
y' = \frac{x}{2} - \frac{1}{2x},
\]

and

\[
1 + (y')^2 = 1 + \left( \frac{x}{2} - \frac{1}{2x} \right)^2
\]

Hence,

\[
s = \int_{1}^{2e} \sqrt{1 + (y')^2} \, dx = \int_{1}^{2e} \sqrt{\frac{x}{2} + \frac{1}{2x}} \, dx
\]

10. \( y = \ln(\cos x) \), \([0, \frac{\pi}{4}]\)

**SOLUTION**  Let \( y = \ln(\cos x) \). Then \( y' = -\tan x \) and \( 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x \). Hence,

\[
s = \int_{0}^{\pi/4} \sqrt{1 + (y')^2} \, dx = \int_{0}^{\pi/4} \sec^2 x \, dx = \int_{0}^{\pi/4} \sec x \, dx
\]

Since \( \sec x > 0 \) on \([0, \frac{\pi}{4}]\),

\[
= \ln |\sec x + \tan x| \bigg|_{0}^{\pi/4} = \ln(\sqrt{2} + 1).
\]
In Exercises 11–16, approximate the arc length of the curve over the interval using the Trapezoidal Rule $T_N$, the Midpoint Rule $M_N$, or Simpson's Rule $S_N$ as indicated.

11. $y = \frac{1}{4}x^4$, $[1, 2]$, $T_5$

**SOLUTION** Let $y = \frac{1}{4}x^4$. Then

$$1 + (y')^2 = 1 + (x^3)^2 = 1 + x^6.$$ Therefore, the arc length over $[1, 2]$ is

$$\int_1^2 \sqrt{1 + x^6} \, dx.$$ Now, let $f(x) = \sqrt{1 + x^6}$. With $n = 5$,

$$\Delta x = \frac{2 - 1}{5} = \frac{1}{5}$$ and \( \{x_i\}_{i=0}^5 = \{1, 6/5, 7/5, 8/5, 9/5, 2\} \).

Using the Trapezoidal Rule,

$$\int_1^2 \sqrt{1 + x^6} \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2 \sum_{i=1}^{4} f(x_i) + f(x_5) \right] \approx 3.957736.$$ The arc length is approximately 3.957736 units.

12. $y = \sin x$, $[0, \frac{\pi}{2}]$, $M_8$

**SOLUTION** Let $y = \sin x$. Then

$$1 + (y')^2 = 1 + \cos^2 x.$$ Therefore, the arc length over $[0, \frac{\pi}{2}]$ is

$$\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx.$$ Now, let $f(x) = \sqrt{1 + \cos^2 x}$. With $n = 8$, we have:

$$\Delta x = \frac{\pi/2}{8} = \frac{\pi}{16}$$ and \( \{x_i^*\}_{i=1}^8 = \{\pi/32, \frac{9\pi}{32}, \frac{5\pi}{32}, \frac{7\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}, \frac{13\pi}{32}, \frac{15\pi}{32}\} \).

Using the Midpoint Rule,

$$\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \Delta x \sum_{i=1}^{8} f(x_i^*) \approx 1.910099.$$ The arc length is approximately 1.910099 units.

13. $y = x^{-1}$, $[1, 2]$, $S_8$

**SOLUTION** Let $y = x^{-1}$. Then $y' = -x^{-2}$ and

$$1 + (y')^2 = 1 + \frac{1}{x^4}.$$ Therefore, the arc length over $[1, 2]$ is

$$\int_1^2 \sqrt{1 + \frac{1}{x^4}} \, dx.$$ Now, let $f(x) = \sqrt{1 + \frac{1}{x^4}}$. With $n = 8$,

$$\Delta x = \frac{2 - 1}{8} = \frac{1}{8}$$ and \( \{x_i\}_{i=0}^8 = \{1, 9/8, 5/8, 11/8, 3/8, 7/8, 13/8, 15/8\} \).

Using Simpson's Rule,

$$\int_1^2 \sqrt{1 + \frac{1}{x^4}} \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4 \sum_{i=1}^{4} f(x_{2i-1}) + 2 \sum_{i=1}^{4} f(x_{2i}) + f(x_8) \right] \approx 1.132123.$$ The arc length is approximately 1.132123 units.
14. \( y = e^{-x^2}, \quad [0, 2], \quad S_8 \)

**SOLUTION**  Let \( y = e^{-x^2} \), then

\[
1 + (y')^2 = 1 + 4x^2 e^{-2x^2}.
\]

Therefore, the arc length over \([0, 2]\) is

\[
\int_0^2 \sqrt{1 + 4x^2 e^{-2x^2}} \, dx.
\]

Now, let \( f(x) = \sqrt{1 + 4x^2 e^{-2x^2}} \). With \( n = 8 \),

\[
\Delta x = \frac{2 - 0}{8} = \frac{1}{4}
\]

and \( \{x_i\}_{i=0}^8 = \left\{0, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, 2\right\} \).

Using Simpson’s Rule,

\[
\int_0^2 \sqrt{1 + 4x^2 e^{-2x^2}} \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4 \sum_{i=1}^4 f(x_{2i-1}) + 2 \sum_{i=1}^3 f(x_{2i}) + f(x_8) \right] \approx 2.280718.
\]

The arc length is approximately 2.280718 units.

15. \( y = \ln x, \quad [1, 3], \quad M_6 \)

**SOLUTION**  Let \( y = \ln x \), then

\[
1 + (y')^2 = 1 + \frac{1}{x^2}.
\]

Therefore, the arc length over \([1, 3]\) is

\[
\int_1^3 \sqrt{1 + \frac{1}{x^2}} \, dx.
\]

Now, let \( f(x) = \sqrt{1 + \frac{1}{x^2}} \). With \( n = 6 \), we have:

\[
\Delta x = \frac{3 - 1}{6} = \frac{1}{3}
\]

and \( \{x_i^*\}_{i=1}^6 = \left\{\frac{7}{6}, \frac{3}{2}, \frac{11}{6}, \frac{13}{6}, \frac{5}{2}, \frac{17}{6}\right\} \).

Using the Midpoint Rule,

\[
\int_1^3 \sqrt{1 + \frac{1}{x^2}} \, dx \approx \Delta x \sum_{i=1}^6 f(x_i^*) \approx 2.298959.
\]

The arc length is approximately 2.298959 units.

16. \( y = \cos x, \quad [0, 2], \quad T_8 \)

**SOLUTION**  Let \( y = \cos x \), then

\[
1 + (y')^2 = 1 + \sin^2 x = 1 + \sin^2 x.
\]

Therefore, the arc length over \([0, 2]\) is

\[
\int_0^2 \sqrt{1 + \sin^2 x} \, dx.
\]

Now, let \( f(x) = \sqrt{1 + \sin^2 x} \). With \( n = 8 \),

\[
\Delta x = \frac{2 - 0}{8} = \frac{1}{4}
\]

and \( \{x_i\}_{i=0}^8 = \left\{0, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, 2\right\} \).

Using the Trapezoidal Rule,

\[
\int_0^2 \sqrt{1 + \sin^2 x} \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2 \sum_{i=1}^7 f(x_i) + f(x_8) \right] \approx 2.506520.
\]

The arc length is approximately 2.506520 units.
17. Calculate the length of the astroid \( x^{2/3} + y^{2/3} = 1 \) (Figure 14).

\[ \text{FIGURE 14 Graph of } x^{2/3} + y^{2/3} = 1. \]

**SOLUTION** We will calculate the arc length of the portion of the astroid in the first quadrant and then multiply by 4. By implicit differentiation

\[ \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0, \]

so

\[ y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}. \]

Thus

\[ 1 + (y')^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}, \]

and

\[ s = \int_{0}^{1} \frac{1}{x^{1/3}} \, dx = \lim_{R \to 0^+} \int_{R}^{1} \frac{1}{x^{1/3}} \, dx = \lim_{R \to 0^+} \left( 3 \left[ -\frac{3}{2} R^{2/3} \right] \right) = \frac{3}{2}. \]

The total arc length is therefore \( 4 \cdot \frac{3}{2} = 6 \).

18. Show that the arc length of the astroid \( x^{2/3} + y^{2/3} = a^{2/3} \) (for \( a > 0 \)) is proportional to \( a \).

**SOLUTION** We will calculate the arc length of the portion of the astroid in the first quadrant and then multiply by 4. By implicit differentiation

\[ \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0, \]

so

\[ y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}. \]

Thus

\[ 1 + (y')^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}, \]

and

\[ s = \int_{0}^{a} \frac{a^{1/3}}{x^{1/3}} \, dx = a^{1/3} \left( 3 \left[ -\frac{3}{2} a^{2/3} \right] \right) = \frac{3}{2} a. \]

The total arc length is therefore \( 4 \cdot \frac{3}{2} a = 6a \), which is proportional to \( a \).

19. Find the length of the arc of the curve \( x^2 = (y - 2)^3 \) from \( P(1, 3) \) to \( Q(8, 6) \).

**SOLUTION** Solving \( x^2 = (y - 2)^3 \) for \( x \) and noting that we are interested in positive \( x \) values yields

\[ x = (y - 2)^{3/2}. \]

Then,

\[ \frac{dx}{dy} = \frac{3}{2} (y - 2)^{1/2}, \]

\[ 1 + \left( \frac{dx}{dy} \right)^2 = 1 + \frac{9}{4} (y - 2) = \frac{9}{4} - \frac{7}{2}, \]

and the length of the curve from the point \( (1, 3) \) to the point \( (8, 6) \) is

\[ \int_{3}^{6} \sqrt{\frac{9}{4} y - \frac{7}{2}} \, dy = \left[ \frac{8}{27} \left( \frac{9}{4} y - \frac{7}{2} \right)^{3/2} \right]_{3}^{6} = \frac{8}{27} \left( 10^{3/2} - \left( \frac{13}{4} \right)^{3/2} \right) \]

\[ = \frac{1}{27} ( 80\sqrt{10} - 13\sqrt{13}). \]
20. Find the arc length of the curve shown in Figure 15.

![Graph of 9y^2 = x(x - 3)^2.](image)

**FIGURE 15**

**SOLUTION** Using implicit differentiation,

\[ 18yy' = x(2)(x - 3) + (x - 3)^2 = 3(x - 3)(x - 1) \]

Hence,

\[ (y')^2 = \frac{(x - 3)^2(x - 1)^2}{36y^2} = \frac{(x - 3)^2(x - 1)^2}{4(9y^2)} = \frac{(x - 3)^2(x - 1)^2}{4x} \]

and

\[ 1 + (y')^2 = \frac{(x - 1)^2 + 4x}{4x} = \frac{(x + 1)^2}{4x}. \]

Finally,

\[ s = \int_{3}^{0} \sqrt{\frac{(x + 1)^2}{4x} - x} \, dx = \int_{0}^{3} |x + 1| \, dx \]

\[ = \int_{0}^{3} \frac{x + 1}{2\sqrt{x}} \, dx \quad \text{since} \quad x + 1 > 0 \text{ on } [0, 3] \]

\[ = \int_{0}^{3} \left( \frac{1}{2} x^{1/2} + \frac{1}{2} x^{-1/2} \right) \, dx = \left[ \frac{1}{3} x^{3/2} + x^{1/2} \right]_{0}^{3} = 2\sqrt{3}. \]

21. Find the value of \( a \) such that the arc length of the catenary \( y = \cosh x \) for \(-a \leq x \leq a\) equals 10.

**SOLUTION** Let \( y = \cosh x \). Then \( y' = \sinh x \) and

\[ 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x. \]

Thus,

\[ s = \int_{-a}^{a} \cosh x \, dx = \sinh(a) - \sinh(-a) = 2 \sinh a. \]

Setting this expression equal to 10 and solving for \( a \) yields \( a = \sinh^{-1}(5) = \ln(5 + \sqrt{26}) \).

22. Calculate the arc length of the graph of \( f(x) = mx + r \) over \([a, b]\) in two ways: using the Pythagorean theorem (Figure 16) and using the arc length integral.

![Graph of y = mx + r.](image)

**FIGURE 16**

**SOLUTION** Let \( h \) denote the length of the hypotenuse. Then, by Pythagoras’ Theorem,

\[ h^2 = (b - a)^2 + m^2(b - a)^2 = (b - a)^2(1 + m^2), \]

or

\[ h = (b - a)\sqrt{1 + m^2} \]

since \( b > a \). Moreover, \((f'(x))^2 = m^2\), so

\[ s = \int_{a}^{b} \sqrt{1 + m^2} \, dx = (b - a)\sqrt{1 + m^2} = h. \]
23. Show that the circumference of the unit circle is equal to

\[ 2 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \]  

(an improper integral)

Evaluate, thus verifying that the circumference is \(2\pi\).

**Solution**  
Note the circumference of the unit circle is twice the arc length of the upper half of the curve defined by \(x^2 + y^2 = 1\). Thus, let \(y = \sqrt{1-x^2}\). Then

\[ y' = -\frac{x}{\sqrt{1-x^2}}, \quad 1 + (y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}, \]

and the circumference of the unit circle is

\[ 2 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2 \left( \int_{0}^{\pi} \frac{dx}{\sqrt{1-x^2}} + \int_{\pi}^{0} \frac{dx}{\sqrt{1-x^2}} \right). \]

Now,

\[ \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}} = \lim_{R \to 1+} \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}} = -\lim_{R \to 1+} \sin^{-1} R = -\left( -\frac{\pi}{2} \right) = \frac{\pi}{2}; \]

and

\[ \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{R \to 1-} \int_{0}^{R} \frac{dx}{\sqrt{1-x^2}} = \lim_{R \to 1-} \sin^{-1} R = \frac{\pi}{2}. \]

Thus, the circumference of the unit circle is

\[ 2 \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi. \]

24. Generalize the result of Exercise 23 to show that the circumference of the circle of radius \(r\) is \(2\pi r\).

**Solution**  
Let \(y = \sqrt{r^2 - x^2}\) denote the upper half of a circle of radius \(r\) centered at the origin. Then

\[ 1 + (y')^2 = 1 + \frac{x^2}{r^2-x^2} = \frac{1}{1-x^2}, \]

and the circumference of the circle is given by

\[ C = 2 \int_{-r}^{r} \frac{dx}{\sqrt{1-x^2/r^2}}. \]

Using the substitution \(u = x/r, \ du = dx/r\), we find

\[ C = 2r \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = 2r \sin^{-1} u \bigg|_{-1}^{1} = 2r \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = 2\pi r. \]

25. Calculate the arc length of \(y = x^2\) over \([0, a]\). *Hint:* Use trigonometric substitution. Evaluate for \(a = 1\).

**Solution**  
Let \(y = x^2\). Then \(y' = 2x\) and

\[ s = \int_{0}^{a} \sqrt{1 + 4x^2} \, dx. \]

Using the substitution \(2x = \tan \theta, \ 2 \, dx = \sec^2 \theta \, d\theta\), we find

\[ s = \frac{1}{2} \int_{x=0}^{x=a} \sec^3 \theta \, d\theta. \]

Next, using a reduction formula for the integral of \(\sec^3 \theta\), we see that

\[ s = \left( \frac{1}{4} \sec \theta \tan \theta + \frac{1}{4} \ln |\sec \theta + \tan \theta| \right) \bigg|_{x=0}^{x=a} = \left( \frac{1}{2} \sqrt{1+4a^2} + \frac{1}{4} \ln |\sqrt{1+4a^2} + 2a| \right) - \left( \frac{1}{2} \sqrt{1+4a^2} + \frac{1}{4} \ln |\sqrt{1+4(0)^2} + 2(0)| \right) = \frac{a}{2} \sqrt{1+4a^2} + \frac{1}{4} \ln |\sqrt{1+4a^2} + 2a|. \]
Thus, when \(a = 1\),
\[
s = \frac{1}{2} \sqrt{5} + \frac{1}{4} \ln(\sqrt{5} + 2) \approx 1.478943.
\]

26. Express the arc length of \(g(x) = \sqrt{x}\) over \([0, 1]\) as a definite integral. Then use the substitution \(u = \sqrt{x}\) to show that this arc length is equal to the arc length of \(y = x^2\) over \([0, 1]\) (but do not evaluate the integrals). Explain this result graphically.

**Solution**

Let \(g(x) = \sqrt{x}\). Then
\[
1 + g'(x)^2 = \frac{1 + 4x}{4x} \quad \text{and} \quad s = \int_0^1 \sqrt{\frac{1 + 4x}{4x}} \, dx = \int_0^1 \frac{\sqrt{1 + 4x}}{2\sqrt{x}} \, dx.
\]

With the substitution \(u = \sqrt{x}\), \(du = \frac{1}{2\sqrt{x}} \, dx\), this becomes
\[
s = \int_0^1 \sqrt{1 + 4u^2} \, du.
\]

Now, let \(f(x) = x^2\). Then \(1 + f'(x)^2 = 1 + 4x^2\), and
\[
s = \int_0^1 \sqrt{1 + 4x^2} \, dx.
\]

Thus, the two arc lengths are equal. This is explained graphically by the fact that for \(x \geq 0\), \(x^2\) and \(\sqrt{x}\) are inverses of each other. This means that the two graphs are symmetric with respect to the line \(y = x\). Moreover, the graphs of \(x^2\) and \(\sqrt{x}\) intersect at \(x = 0\) and at \(x = 1\). Thus, it is clear that the arc length of the two graphs on \([0, 1]\) are equal.

27. Find the arc length of \(y = e^x\) over \([0, a]\). Hint: Try the substitution \(u = \sqrt{1 + e^{2x}}\) followed by partial fractions.

**Solution**

Let \(y = e^x\). Then \(1 + (y')^2 = 1 + e^{2x}\), and the arc length over \([0, a]\) is
\[
\int_0^a \sqrt{1 + e^{2x}} \, dx.
\]

Now, let \(u = \sqrt{1 + e^{2x}}\). Then
\[
du = \frac{1}{2} \cdot \frac{2e^{2x}}{\sqrt{1 + e^{2x}}} \, dx = \frac{u^2 - 1}{u} \, dx
\]

and the arc length is
\[
\int_0^a \sqrt{1 + e^{2x}} \, dx = \int_{x=0}^{x=a} u \cdot \frac{u}{u^2 - 1} \, du = \int_{x=0}^{x=a} \frac{u^2}{u^2 - 1} \, du = \left. \int_{x=0}^{x=a} \left( 1 + \frac{1}{u^2 - 1} \right) \, du \right|\bigg|_{x=0}^{x=a}
\]
\[
= \left. \left[ \sqrt{1 + e^{2x}} + \frac{1}{2} \ln \left( \frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} \right) \right] \right|_{0}^{a}
\]
\[
= \sqrt{1 + e^{2a}} + \frac{1}{2} \ln \left( \frac{\sqrt{1 + e^{2a}} - 1}{\sqrt{1 + e^{2a}} + 1} \right) - \sqrt{2} + \frac{1}{2} \ln \frac{1 + \sqrt{2}}{\sqrt{2} - 1}
\]
\[
= \sqrt{1 + e^{2a}} + \frac{1}{2} \ln \left( \frac{\sqrt{1 + e^{2a}} - 1}{\sqrt{1 + e^{2a}} + 1} \right) - \sqrt{2} + \ln(1 + \sqrt{2}).
\]

28. Show that the arc length of \(y = \ln(f(x))\) for \(a \leq x \leq b\) is
\[
\int_a^b \sqrt{\frac{f(x)^2 + f'(x)^2}{f(x)^2}} \, dx
\]

**Solution**

Let \(y = \ln(f(x))\). Then
\[
y' = \frac{f'(x)}{f(x)} \quad \text{and} \quad 1 + (y')^2 = \frac{f(x)^2 + f'(x)^2}{f(x)^2}.
\]
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Therefore,
\[ s = \int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)} \, dx \]
since \( f(x) > 0 \) in order for \( y = \ln(f(x)) \) to be defined on \([a, b] \).

29. Use Eq. (4) to compute the arc length of \( y = \ln(x) \) for \( \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \).

Solution  With \( f(x) = \sin x \), Eq. (4) yields
\[ s = \int_{\pi/4}^{\pi/2} \frac{\sqrt{\sin^2 x + \cos^2 x}}{\sin x} \, dx = \int_{\pi/4}^{\pi/2} \csc x \, dx = \ln(\csc x - \cot x)|_{\pi/4}^{\pi/2} \]
\[ = \ln 1 - \ln(\sqrt{2} - 1) = \ln \frac{1}{\sqrt{2} - 1} = \ln(\sqrt{2} + 1). \]

30. Use Eq. (4) to compute the arc length of \( y = \ln \left( \frac{e^x + 1}{e^x - 1} \right) \) over \([1, 3]\).

Solution  With \( f(x) = \frac{e^x + 1}{e^x - 1} \),
\[ f'(x) = \frac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2} = -\frac{2e^x}{(e^x - 1)^2} \]
and
\[ f(x)^2 + f'(x)^2 = \left( \frac{e^x + 1}{e^x - 1} \right)^2 + \frac{4e^x}{(e^x - 1)^2} = \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^x - 1)^2} = \frac{(e^{2x} + 1)^2}{(e^x - 1)^4}. \]
Thus, by Eq. (4),
\[ s = \int_1^3 \frac{e^{2x} + 1}{e^x - 1} \cdot \frac{e^x - 1}{e^{2x} - 1} \, dx = \int_1^3 \frac{e^{2x} + 1}{e^{2x} - 1} \, dx. \]

Observe that
\[ \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{(e^x + e^{-x})/2}{\sinh x} = \frac{\cosh x}{\sinh x}. \]
Therefore,
\[ s = \int_1^3 \frac{\cosh x}{\sinh x} \, dx = \ln(\sinh x)|_1^3 = \ln(\sinh 3) - \ln(\sinh 1). \]

31. Show that if \( 0 \leq f'(x) \leq 1 \) for all \( x \), then the arc length of \( y = f(x) \) over \([a, b] \) is at most \( \sqrt{2}(b - a) \). Show that for \( f(x) = x \), the arc length equals \( \sqrt{2}(b - a) \).

Solution  If \( 0 \leq f'(x) \leq 1 \) for all \( x \), then
\[ s = \int_a^b \sqrt{1 + f'(x)^2} \, dx \leq \int_a^b \sqrt{1 + 1} \, dx = \sqrt{2}(b - a). \]

If \( f(x) = x \), then \( f'(x) = 1 \) and
\[ s = \int_a^b \sqrt{1 + 1} \, dx = \sqrt{2}(b - a). \]

32. Use the Comparison Theorem (Section 5.2) to prove that the arc length of \( y = x^{4/3} \) over \([1, 2]\) is not less than \( \frac{5}{3} \).

Solution  Note that \( f'(x) = \frac{4}{3} x^{1/3} \); for \( x \in [1, 2] \), we have \( x^{1/3} \geq 1 \) so that \( f'(x) \geq \frac{4}{3} \). Then
\[ \sqrt{1 + f'(x)^2} \geq \sqrt{1 + \left( \frac{4}{3} \right)^2} = \sqrt{\frac{25}{9}} = \frac{5}{3} \]
and then the arc length is
\[ \int_1^2 \sqrt{1 + f'(x)^2} \, dx \geq \int_1^2 \frac{5}{3} \, dx = \frac{5}{3}. \]
33. Approximate the arc length of one-quarter of the unit circle (which we know is $\frac{\pi}{2}$) by computing the length of the polygonal approximation with $N = 4$ segments (Figure 17).

\[ \text{FIGURE 17 One-quarter of the unit circle.} \]

**Solution** With $y = \sqrt{1 - x^2}$, the five points along the curve are

- $P_0(0, 1)$, $P_1(1/4, \sqrt{15}/4)$, $P_2(1/2, \sqrt{3}/2)$, $P_3(3/4, \sqrt{7}/4)$, $P_4(1, 0)$

Then

\[
\begin{align*}
P_0P_1 &= \frac{1}{\sqrt{16 + \left(\frac{4 - \sqrt{15}}{4}\right)^2}} \approx 0.252009 \\
P_1P_2 &= \frac{1}{\sqrt{16 + \left(\frac{2\sqrt{3} - \sqrt{15}}{4}\right)^2}} \approx 0.270091 \\
P_2P_3 &= \frac{1}{\sqrt{16 + \left(\frac{2\sqrt{3} - \sqrt{7}}{4}\right)^2}} \approx 0.323042 \\
P_3P_4 &= \frac{1}{\sqrt{\frac{1}{16} + \frac{7}{16}}} \approx 0.707108
\end{align*}
\]

and the total approximate distance is 1.552250 whereas $\frac{\pi}{2} \approx 1.570796$.

34. **CRA** A merchant intends to produce specialty carpets in the shape of the region in Figure 18, bounded by the axes and graph of $y = 1 - x^n$ (units in yards). Assume that material costs $50/\text{yd}^2$ and that it costs $50L$ dollars to cut the carpet, where $L$ is the length of the curved side of the carpet. The carpet can be sold for $150A$ dollars, where $A$ is the carpet’s area. Using numerical integration with a computer algebra system, find the whole number $n$ for which the merchant’s profits are maximal.

\[ \text{FIGURE 18} \]

**Solution** The area of the carpet is

\[
A = \int_0^1 (1 - x^n)dx = \left[ x - \frac{x^{n+1}}{n+1} \right]_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1},
\]

while the length of the curved side of the carpet is

\[
L = \int_0^1 \sqrt{1 + (nx^{n-1})^2} \, dx = \int_0^1 \sqrt{1 + n^2x^{2n-2}} \, dx.
\]

Using these formulas, we find that the merchant’s profit is given by

\[
150A - (50A + 50L) = 100A - 50L = \frac{100n}{n+1} - 50 \int_0^1 \sqrt{1 + n^2x^{2n-2}} \, dx.
\]
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Using a CAS, we find that the merchant’s profit is maximized (approximately $3.31 per carpet) when \( n = 13. \) The table below lists the profit for \( 1 \leq n \leq 15. \)

<table>
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<tr>
<th>( n )</th>
<th>( \text{Profit} )</th>
<th>( n )</th>
<th>( \text{Profit} )</th>
</tr>
</thead>
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<tr>
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<td>9</td>
<td>3.06855532</td>
</tr>
<tr>
<td>2</td>
<td>−7.28047621</td>
<td>10</td>
<td>3.18862208</td>
</tr>
<tr>
<td>3</td>
<td>−2.39328273</td>
<td>11</td>
<td>3.25953632</td>
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<td>4</td>
<td>−0.01147138</td>
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<td>3.31024566</td>
</tr>
<tr>
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</tr>
<tr>
<td>8</td>
<td>2.87539525</td>
<td>&gt;</td>
<td>&gt;</td>
</tr>
</tbody>
</table>

In Exercises 35–42, compute the surface area of revolution about the x-axis over the interval.

35. \( y = x, \quad [0, 4] \)

**SOLUTION** Let \( y = x. \) Then \( 1 + (y')^2 = 2 \) and

\[
SA = 2\pi \int_0^4 x\sqrt{2} \, dx = 2\pi \sqrt{2} \left[ \frac{1}{2} x^2 \right]_0^4 = 16\pi \sqrt{2}.
\]

36. \( y = 4x + 3, \quad [0, 1] \)

**SOLUTION** Let \( y = 4x + 3. \) Then \( 1 + (y')^2 = 17 \) and

\[
SA = 2\pi \int_0^1 (4x + 3)\sqrt{17} \, dx = 2\pi \sqrt{17} \left[ \frac{2}{3} (2x^2 + 3x) \right]_0^1 = 10\pi \sqrt{17}.
\]

37. \( y = x^3, \quad [0, 2] \)

**SOLUTION** Let \( y = x^3. \) Then \( 1 + (y')^2 = 1 + 9x^4, \) and

\[
SA = 2\pi \int_0^2 x^3\sqrt{1 + 9x^4} \, dx = \frac{2\pi}{36} \int_0^2 36x^3\sqrt{1 + 9x^4} \, dx = \frac{\pi}{27} [1 + 9x^4]^{3/2} \bigg|_0^1 = \frac{\pi}{27} (145^{3/2} - 1).
\]

38. \( y = x^2, \quad [0, 4] \)

**SOLUTION** Let \( y = x^2. \) Then \( y' = 2x \) and

\[
SA = 2\pi \int_0^4 x^2\sqrt{1 + 4x^2} \, dx.
\]

Using the substitution \( 2x = \tan \theta, \) \( 2dx = \sec^2 \theta \, d\theta, \) we find that

\[
\int x^2\sqrt{1 + 4x^2} \, dx = \frac{1}{8} \int \sec^3 \theta \tan^2 \theta \, d\theta = \frac{1}{8} \int \left( \sec^5 \theta - \sec^3 \theta \right) \, d\theta = \frac{1}{8} \left( \frac{1}{4} \sec^3 \theta \tan \theta + \frac{\ln |\sec \theta + \tan \theta|}{8} - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C
\]

\[= \frac{x}{16} (1 + 4x^2)^{3/2} - \frac{x}{32} \sqrt{1 + 4x^2} - \frac{1}{64} \ln |\sqrt{1 + 4x^2} + 2x| + C.\]

Finally,

\[
SA = 2\pi \left( \frac{x}{16} (1 + 4x^2)^{3/2} - \frac{x}{32} \sqrt{1 + 4x^2} - \frac{1}{64} \ln |\sqrt{1 + 4x^2} + 2x| \right) \bigg|_0^4 = 2\pi \left( \frac{1}{4} 65^{3/2} - \frac{\sqrt{65}}{8} - \frac{1}{64} \ln (8 + \sqrt{65}) \right) = \frac{129\sqrt{65}}{4} - \frac{\pi}{32} \ln (8 + \sqrt{65}).
\]

39. \( y = (4 - x^2)^{3/2}, \quad [0, 8] \)

**SOLUTION** Let \( y = (4 - x^2)^{3/2}. \) Then

\[
y' = -x^{-1/3} (4 - x^2)^{1/2},\]

\[
\int x^2\sqrt{1 + 4x^2} \, dx = \frac{1}{8} \int \sec^3 \theta \tan^2 \theta \, d\theta = \frac{1}{8} \int \left( \sec^5 \theta - \sec^3 \theta \right) \, d\theta = \frac{1}{8} \left( \frac{1}{4} \sec^3 \theta \tan \theta + \frac{\ln |\sec \theta + \tan \theta|}{8} - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C
\]

\[= \frac{x}{16} (1 + 4x^2)^{3/2} - \frac{x}{32} \sqrt{1 + 4x^2} - \frac{1}{64} \ln |\sqrt{1 + 4x^2} + 2x| + C.\]

Finally,

\[
SA = 2\pi \left( \frac{x}{16} (1 + 4x^2)^{3/2} - \frac{x}{32} \sqrt{1 + 4x^2} - \frac{1}{64} \ln |\sqrt{1 + 4x^2} + 2x| \right) \bigg|_0^4 = 2\pi \left( \frac{1}{4} 65^{3/2} - \frac{\sqrt{65}}{8} - \frac{1}{64} \ln (8 + \sqrt{65}) \right) = \frac{129\sqrt{65}}{4} - \frac{\pi}{32} \ln (8 + \sqrt{65}).
\]
and

\[1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}.\]

Therefore,

\[SA = 2\pi \int \frac{2}{x^{1/3}} \left(4 - x^{2/3}\right)^{3/2} dx.\]

Using the substitution \(u = 4 - x^{2/3}, \ du = -\frac{2}{3}x^{-1/3} dx\), we find

\[SA = 2\pi \int_0^0 u^{3/2}(-3) du = 6\pi \int_0^4 u^{3/2} du = \frac{12\pi}{5} - \frac{144\pi}{5} = \frac{384\pi}{5}.

40. \(y = e^{-x}\), \([0, 1] \]

**SOLUTION** Let \(y = e^{-x}\). Then \(y' = -e^{-x}\) and

\[SA = 2\pi \int_0^1 e^{-x}\sqrt{1 + e^{-2x}} dx.\]

Using the substitution \(e^{-x} = \tan \theta\), \(-e^{-x} dx = \sec^2 \theta d\theta\), we find that

\[
\int e^{-x}\sqrt{1 + e^{-2x}} dx = -\int \sec^3 \theta d\theta = -\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C
\]

\[= -\frac{1}{2} e^{-x}\sqrt{1 + e^{-2x}} - \frac{1}{2} \ln |\sqrt{1 + e^{-2x}} + e^{-x}| + C.\]

Finally,

\[SA = \left(-\pi e^{-x}\sqrt{1 + e^{-2x}} - \pi \ln |\sqrt{1 + e^{-2x}} + e^{-x}|\right)|_0^1
\]

\[= -\pi e^{-1}\sqrt{1 + e^{-2}} - \pi \ln (\sqrt{1 + e^{-2}} + e^{-1}) + \pi \sqrt{2} + \pi \ln (\sqrt{2} + 1)
\]

\[= \pi \sqrt{2} - \pi e^{-1}\sqrt{1 + e^{-2}} + \pi \ln \left(\frac{\sqrt{2} + 1}{\sqrt{1 + e^{-2}} + e^{-1}}\right).
\]

41. \(y = \frac{1}{4}x^2 - \frac{1}{2}\ln x\), \([1, e] \]

**SOLUTION** We have \(y' = \frac{x}{2} - \frac{1}{2x}\), and

\[1 + (y')^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{4} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.
\]

Thus,

\[SA = 2\pi \int_1^e \left(\frac{x}{4} - \ln x\frac{x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^e \left(\frac{x^3}{8} + \frac{x}{8} - \frac{ln x}{4} - \frac{ln x}{4x}\right) dx
\]

\[= 2\pi \left(\frac{x^4}{32} - \frac{1}{16} - \frac{x^2}{8} + \frac{x^2}{16} - \frac{ln x}{8} - \frac{(ln x)^2}{16}\right)|_1^e
\]

\[= 2\pi \left(\frac{e^4}{32} - \frac{1}{8} - \frac{1}{32} - \frac{1}{16} - \frac{1}{16} - \frac{1}{16}\right)
\]

\[= \pi (e^4 - 9).
\]
42. \( y = \sin x, \quad [0, \pi] \)

**Solution** Let \( y = \sin x \). Then \( y' = \cos x \), and

\[
SA = 2\pi \int_{0}^{\pi} \sin x \sqrt{1 + \cos^2 x} \, dx.
\]

Using the substitution \( \cos x = \tan \theta, \ -\sin x \, dx = \sec^2 \theta \, d\theta \), we find that

\[
\int \sin x \sqrt{1 + \cos^2 x} \, dx = -\int \sec^3 \theta \, d\theta = -\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C
\]

\[
= -\frac{1}{2} \cos x \sqrt{1 + \cos^2 x} - \frac{1}{2} \ln |\sqrt{1 + \cos^2 x} + \cos x| + C.
\]

Finally,

\[
SA = 2\pi \left( -\frac{1}{2} \cos x \sqrt{1 + \cos^2 x} - \frac{1}{2} \ln |\sqrt{1 + \cos^2 x} + \cos x| \right) \bigg|_{0}^{\pi}
\]

\[
= 2\pi \left( \frac{1}{2} \sqrt{2} - \frac{1}{2} \ln(\sqrt{2} - 1) + \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(\sqrt{2} + 1) \right) = 2\pi \left( \sqrt{2} + \ln(\sqrt{2} + 1) \right).
\]

**CAS** In Exercises 43–46, use a computer algebra system to find the approximate surface area of the solid generated by rotating the curve about the \( x \)-axis.

43. \( y = x^{-1}, \quad [1, 3] \)

**Solution**

\[
SA = 2\pi \int_{1}^{3} \sqrt{1 + \left( -\frac{1}{x^2} \right)^2} \, dx = 2\pi \int_{1}^{3} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx.
\]

Using a computer algebra system to approximate the value of the definite integral, we find

\( SA \approx 7.60306 \).

44. \( y = x^4, \quad [0, 1] \)

**Solution**

\[
SA = 2\pi \int_{0}^{1} x^4 \sqrt{1 + (4x^3)^2} \, dx = 2\pi \int_{0}^{1} x^4 \sqrt{1 + 16x^6} \, dx
\]

Using a computer algebra system to approximate the value of the definite integral, we find

\( SA \approx 3.43657 \).

45. \( y = e^{-x^2/2}, \quad [0, 2] \)

**Solution**

\[
SA = 2\pi \int_{0}^{2} e^{-x^2/2} \sqrt{1 + (-xe^{-x^2/2})^2} \, dx = 2\pi \int_{0}^{2} e^{-x^2/2} \sqrt{1 + x^2 e^{-x^2}} \, dx
\]

Using a computer algebra system to approximate the value of the definite integral, we find

\( SA \approx 8.22270 \).

46. \( y = \tan x, \quad [0, \frac{\pi}{4}] \)

**Solution** Let \( y = \tan x \). Then \( y' = \sec^2 x, \ 1 + (y')^2 = 1 + \sec^4 x, \) and

\[
SA = 2\pi \int_{0}^{\pi/4} \tan x \sqrt{1 + \sec^4 x} \, dx.
\]

Using a computer algebra system to approximate the value of the definite integral, we find

\( SA \approx 3.83908 \).

47. Find the area of the surface obtained by rotating \( y = \cosh x \) over \([-\ln 2, \ln 2]\) around the \( x \)-axis.

**Solution** Let \( y = \cosh x \). Then \( y' = \sinh x \), and

\[
\sqrt{1 + (y')^2} = \sqrt{1 + \sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x.
\]
Therefore,
\[
SA = 2\pi \int_{-\ln 2}^{\ln 2} \cosh^2 x \, dx = \pi \int_{-\ln 2}^{\ln 2} (1 + \cosh 2x) \, dx = \pi \left( x + \frac{1}{2} \sinh 2x \right) \bigg|_{-\ln 2}^{\ln 2}
\]
\[
= \pi \left( \ln 2 + \frac{1}{2} \sinh(2\ln 2) + \ln 2 - \frac{1}{2} \sinh(-2\ln 2) \right) = 2\pi \ln 2 + \pi \sinh(2\ln 2).
\]

We can simplify this answer by recognizing that
\[
\sinh(2\ln 2) = \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} = \frac{4 - \frac{1}{4}}{4} = \frac{15}{8}.
\]

Thus,
\[
SA = 2\pi \ln 2 + \frac{15\pi}{8}.
\]

48. Show that the surface area of a spherical cap of height \( h \) and radius \( R \) (Figure 19) has surface area \( 2\pi Rh \).

![](FIGURE 19)

**SOLUTION** To determine the surface area of the cap, we will rotate a portion of a circle of radius \( R \), centered at the origin, about the \( y \)-axis. Since the equation of the right half of the circle is \( x = \sqrt{R^2 - y^2} \),
\[
1 + (x')^2 = 1 + \frac{y^2}{R^2 - y^2} = \frac{R^2}{R^2 - y^2},
\]
and
\[
SA = 2\pi \int_{R-h}^{R} \sqrt{R^2 - y^2} \left( \frac{R}{\sqrt{R^2 - y^2}} \right) \, dy = 2\pi R (R - (R - h)) = 2\pi Rh.
\]

49. Find the surface area of the torus obtained by rotating the circle \( x^2 + (y - b)^2 = r^2 \) about the \( x \)-axis (Figure 20).

![](FIGURE 20)

**SOLUTION** \( y = b + \sqrt{a^2 - x^2} \) gives the top half of the circle and \( y = b - \sqrt{a^2 - x^2} \) gives the bottom half. Note that in each case,
\[
1 + (y')^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.
\]

Rotating the two halves of the circle around the \( x \)-axis then yields
\[
SA = 2\pi \int_{-a}^{a} (b + \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} \, dx + 2\pi \int_{-a}^{a} (b - \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} \, dx
\]
\[
= 2\pi \int_{-a}^{a} 2b \frac{a}{\sqrt{a^2 - x^2}} \, dx = 4\pi ba \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}} \, dx
\]
\[
= 4\pi ba \cdot \sin^{-1}\left( \frac{x}{a} \right) \bigg|_{-a}^{a} = 4\pi ba \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = 4\pi^2 ba.
\]
FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

50. By rotating the line \( y = mx \) about the \( x \)-axis for \( 0 \leq a \leq x \leq b \), we obtain a truncated cone. In this section, we used geometry to show that the surface area of such a truncated cone is \( \pi (r_1 + r_2)l \) where \( r_1 \) and \( r_2 \) are the radii of the boundary circles of the cone and \( l \) is the slant length of the cone.

(a) Show that the slant height of the cone is given by \( l = \sqrt{1 + m^2(b - a)} \) and that the surface area then becomes \( \pi (r_1 + r_2)l = \pi m\sqrt{1 + m^2(b^2 - a^2)} \).

(b) Apply Equation 3 to the surface of revolution to obtain the same result for the surface area.

**SOLUTION**

(a) The slant height of the cone is the distance between the points \( (a, ma) \) and \( (b, mb) \). Thus,

\[
l = \sqrt{(b - a)^2 + (mb - ma)^2} = \sqrt{(b - a)^2(1 + m^2)} = (b - a)\sqrt{1 + m^2}.
\]

Additionally,

\[
r_1 + r_2 = ma + mb = m(a + b),
\]

so the surface area becomes

\[
\pi (r_1 + r_2)l = \pi \cdot m(a + b) \cdot (b - a)\sqrt{1 + m^2} = \pi m\sqrt{1 + m^2(b^2 - a^2)}.
\]

(b) Let \( y = mx \). Then \( y' = m \) and \( 1 + (y')^2 = 1 + m^2 \). Applying Equation 3 then yields

\[
2\pi \int_a^b mx\sqrt{1 + m^2} \, dx = 2\pi m\sqrt{1 + m^2} \int_a^b x \, dx = 2\pi m\sqrt{1 + m^2} \cdot \frac{1}{2} \int_a^b x^2 \, dx = \pi m\sqrt{1 + m^2(b^2 - a^2)}
\]

as the surface area for the surface of revolution.

**Further Insights and Challenges**

51. Find the surface area of the ellipsoid obtained by rotating the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) about the \( x \)-axis.

**SOLUTION**

Taking advantage of symmetry, we can find the surface area of the ellipsoid by doubling the surface area obtained by rotating the portion of the ellipse in the first quadrant about the \( x \)-axis. The equation for the portion of the ellipse in the first quadrant is

\[
y = \frac{b}{a}\sqrt{a^2 - x^2},
\]

Thus,

\[
1 + (y')^2 = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 + (b^2 - a^2)x^2}{a^2(a^2 - x^2)},
\]

and

\[
SA = 4\pi \int_0^a \frac{b}{a}\sqrt{a^2 - x^2} \frac{\sqrt{a^4 + (b^2 - a^2)x^2}}{a\sqrt{a^2 - x^2}} \, dx = 4\pi b \int_0^a \sqrt{1 + \left( \frac{b^2 - a^2}{a^4} \right)x^2} \, dx.
\]

We now consider two cases. If \( b^2 > a^2 \), then we make the substitution

\[
\frac{\sqrt{b^2 - a^2}}{a^2} x = \tan \theta, \quad dx = \frac{a^2}{\sqrt{b^2 - a^2}} \sec^2 \theta \, d\theta,
\]

and find that

\[
SA = 4\pi b \frac{a^2}{\sqrt{b^2 - a^2}} \int_0^{x=a} \frac{b^2 - a^2}{a^4} \sec^3 \theta \, d\theta = 2\pi b \frac{a^2}{\sqrt{b^2 - a^2}} \left( \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \bigg|_{x=0}^{x=a}
\]

\[
= \left( 2\pi b \sqrt{1 + \left( \frac{b^2 - a^2}{a^4} \right)x^2} + 2\pi b \frac{a^2}{\sqrt{b^2 - a^2}} \ln \left| 1 + \left( \frac{b^2 - a^2}{a^4} \right)x^2 + \frac{\sqrt{b^2 - a^2}}{a^2} \right| \right) \bigg|_0^a
\]

\[
= 2\pi b^2 + 2\pi b \frac{a^2}{\sqrt{b^2 - a^2}} \ln \left( \frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a} \right).
\]

On the other hand, if \( a^2 > b^2 \), then we make the substitution

\[
\frac{\sqrt{a^2 - b^2}}{a^2} x = \sin \theta, \quad dx = \frac{a^2}{\sqrt{a^2 - b^2}} \cos \theta \, d\theta,
\]

and find that

\[
SA = 4\pi b \frac{a^2}{\sqrt{a^2-b^2}} \int_{x=0}^{x=a} \cos^2 \theta \, d\theta = 2\pi b \frac{a^2}{\sqrt{a^2-b^2}} (\theta + \sin \theta \cos \theta) \int_{x=0}^{x=a}
\]

\[
= \left[ 2\pi bx \left( 1 - \left( \frac{a^2-b^2}{a^4} \right) x^2 \right) \right]_0^a + 2\pi b \frac{a^2}{\sqrt{a^2-b^2}} \sin^{-1} \left( \frac{\sqrt{a^2-b^2}}{a} x \right) \bigg|_0^a
\]

\[
= 2\pi b^2 + 2\pi b \frac{a^2}{\sqrt{a^2-b^2}} \sin^{-1} \left( \frac{\sqrt{a^2-b^2}}{a} \right).
\]

Observe that in both cases, as \(a\) approaches \(b\), the value of the surface area of the ellipsoid approaches \(4\pi b^2\), the surface area of a sphere of radius \(b\).

52. Show that if the arc length of \(y = f(x)\) over \([0, a]\) is proportional to \(a\), then \(y = f(x)\) must be a linear function.

**Solution**

\[
s = \int_0^a \sqrt{1 + f'(x)^2} \, dx
\]

For \(s\) to be proportional to \(a\), \(\sqrt{1 + f'(x)^2}\) must be a constant, which implies \(f'(x)\) is a constant. This, in turn, requires \(f(x)\) be linear.

53. **Let \(L\) be the arc length of the upper half of the ellipse with equation**

\[
y = \frac{b}{a} \sqrt{a^2 - x^2}
\]

(Figure 21) and let \(\eta = \sqrt{1 - \left( \frac{b^2}{a^2} \right)^2}\). Use substitution to show that

\[
L = a \int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2 \sin^2 \theta} \, d\theta
\]

Use a computer algebra system to approximate \(L\) for \(a = 2, b = 1\).

**Solution**

Let \(y = \frac{b}{a} \sqrt{a^2 - x^2}\). Then

\[
1 + (y')^2 = \frac{b^2 x^2 + a^2 (a^2 - x^2)}{a^4 (a^2 - x^2)}
\]

and

\[
s = \int_{-a}^{a} \sqrt{\frac{b^2 x^2 + a^2 (a^2 - x^2)}{a^4 (a^2 - x^2)}} \, dx.
\]

With the substitution \(x = a \sin t, dx = a \cos t \, dt, a^2 - x^2 = a^2 \cos^2 t\) and

\[
s = a \int_{-\pi/2}^{\pi/2} \sqrt{\frac{a^2 b^2 \sin^2 t + a^2 a^2 \cos^2 t}{a^4 (a^2 \cos^2 t)}} \, dt = a \int_{-\pi/2}^{\pi/2} \sqrt{\frac{b^2 \sin^2 t}{a^2} + \cos^2 t} \, dt
\]

Because

\[
\eta = \sqrt{1 - \left( \frac{b^2}{a^2} \right)^2}, \quad \eta^2 = 1 - \left( \frac{b^2}{a^2} \right)^2
\]

we then have

\[
1 - \eta^2 \sin^2 t = 1 - \left( 1 - \left( \frac{b^2}{a^2} \right) \sin^2 t \right) = 1 - \sin^2 t + \frac{b^2}{a^2} \sin^2 t = \cos^2 t + \frac{b^2}{a^2} \sin^2 t
\]
which is the same as the expression under the square root above. Substituting, we get
\[ s = a \int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2 \sin^2 t} \, dt \]

When \( a = 2 \) and \( b = 1 \), \( \eta^2 = \frac{3}{4} \). Using a computer algebra system to approximate the value of the definite integral, we find \( s \approx 4.84422 \).

54. Prove that the portion of a sphere of radius \( R \) seen by an observer located at a distance \( d \) above the North Pole has area \( A = 2\pi dR^2 / (d + R) \). Hint: According to Exercise 48, the cap has surface area \( 2\pi Rh \). Show that \( h = dR / (d + R) \) by applying the Pythagorean Theorem to the three right triangles in Figure 22.

**SOLUTION** Label distances as shown in the figure below.

\[
\begin{align*}
(d + R)^2 & = R^2 + k^2 = R^2 + (d + h)^2 + h^2 = R^2 + (d + h)^2 + R^2 - (R - h)^2. \\
\text{Solving for } h \text{ yields} \\
\frac{d^2 + 2dh + h^2}{2dR} & = \frac{2dh + 2Rh}{2R} \\
h & = \frac{dR}{d + R}
\end{align*}
\]

and thus
\[
SA = 2\pi R \left( \frac{dR}{d + R} \right).
\]

55. Suppose that the observer in Exercise 54 moves off to infinity—that is, \( d \to \infty \). What do you expect the limiting value of the observed area to be? Check your guess by calculating the limit using the formula for the area in the previous exercise.

**SOLUTION** We would assume the observed surface area would approach \( 2\pi R^2 \) which is the surface area of a hemisphere of radius \( R \). To verify this, observe:
\[
\lim_{d \to \infty} SA = \lim_{R \to \infty} \frac{2\pi R^2 d}{R + d} = \lim_{R \to \infty} \frac{2\pi R^2}{1} = 2\pi R^2.
\]

56. Let \( M \) be the total mass of a metal rod in the shape of the curve \( y = f(x) \) over \([a, b]\) whose mass density \( \rho(x) \) varies as a function of \( x \). Use Riemann sums to justify the formula
\[
M = \int_a^b \rho(x) \sqrt{1 + f'(x)^2} \, dx
\]
SOLUTION  Divide the interval \([a, b]\) into \(n\) subintervals, which we shall denote by \([x_{j-1}, x_j]\) for \(j = 1, 2, 3, \ldots, n\). On each subinterval, we will assume that the mass density of the rod is constant; hence, the mass of the corresponding segment of the rod will be approximately equal to the product of the mass density of the segment and the length of the segment. Specifically, let \(c_j\) be any point in the \(j\)th subinterval and approximate the mass of the segment by 

\[
\rho(c_j)\sqrt{1 + \left(f'(c_j)^2\right)} \Delta x,
\]

where \(\sqrt{1 + f'(c_j)^2} \Delta x\) is the approximate length of the segment. Thus,

\[
M \approx \sum_{j=1}^{n} \rho(c_j)\sqrt{1 + f'(c_j)^2} \Delta x.
\]

As \(n \to \infty\), this Riemann sum approaches a definite integral, and we have

\[
M = \int_{a}^{b} \rho(x)\sqrt{1 + f'(x)^2} \, dx.
\]

57. Let \(f\) be an increasing function on \([a, b]\) and let \(g\) be its inverse. Argue on the basis of arc length that the following equality holds:

\[
\int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx = \int_{f(a)}^{f(b)} \sqrt{1 + g'(y)^2} \, dy
\]

Then use the substitution \(u = f(x)\) to prove Eq. (5).

SOLUTION  Since the graphs of \(f(x)\) and \(g(x)\) are symmetric with respect to the line \(y = x\), the arc length of the curves will be equal on the respective domains. Since the domain of \(g\) is the range of \(f\), on \(f(a)\) to \(f(b)\), \(g(x)\) will have the same arc length as \(f(x)\) on \(a\) to \(b\). If \(g(x) = f^{-1}(x)\) and \(u = f(x)\), then \(x = g(u)\) and \(du = f'(x) \, dx\). But

\[
g'(u) = \frac{1}{f'(g(u))} = \frac{1}{f'(x)} \Rightarrow f'(x) = \frac{1}{g'(u)}
\]

Now substituting \(u = f(x)\),

\[
s = \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{1}{g'(u)}\right)^2} g'(u) \, du = \int_{f(a)}^{f(b)} \sqrt{g'(u)^2 + 1} \, du
\]

8.2 Fluid Pressure and Force

Preliminary Questions
1. How is pressure defined?

SOLUTION  Pressure is defined as force per unit area.

2. Fluid pressure is proportional to depth. What is the factor of proportionality?

SOLUTION  The factor of proportionality is the weight density of the fluid, \(w = \rho g\), where \(\rho\) is the mass density of the fluid.

3. When fluid force acts on the side of a submerged object, in which direction does it act?

SOLUTION  Fluid force acts in the direction perpendicular to the side of the submerged object.

4. Why is fluid pressure on a surface calculated using thin horizontal strips rather than thin vertical strips?

SOLUTION  Pressure depends only on depth and does not change horizontally at a given depth.

5. If a thin plate is submerged horizontally, then the fluid force on one side of the plate is equal to pressure times area. Is this true if the plate is submerged vertically?

SOLUTION  When a plate is submerged vertically, the pressure is not constant along the plate, so the fluid force is not equal to the pressure times the area.
Exercises

1. A box of height 6 m and square base of side 3 m is submerged in a pool of water. The top of the box is 2 m below the surface of the water.
   (a) Calculate the fluid force on the top and bottom of the box.
   (b) Write a Riemann sum that approximates the fluid force on a side of the box by dividing the side into $N$ horizontal strips of thickness $\Delta y = 6/N$.
   (c) To which integral does the Riemann sum converge?
   (d) Compute the fluid force on a side of the box.

**solution**

(a) At a depth of 2 m, the pressure on the top of the box is $\rho gh = 10^3 \cdot 9.8 \cdot 2 = 19,600$ Pa. The top has area 9 m$^2$, and the pressure is constant, so the force on the top of the box is $19,600 \cdot 9 = 176,400$ N. At a depth of 8 m, the pressure on the bottom of the box is $\rho gh = 10^3 \cdot 9.8 \cdot 8 = 78,400$ Pa, so the force on the bottom of the box is $78,400 \cdot 9 = 705,600$ N.

(b) Let $y_j$ denote the depth of the $j$th strip, for $j = 1, 2, 3, \ldots, N$; the pressure at this depth is $10^3 \cdot 9.8 \cdot y_j = 9800y_j$ Pa. The strip has thickness $\Delta y$ m and length 3 m, so has area $3\Delta y$ m$^2$. Thus the force on the strip is $29,400y_j \Delta y$ N. Sum over all the strips to conclude that the force on one side of the box is approximately

$$F \approx \sum_{j=1}^{N} 29,400y_j \Delta y.$$

(c) As $N \to \infty$, the Riemann sum in part (b) converges to the definite integral $29,400 \int_2^8 y \, dy$.

(d) Using the result from part (c), the fluid force on one side of the box is

$$29,400 \int_2^8 y \, dy = 14,700y^2 \bigg|_2^8 = 882,000$$ N.

2. A square plate that is 2 meters by 2 meters is submerged in water so that its top edge is level with the surface of the water. Calculate the fluid force on one side of it.

**solution** Divide the side of the plate into $N$ horizontal strips, each of thickness $\Delta y = 2/N$. Moreover, let $y_j$ denote the depth of the $j$th strip. The pressure on the side of the plate at this depth is $\rho gy_j = 10^3 (9.8)y_j = 9800y_j$ Pa.

The strip has thickness $\Delta y$ m and length 2 m, so has area $2\Delta y$ m$^2$. Consequently, the fluid force on this strip is $19,600y_j \Delta y$ N. Summing over all of the strips yields the fluid force on the side of the plate

$$F \approx \sum_{j=1}^{N} 19,600y_j \Delta y.$$

As $N \to \infty$, this Riemann sum converges to the definite integral

$$19,600 \int_0^2 y \, dy.$$

Finally,

$$F = 19,600 \int_0^2 y \, dy = 9800y^2 \bigg|_0^2 = 39,200$$ N.

3. If a rectangular plate that is 1 meter by 2 meters is dipped into a pool of water so that initially its top edge of length 1 is even with the surface of the water, and then it is lowered so that its top edge is at a depth of 1 m, calculate the increase in fluid force on one side of it.

**solution** Start with the initial configuration, with the top edge of the plate even with the surface of the water. Divide the side of the plate into $N$ horizontal strips, each of thickness $\Delta y = 2/N$. Moreover, let $y_j$ denote the depth of the $j$th strip. The pressure on the side of the plate at this depth is $\rho gy_j = 10^3 (9.8)y_j = 9800y_j$ Pa.
The strip has thickness \( \Delta y \) and length 1 m, so the area is \( \Delta y \) m\(^2\). Consequently, the fluid force on this strip is \( 9800y_j \Delta y \) N. Summing over all of the strips yields the fluid force on the side of the plate

\[
F \approx \sum_{j=1}^{N} 9800y_j \Delta y.
\]

As \( N \to \infty \), this Riemann sum converges to the definite integral

\[
9800 \int_{0}^{2} y \, dy.
\]

Thus, the initial fluid force on one side of the plate is

\[
F = 9800 \int_{0}^{2} y \, dy = 4900y^2 \bigg|_{0}^{2} = 19,600 \text{ N}.
\]

Lowering the plate so that its top edge is at a depth of 1 m, the fluid force on one side of the plate becomes

\[
F = 9800 \int_{1}^{3} y \, dy = 4900y^2 \bigg|_{1}^{3} = 39,200 \text{ N}.
\]

Finally, the increase in the fluid force on one side of the plate is 19,600 N.

4. A plate in the shape of an isosceles triangle with base 1 m and height 2 m is submerged vertically in a tank of water so that its vertex touches the surface of the water (Figure 7).

(a) Show that the width of the triangle at depth \( y \) is \( f(y) = \frac{1}{2}y \).

(b) Consider a thin strip of thickness \( \Delta y \) at depth \( y \). Explain why the fluid force on a side of this strip is approximately equal to \( \rho g \frac{1}{2} y^2 \Delta y \).

(c) Write an approximation for the total fluid force \( F \) on a side of the plate as a Riemann sum and indicate the integral to which it converges.

(d) Calculate \( F \).

\[
\text{FIGURE 7}
\]

\[
\text{SOLUTION}
\]

(a) By similar triangles, \( \frac{y}{2} = \frac{f(y)}{1} \) so \( f(y) = \frac{y}{2} \).

(b) The pressure at a depth of \( y \) feet is \( \rho g y \) Pa, and the area of the strip is approximately \( f(y) \Delta y = \frac{1}{2} y \Delta y \) m\(^2\). Therefore, the fluid force on this strip is approximately

\[
\rho g \left( \frac{1}{2} y \Delta y \right) = \frac{1}{2} \rho g y^2 \Delta y \text{ N}.
\]

(c) \( F \approx \sum_{j=1}^{N} \rho g \frac{y_j^2}{2} \Delta y \). As \( N \to \infty \), the Riemann sum converges to the definite integral

\[
\frac{\rho g}{2} \int_{0}^{2} y^2 \, dy.
\]

(d) Using the result of part (c),

\[
F = \frac{\rho g}{2} \int_{0}^{2} y^2 \, dy = \frac{\rho g}{2} \left( \frac{y^3}{3} \right) \bigg|_{0}^{2} = 9800 \cdot \frac{8}{3} = \frac{39200}{3} \text{ N}.
\]
5. Repeat Exercise 4, but assume that the top of the triangle is located 3 m below the surface of the water.

SOLUTION

(a) Examine the figure below. By similar triangles, \( \frac{y - 3}{2} = \frac{f(y)}{1} \) so \( f(y) = \frac{y - 3}{2} \).

\[
\begin{align*}
\frac{y - 3}{2} & = \frac{f(y)}{1} \\
\Rightarrow f(y) & = \frac{y - 3}{2}.
\end{align*}
\]

(b) The pressure at a depth of \( y \) feet is \( \rho gy \) Pa, and the area of the strip is approximately \( f(y)/\Delta y \) m². Therefore, the fluid force on this strip is approximately

\[
\rho g \left( \frac{1}{2} (y - 3) \Delta y \right) = \frac{1}{2} \rho g y (y - 3) \Delta y \text{ N.}
\]

(c) \( F \approx \sum_{j=1}^{N} \rho g \frac{y_j^2 - 3y_j}{2} \Delta y \). As \( N \to \infty \), the Riemann sum converges to the definite integral

\[
\frac{\rho g}{2} \int_{3}^{5} (y^2 - 3y) \, dy.
\]

(d) Using the result of part (c),

\[
F = \frac{\rho g}{2} \int_{3}^{5} (y^2 - 3y) \, dy = \frac{\rho g}{2} \left( \frac{3}{3} - \frac{3y^2}{2} \right) \bigg|_{3}^{5} = \frac{9800}{2} \left[ \left( \frac{125}{3} - \frac{75}{2} \right) - \left( \frac{9}{2} - \frac{27}{2} \right) \right] = \frac{127,400}{3} \text{ N.}
\]

6. The plate \( R \) in Figure 8, bounded by the parabola \( y = x^2 \) and \( y = 1 \), is submerged vertically in water (distance in meters).

(a) Show that the width of \( R \) at height \( y \) is \( f(y) = 2\sqrt{y} \) and the fluid force on a side of a horizontal strip of thickness \( \Delta y \) at height \( y \) is approximately \( (\rho g)2y^{1/2}(1 - y)\Delta y \).

(b) Write a Riemann sum that approximates the fluid force \( F \) on a side of \( R \) and use it to explain why

\[
F = \rho g \int_{0}^{1} 2y^{1/2}(1 - y) \, dy
\]

(c) Calculate \( F \).

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Figure 8}
\end{figure}
\]

SOLUTION

(a) At height \( y \), the thin plate \( R \) extends from the point \((-\sqrt{y}, y)\) on the left to the point \((\sqrt{y}, y)\) on the right; thus, the width of the plate is \( f(y) = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y} \). Moreover, the area of a horizontal strip of thickness \( \Delta y \) at height \( y \) is \( f(y) \Delta y = 2\sqrt{y} \Delta y \). Because the water surface is at height \( y = 1 \), the horizontal strip at height \( y \) is at a depth of \( 1 - y \). Consequently, the fluid force on the strip is approximately

\[
\rho g (1 - y) \times 2\sqrt{y} \Delta y = 2\rho g y^{1/2}(1 - y) \Delta y.
\]
(b) If the plate is divided into $N$ strips with $y_j$ being the representative height of the $j$th strip (for $j = 1, 2, 3, \ldots, N$), then the total fluid force exerted on the plate is

$$F \approx 2\rho g \sum_{j=1}^{N} (1 - y_j) \sqrt{y_j} \Delta y.$$  

As $N \to \infty$, the Riemann sum converges to the definite integral

$$2\rho g \int_{0}^{1} (1 - y) \sqrt{y} \, dy.$$  

(c) Using the result from part (b),

$$F = 2\rho g \int_{0}^{1} (1 - y) \sqrt{y} \, dy = 2\rho g \left( \frac{2}{3} y^{3/2} - \frac{2}{5} y^{5/2} \right) \Bigg|_{0}^{1} = \frac{8}{15} \rho g.$$  

Now, $\rho g = 9800$ N/m$^3$ so that $F = \frac{15680}{3}$ N.

7. Let $F$ be the fluid force on a side of a semicircular plate of radius $r$ meters, submerged vertically in water so that its diameter is level with the water’s surface (Figure 9).

(a) Show that the width of the plate at depth $y$ is $2\sqrt{r^2 - y^2}$.

(b) Calculate $F$ as a function of $r$ using Eq. (2).

![Figure 9](image_url)

SOLUTION

(a) Place the origin at the center of the semicircle and point the positive $y$-axis downward. The equation for the edge of the semicircular plate is then $x^2 + y^2 = r^2$. At a depth of $y$, the plate extends from the point $(-\sqrt{r^2 - y^2}, y)$ on the left to the point $(\sqrt{r^2 - y^2}, y)$ on the right. The width of the plate at depth $y$ is then

$$\sqrt{r^2 - y^2} - (-\sqrt{r^2 - y^2}) = 2\sqrt{r^2 - y^2}.$$  

(b) With $\rho g = 9800$ N/m$^3$,

$$F = 2\rho g \int_{0}^{r} y \sqrt{r^2 - y^2} \, dy = -\frac{19600}{3} (r^2 - y^2)^{3/2} \Bigg|_{0}^{r} = \frac{19600r^3}{3} \text{ N}.$$  

8. Calculate the force on one side of a circular plate with radius 2 m, submerged vertically in a tank of water so that the top of the circle is tangent to the water surface.

SOLUTION Place the origin at the point where the top of the circle is tangent to the water surface and orient the positive $y$-axis pointing downward. The equation of the circle is then $x^2 + (y - 2)^2 = 4$, and the width at any depth $y$ is $2\sqrt{4 - (y - 2)^2}$. Thus,

$$F = 2\rho g \int_{0}^{4} y \sqrt{4 - (y - 2)^2} \, dy.$$  

Using the substitution $y - 2 = 2 \sin \theta$, $dy = 2 \cos \theta \, d\theta$, the limits of integration become $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so we find

$$F = 2\rho g \int_{0}^{4} y \sqrt{4 - (y - 2)^2} \, dy$$  

$$= 2\rho g \int_{-\pi/2}^{\pi/2} (2 + 2 \sin \theta)(2 \cos \theta) 2 \cos \theta \, d\theta$$  

$$= 16\rho g \left( \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta - \frac{1}{3} \cos^3 \theta \right) \Bigg|_{-\pi/2}^{\pi/2}$$  

$$= 16\rho g \left( \frac{\pi}{4} + 0 - 0 - \left( -\frac{\pi}{4} + 0 - 0 \right) \right) = 8\rho g \pi = 78,400 \pi \text{ N}.$$  

9. A semicircular plate of radius \( r \) meters, oriented as in Figure 9, is submerged in water so that its diameter is located at a depth of \( m \) meters. Calculate the fluid force on one side of the plate in terms of \( m \) and \( r \).

**SOLUTION** Place the origin at the center of the semicircular plate with the positive \( y \)-axis pointing downward. The water surface is then at \( y = -m \). Moreover, at location \( y \), the width of the plate is \( 2\sqrt{r^2 - y^2} \) and the depth is \( y + m \). Thus,

\[
F = 2 \rho g \int_0^r (y + m)\sqrt{r^2 - y^2} \, dy.
\]

Now,

\[
\int_0^r y\sqrt{r^2 - y^2} \, dy = -\frac{1}{3}(r^2 - y^2)^{3/2}\bigg|_0^r = \frac{1}{3}r^3.
\]

Geometrically,

\[
\int_0^r \sqrt{r^2 - y^2} \, dy
\]

represents the area of one quarter of a circle of radius \( r \), and thus has the value \( \frac{\pi r^2}{4} \). Bringing these results together, we find that

\[
F = 2 \rho g \left( \frac{1}{3}r^3 + \frac{\pi}{4}r^2 \right) = \frac{19,600}{3}r^3 + 4900mr^2 \text{ N}.
\]

10. A plate extending from depth \( y = 2 \) m to \( y = 5 \) m is submerged in a fluid of density \( \rho = 850 \text{ kg/m}^3 \). The horizontal width of the plate at depth \( y \) is \( f(y) = 2(1 + y^2)^{-1} \). Calculate the fluid force on one side of the plate.

**SOLUTION** The fluid force on one side of the plate is given by

\[
F = \rho g \int_2^5 yf(y) \, dy = \rho g \int_2^5 2(1 + y^2)^{-1} \, dy = \rho g \ln(1 + y^2)\bigg|_2^5 = \rho g(\ln 26 - \ln 5)
\]

\[
= 8330 \ln 26 \approx 13733.32 \text{ N}.
\]

11. Figure 10 shows the wall of a dam on a water reservoir. Use the Trapezoidal Rule and the width and depth measurements in the figure to estimate the fluid force on the wall.

**SOLUTION** Let \( f(y) \) denote the width of the dam wall at depth \( y \) feet. Then the force on the dam wall is

\[
F = w \int_0^{100} yf(y) \, dy.
\]

Using the Trapezoidal Rule and the width and depth measurements in the figure,

\[
F \approx 10w \left[ \frac{40 \cdot f(0)}{2} + 2 \cdot 20 \cdot f(20) + 2 \cdot 40 \cdot f(40) + 2 \cdot 60 \cdot f(60) + 2 \cdot 80 \cdot f(80) + 100 \cdot f(100) \right]
\]

\[
= 10w(0 + 66000 + 112000 + 132000 + 144000 + 60000) = 321,250,000 \text{ lb}.
\]

12. Calculate the fluid force on a side of the plate in Figure 11(A), submerged in water.

**SOLUTION** The width of the plate varies linearly from 4 meters at a depth of 3 meters to 7 meters at a depth of 5 meters. Thus, at depth \( y \), the width of the plate is

\[
4 + \frac{3}{2}(y - 3) = \frac{3}{2}y - \frac{1}{2}.
\]

Finally, the force on a side of the plate is

\[
F = w \int_3^5 \left( \frac{3}{2}y - \frac{1}{2} \right) \, dy = w \left( \frac{1}{2}y^3 - \frac{1}{4}y^2 \right)\bigg|_3^5 = 45w = 441,000 \text{ N}.
\]
13. Calculate the fluid force on a side of the plate in Figure 11(B), submerged in a fluid of mass density \( \rho = 800 \text{ kg/m}^3 \).

![Figure 11](image)

**Solution** Because the fluid has a mass density of \( \rho = 800 \text{ kg/m}^3 \),

\[
\begin{align*}
\text{the fluid force can be calculated using the formula:} \\
F = \rho g \int_0^2 2y \, dy + \rho g \int_2^6 2y \, dy \\
&= \rho g \left[ \frac{2}{3} y^3 \right]_0^2 + \rho g \left[ \frac{2}{3} y^3 \right]_2^6 \\
&= \frac{8}{3} \rho g + \frac{32}{3} \rho g \\
&= 8 \rho g ,
\end{align*}
\]

Thus, we can calculate the force as

\[
F = 8 \rho g = 8 \cdot 800 \cdot 9.8 = 6153.6 \text{ N}.
\]

14. Find the fluid force on the side of the plate in Figure 12, submerged in a fluid of density \( \rho = 1200 \text{ kg/m}^3 \). The edges of the plate are the curves \( y = x^{1/3} \) and \( y = -x^{1/3} \).

![Figure 12](image)

**Solution** At height \( y \), the plate extends from the point \((-y^3, y)\) on the left to the point \((y^3, y)\) on the right; thus, the width of the plate is \( f(y) = 2y^{1/3} \). Because the water surface is at height \( y = 2 \), the horizontal strip at height \( y \) is at a depth of \( 2 - y \). Consequently,

\[
\begin{align*}
F &= \rho g \int_0^2 (2 - y)(2y^3) \, dy \\
&= 2 \rho g \left[ \frac{1}{2} y^4 - \frac{1}{5} y^5 \right]_0^2 \\
&= \frac{16}{5} \rho g \approx 37,632 \text{ N}.
\end{align*}
\]

15. Let \( R \) be the plate in the shape of the region under \( y = \sin x \) for \( 0 \leq x \leq \frac{\pi}{2} \) in Figure 13(A). If \( R \) is rotated counterclockwise by \( 90^\circ \) and then submerged in a fluid of density \( 1100 \text{ kg/m}^3 \) with its top edge level with the surface of the fluid as in (B), find the fluid force on a side of \( R \).

![Figure 13](image)

**Solution** Place the origin at the bottom corner of the plate with the positive y-axis pointing upward. The fluid surface is then at height \( y = \frac{\pi}{2} \), and the horizontal strip of the plate at height \( y \) is at a depth of \( \frac{\pi}{2} - y \) and has a width of \( \sin y \). Now, using integration by parts we find

\[
\begin{align*}
F &= \rho g \int_0^{\pi/2} \left( \frac{\pi}{2} - y \right) \sin y \, dy \\
&= \rho g \left[ - \left( \frac{\pi}{2} - y \right) \cos y - \sin y \right]_0^{\pi/2} \\
&= \rho g \left( \frac{\pi}{2} - 1 \right) \\
&= 1100 \cdot 9.8 \left( \frac{\pi}{2} - 1 \right) \\
&\approx 6153.184 \text{ N}.
\end{align*}
\]
16. In the notation of Exercise 15, calculate the fluid force on a side of the plate $R$ if it is oriented as in Figure 13(A). You may need to use Integration by Parts and trigonometric substitution.

**Solution** Place the origin at the lower left corner of the plate. Because the fluid surface is at height $y = 1$, the horizontal strip at height $y$ is at a depth of $1 - y$. Moreover, this strip has a width of

$$\frac{\pi}{2} - \sin^{-1} y = \cos^{-1} y.$$

Thus,

$$F = \rho g \int_0^1 (1 - y) \cos^{-1} y \, dy.$$

Starting with integration by parts, we find

$$\int_0^1 (1 - y) \cos^{-1} y \, dy = \left( y - \frac{1}{2} y^2 \right) \cos^{-1} y \bigg|_0^1 + \int_0^1 \frac{y - \frac{1}{2} y^2}{\sqrt{1 - y^2}} \, dy$$

$$= \frac{1}{2} \cos^{-1} 1 + \int_0^1 \frac{y - \frac{1}{2} y^2}{\sqrt{1 - y^2}} \, dy = \int_0^1 \frac{y}{\sqrt{1 - y^2}} \, dy - \frac{1}{2} \int_0^1 \frac{y^2}{\sqrt{1 - y^2}} \, dy.$$

Now,

$$\int_0^1 \frac{y}{\sqrt{1 - y^2}} \, dy = - \sqrt{1 - y^2} \bigg|_0^1 = 1.$$

For the remaining integral, we use the trigonometric substitution $y = \sin \theta$, $dy = \cos \theta \, d\theta$ and find

$$\int_0^1 \frac{y^2}{\sqrt{1 - y^2}} \, dy = \frac{1}{2} \int_0^1 \sin^2 \theta \, d\theta = \frac{1}{4} \left( \theta - \sin \theta \cos \theta \right) \bigg|_{\theta=0}^{\theta=1}$$

$$= \frac{1}{4} \left( \sin^{-1} y - y \sqrt{1 - y^2} \right) \bigg|_0^1 = \frac{\pi}{8}.$$

Finally,

$$F = \rho g \left( 1 - \frac{\pi}{8} \right) = 1100 \cdot 9.8 \left( 1 - \frac{\pi}{8} \right) \approx 6546.70 \text{ N}.$$

17. Calculate the fluid force on one side of a plate in the shape of region $A$ shown Figure 14. The water surface is at $y = 1$, and the fluid has density $\rho = 900 \text{ kg/m}^3$.

**Solution** Because the fluid surface is at height $y = 1$, the horizontal strip at height $y$ is at a depth of $1 - y$. Moreover, this strip has a width of $e - e^y$. Thus,

$$F = \rho g \int_0^1 (1 - y)(e - e^y) \, dy = \rho g \int_0^1 (1 - y) \, dy - \rho g \int_0^1 (1 - y)e^y \, dy.$$

Now,

$$\int_0^1 (1 - y) \, dy = \left( y - \frac{1}{2} y^2 \right) \bigg|_0^1 = \frac{1}{2},$$

and using integration by parts

$$\int_0^1 (1 - y)e^y \, dy = \left( (1 - y)e^y + e^y \right) \bigg|_0^1 = e - 2.$$
Combining these results, we find that

\[ F = \rho g \left( \frac{1}{2} e - (e - 2) \right) = \rho g \left( 2 - \frac{1}{2} e \right) = 900 \cdot 9.8 \left( 2 - \frac{1}{2} e \right) \approx 5652.37 \text{ N}. \]

18. Calculate the fluid force on one side of the “infinite” plate \( B \) in Figure 14, assuming the fluid has density \( \rho = 900 \text{ kg/m}^3 \).

**Solution** Because the fluid surface is at height \( y = 1 \), the horizontal strip at height \( y \) is at a depth of \( 1 - y \). Moreover, this strip has a width of \( e^y \). Thus,

\[ F = \rho g \int_{-\infty}^{0} (1 - y)e^y \, dy. \]

Using integration by parts, we find

\[ \int_{-\infty}^{0} (1 - y)e^y \, dy = [(1 - y)e^y + e^y]|_{-\infty}^{0} = 2. \]

Thus, \( F = 2\rho g = 2 \cdot 900 \cdot 9.8 = 17,640 \text{ N}. \)

19. Figure 15(A) shows a ramp inclined at 30° leading into a swimming pool. Calculate the fluid force on the ramp.

**Solution** A horizontal strip at depth \( y \) has length \( 6 \) and width

\[ \Delta y \sin 30^\circ = 2\Delta y. \]

Thus,

\[ F = 2w \int_{0}^{4} 6y \, dy = 96w. \]

If distances are in feet, then \( w = 62.5 \text{ lb/ft}^3 \) and \( F = 6000 \text{ lb} \); if distances are in meters, then \( w = 9800 \text{ N/m}^3 \) and \( F = 940,800 \text{ N} \).

20. Calculate the fluid force on one side of the plate (an isosceles triangle) shown in Figure 15(B).

**Solution** A horizontal strip at depth \( y \) has length \( f(y) = \frac{1}{10} y \) and width

\[ \frac{\Delta y}{\sin 60^\circ} = 2\sqrt{3}\Delta y. \]

Thus,

\[ F = \frac{\sqrt{3}}{3} w \int_{0}^{10} y^2 \, dy = \frac{200\sqrt{3}}{3} w. \]

If distances are in feet, then \( w = 62.5 \text{ lb/ft}^3 \) and \( F \approx 7216.88 \text{ lb} \); if distances are in meters, then \( w = 9800 \text{ N/m}^3 \) and \( F \approx 1,131,606.5 \text{ N} \).

21. The massive Three Gorges Dam on China’s Yangtze River has height 185 m (Figure 16). Calculate the force on the dam, assuming that the dam is a trapezoid of base 2000 m and upper edge 3000 m, inclined at an angle of 55° to the horizontal (Figure 17).
CHAPTER 8  FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

FIGURE 16 Three Gorges Dam on the Yangtze River. (Reuters/STR/Landov)

FIGURE 17

solution

Let \( y = 0 \) be at the bottom of the dam, so that the top of the dam is at \( y = 185 \). Then the width of the dam at height \( y \) is \( 2000 + \frac{1000y}{185} \). The dam is inclined at an angle of 55° to the horizontal, so the height of a horizontal strip is

\[
\frac{\Delta y}{\sin 55^\circ} \approx \Delta y
\]

so that the area of such a strip is

\[
1.221 \left( 2000 + \frac{1000y}{185} \right) \Delta y
\]

Then

\[
F = \rho g \int_0^{185} 1.221 \left( 2000 + \frac{1000y}{185} \right) dy = \rho g \int_0^{185} 2442y + 6.6y^2 dy = \rho g (1221y^2 + 2.2y^3)
\]

\[
= 55,718,300 \rho g = 55,718 \times 9800 = 5.460393400 \times 10^{11} \text{ N}.
\]

22. A square plate of side 3 m is submerged in water at an incline of 30° with the horizontal. Calculate the fluid force on one side of the plate if the top edge of the plate lies at a depth of 6 m.

solution

Because the plate is 3 meters on a side, is submerged at a horizontal angle of 30°, and has its top edge located at a depth of 6 meters, the bottom edge of the plate is located at a depth of \( 6 + 3 \sin 30^\circ = \frac{15}{2} \) meters. Let \( y \) denote the depth at any point of the plate. The width of each horizontal strip of the plate is then

\[
\frac{\Delta y}{\sin 30^\circ} = 2\Delta y,
\]

and

\[
F = \rho g \int_0^{15/2} (2)3y dy = (\rho g) \frac{243}{4} = 595350 \text{ N}.
\]

23. The trough in Figure 18 is filled with corn syrup, whose weight density is 90 lb/ft³. Calculate the force on the front side of the trough.

solution

Place the origin along the top edge of the trough with the positive y-axis pointing downward. The width of the front side of the trough varies linearly from \( b \) when \( y = 0 \) to \( a \) when \( y = h \); thus, the width of the front side of the trough at depth \( y \) feet is given by

\[
b + \frac{a - b}{h} y.
\]

Now,

\[
F = w \int_0^h y \left( b + \frac{a - b}{h} y \right) dy = w \left[ \frac{1}{2} by^2 + \frac{a - b}{3h} y^3 \right]_0^h = w \left( \frac{b}{6} + \frac{a}{3} \right) h^2 = (15b + 30a)h^2 \text{ lb}.
\]

24. Calculate the fluid pressure on one of the slanted sides of the trough in Figure 18 when it is filled with corn syrup as in Exercise 23.
The diagram above displays a side view of the trough. From this diagram, we see that
\[ \sin \theta = \frac{h}{\sqrt{\left(\frac{b-a}{2}\right)^2 + h^2}}. \]

Thus,
\[ F = \frac{w}{\sin \theta} \int_0^h d \cdot y \, dy = \frac{90}{h} \sqrt{\left(\frac{b-a}{2}\right)^2 + h^2} \frac{dh^2}{h} = 45dh \sqrt{\left(\frac{b-a}{2}\right)^2 + h^2}. \]

Further Insights and Challenges

25. The end of the trough in Figure 19 is an equilateral triangle of side 3. Assume that the trough is filled with water to height \( H \). Calculate the fluid force on each side of the trough as a function of \( H \) and the length \( l \) of the trough.

SOLUTION Place the origin at the lower vertex of the trough and orient the positive \( y \)-axis pointing upward. First, consider the faces at the front and back ends of the trough. A horizontal strip at height \( y \) has a length of \( 2y\sqrt{3} \) and is at a depth of \( H - y \). Thus,
\[ F = w \int_0^H (H - y) \frac{2y}{\sqrt{3}} \, dy = w \left( \frac{H}{\sqrt{3}} \frac{y^2}{2} - \frac{2}{3\sqrt{3}} y^3 \right) \bigg|_0^H = \frac{\sqrt{3}}{9} wH^3. \]

For the slanted sides, we note that each side makes an angle of 60° with the horizontal. If we let \( \ell \) denote the length of the trough, then
\[ F = \frac{2w\ell}{\sqrt{3}} \int_0^H (H - y) \, dy = \frac{\sqrt{3}}{3} \ell wH^2. \]

26. A rectangular plate of side \( \ell \) is submerged vertically in a fluid of density \( w \), with its top edge at depth \( h \). Show that if the depth is increased by an amount \( \Delta h \), then the force on a side of the plate increases by \( wA\Delta h \), where \( A \) is the area of the plate.

SOLUTION Let \( F_1 \) be the force on a side of the plate when its top edge is at depth \( h \) and \( F_2 \) be the force on a side of the plate when its top edge is at depth \( h + \Delta h \). Further, let \( b \) denote the width of the rectangular plate. Then
\[ F_1 = w \int_h^{h+\ell} yb \, dy = bw \left( \frac{y^2}{2} \right) \bigg|_h^{h+\ell} = bw \left( \frac{\ell^2 + 2\ell h}{2} \right), \]
\[ F_2 = w \int_{h+\Delta h}^{h+\ell+\Delta h} yb \, dy = bw \left( \frac{y^2}{2} \right) \bigg|_{h+\Delta h}^{h+\ell+\Delta h} = bw \frac{\ell^2 + 2\ell h + 2\ell \Delta h}{2}, \]

and \( F_2 - F_1 = bw\ell \Delta h = wA\Delta h \).
27. Prove that the force on the side of a rectangular plate of area $A$ submerged vertically in a fluid is equal to $p_0 A$, where $p_0$ is the fluid pressure at the center point of the rectangle.

**Solution** Let $\ell$ denote the length of the vertical side of the rectangle, $x$ denote the length of the horizontal side of the rectangle, and suppose the top edge of the rectangle is at depth $y = m$. The pressure at the center of the rectangle is then

$$ p_0 = w \left( m + \frac{\ell}{2} \right), $$

and the force on the side of the rectangular plate is

$$ F = \int_m^{\ell+m} wxy \, dy = \frac{wx}{2} \left[ (\ell + m)^2 - m^2 \right] = \frac{wx\ell}{2} (\ell + 2m) = A w \left( \frac{\ell}{2} + m \right) = Ap_0. $$

28. If the density of a fluid varies with depth, then the pressure at depth $y$ is $p(y)$ (which need not equal $wy$ as in the case of constant density). Use Riemann sums to argue that the total force $F$ on the flat side of a submerged object submerged vertically is

$$ F = \int_a^b x f(y) p(y) \, dy, $$

where $f(y)$ is the width of the flat side at depth $y$.

**Solution** Suppose the object extends from a depth of $y = a$ to a depth of $y = b$. Divide the object into $N$ horizontal strips, each of width $\Delta y$. Let $p(y)$ denote the pressure within the fluid at depth $y$ and $f(y)$ denote the width of the flat side of the submerged object at depth $y$. The approximate force on the $j$th strip ($j = 1, 2, 3, \ldots, N$) is

$$ p(y_j) f(y_j) \Delta y, $$

where $y_j$ is a depth associated with the $j$th strip. Summing over all of the strips,

$$ F \approx \sum_{j=1}^N p(y_j) f(y_j) \Delta y. $$

As $N \to \infty$, this Riemann sum converges to a definite integral, and

$$ F = \int_a^b p(y) f(y) \, dy. $$

### 8.3 Center of Mass

**Preliminary Questions**

1. What are the $x$- and $y$-moments of a lamina whose center of mass is located at the origin?

**Solution** Because the center of mass is located at the origin, it follows that $M_x = M_y = 0$.

2. A thin plate has mass 3. What is the $x$-moment of the plate if its center of mass has coordinates $(2, 7)$?

**Solution** The $x$-moment of the plate is the product of the mass of the plate and the $x$-coordinate of the center of mass. Thus, $M_x = 3(2) = 21$.

3. The center of mass of a lamina of total mass 5 has coordinates $(2, 1)$. What are the lamina’s $x$- and $y$-moments?

**Solution** The $x$-moment of the plate is the product of the mass of the plate and the $x$-coordinate of the center of mass, whereas the $y$-moment is the product of the mass of the plate and the $x$-coordinate of the center of mass. Thus, $M_x = 5(1) = 5$, and $M_y = 5(2) = 10$.

4. Explain how the Symmetry Principle is used to conclude that the centroid of a rectangle is the center of the rectangle.

**Solution** Because a rectangle is symmetric with respect to both the vertical line and the horizontal line through the center of the rectangle, the Symmetry Principle guarantees that the centroid of the rectangle must lie along both of these lines. The only point in common to both lines of symmetry is the center of the rectangle, so the centroid of the rectangle must be the center of the rectangle.

5. Give an example of a plate such that its center of mass does not occur at any point on the plate.

**Solution** Consider the plate shown in the figure below. Symmetry guarantees that the center of mass is at the origin, which is not a point on the plate.
6. Draw a plate such that its center of mass occurs on its boundary. (You do not need to verify this fact. It should just be believable from the drawing.)

**Solution**  Consider the plate shown in the figure below. Because the hole missing from the plate is off-center, it is plausible that the center of mass occurs at the indicated boundary point.

---

**Exercises**

1. Four particles are located at points (1, 1), (1, 2), (4, 0), (3, 1).
   (a) Find the moments $M_x$ and $M_y$ and the center of mass of the system, assuming that the particles have equal mass $m$. 
   (b) Find the center of mass of the system, assuming the particles have masses 3, 2, 5, and 7, respectively.

**Solution**

(a) Because each particle has mass $m$,

\[
M_x = m(1) + m(2) + m(0) + m(1) = 4m;
\]
\[
M_y = m(1) + m(1) + m(4) + m(3) = 9m;
\]

and the total mass of the system is $4m$. Thus, the coordinates of the center of mass are

\[
\left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{9m}{4m}, \frac{4m}{4m} \right) = \left( \frac{9}{4}, 1 \right).
\]

(b) With the indicated masses of the particles,

\[
M_x = 3(1) + 2(2) + 5(0) + 7(1) = 14;
\]
\[
M_y = 3(1) + 2(1) + 5(4) + 7(3) = 46;
\]

and the total mass of the system is 17. Thus, the coordinates of the center of mass are

\[
\left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{46}{17}, \frac{14}{17} \right).
\]

2. Find the center of mass for the system of particles of masses 4, 2, 5, 1 located at (1, 2), (−3, 2), (2, −1), (4, 0).

**Solution**  With the indicated masses and locations of the particles

\[
M_x = 4(2) + 2(2) + 5(−1) + 1(0) = 7;
\]
\[
M_y = 4(1) + 2(−3) + 5(2) + 1(4) = 12;
\]

and the total mass of the system is 12. Thus, the coordinates of the center of mass are

\[
\left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( 1, \frac{7}{12} \right).
\]
3. Point masses of equal size are placed at the vertices of the triangle with coordinates \((a, 0), (b, 0),\) and \((0, c)\). Show that the center of mass of the system of masses has coordinates \(\left(\frac{1}{2}(a + b), \frac{1}{3}c\right)\).

**Solution** Let each particle have mass \(m\). The total mass of the system is then \(3m\) and the moments are

\[
M_x = 0(m) + 0(m) + c(m) = cm; \quad \text{and} \quad M_y = a(m) + b(m) + 0(m) = (a + b)m.
\]

Thus, the coordinates of the center of mass are

\[
\left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{(a + b)m}{3m}, \frac{cm}{3m}\right) = \left(\frac{a + b}{3}, \frac{c}{3}\right).
\]

4. Point masses of mass \(m_1, m_2,\) and \(m_3\) are placed at the points \((-1, 0), (3, 0),\) and \((0, 4)\).

(a) Suppose that \(m_1 = 6\). Find \(m_2\) such that the center of mass lies on the \(y\)-axis.

(b) Suppose that \(m_1 = 6\) and \(m_2 = 4\). Find the value of \(m_3\) such that \(y = 2\).

**Solution** With the given masses and locations, we find

\[
M_x = m_1(0) + m_2(0) + m_3(4) = 4m_3;
\]

\[
M_y = m_1(-1) + m_2(3) + m_3(0) = 3m_2 - m_1;
\]

and the total mass of the system is \(m_1 + m_2 + m_3\). Thus, the coordinates of the center of mass are

\[
\left(\frac{3m_2 - m_1}{m_1 + m_2 + m_3}, \frac{4m_3}{m_1 + m_2 + m_3}\right).
\]

(a) For the center of mass to lie on the \(y\)-axis, we must have \(3m_2 - m_1 = 0\), or \(m_2 = \frac{1}{3}m_1\). Given \(m_1 = 6\), it follows that \(m_2 = 2\).

(b) To have \(y_{CM} = 2\) requires

\[
\frac{4m_3}{m_1 + m_2 + m_3} = 2 \quad \text{or} \quad m_3 = m_1 + m_2.
\]

Given \(m_1 = 6\) and \(m_2 = 4\), it follows that \(m_3 = 10\).

5. Sketch the lamina \(S\) of constant density \(\rho = 3\) \(g/cm^2\) occupying the region beneath the graph of \(y = x^2\) for \(0 \leq x \leq 3\).

(a) Use Eqs. (1) and (2) to compute \(M_x\) and \(M_y\).

(b) Find the area and the center of mass of \(S\).

**Solution** A sketch of the lamina is shown below.

![Sketch of the lamina](image)

(a) Using (2),

\[
M_x = 3 \int_0^3 y(3 - \sqrt{y}) \, dy = \left(\frac{9y^2}{2} - \frac{6y^{5/2}}{5}\right) \bigg|_0^3 = 729/10.
\]

Using (1),

\[
M_y = 3 \int_0^3 x(x^2) \, dx = \frac{3x^4}{4} \bigg|_0^3 = 243/4.
\]

(b) The area of the lamina is

\[
A = \int_0^3 x^2 \, dx = \frac{x^3}{3} \bigg|_0^3 = 9\text{ cm}^2.
\]

With a constant density of \(\rho = 3\) \(g/cm^2\), the mass of the lamina is \(M = 27\) grams, and the coordinates of the center of mass are

\[
\left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{243/4}{27}, \frac{729/10}{27}\right) = \left(9 \frac{27}{10}, \frac{9}{27}\right).
\]
6. Use Eqs. (1) and (3) to find the moments and center of mass of the lamina \( S \) of constant density \( \rho = 2 \text{ g/cm}^2 \) occupying the region between \( y = x^2 \) and \( y = 9x \) over \([0, 3]\). Sketch \( S \), indicating the location of the center of mass.

**SOLUTION** With \( \rho = 2 \text{ g/cm}^2 \),

\[
M_x = \frac{1}{2} \rho \int_0^3 (9x)^2 - (x^2)^2 \, dx = \frac{3402}{5},
\]

and

\[
M_y = 2 \rho \int_0^3 x(9x - x^2) \, dx = \frac{243}{2}.
\]

The mass of the lamina is

\[
M = 2 \int_0^3 (9x - x^2) \, dx = 63 \text{ g},
\]

so the coordinates of the center of mass are

\[
\left( \frac{M_x}{M}, \frac{M_y}{M} \right) = \left( \frac{243}{126}, \frac{3402}{315} \right).
\]

A sketch of the lamina, with the location of the center of mass indicated, is shown below.

7. Find the moments and center of mass of the lamina of uniform density \( \rho \) occupying the region underneath \( y = x^3 \) for \( 0 \leq x \leq 2 \).

**SOLUTION** With uniform density \( \rho \),

\[
M_x = \frac{1}{2} \rho \int_0^2 (x^3)^2 \, dx = \frac{64\rho}{7} \quad \text{and} \quad M_y = \rho \int_0^2 x(x^3) \, dx = \frac{32\rho}{5}.
\]

The mass of the lamina is

\[
M = \rho \int_0^2 x^3 \, dx = 4\rho,
\]

so the coordinates of the center of mass are

\[
\left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{8}{5}, \frac{16}{7} \right).
\]

8. Calculate \( M_x \) (assuming \( \rho = 1 \)) for the region underneath the graph of \( y = 1 - x^2 \) for \( 0 \leq x \leq 1 \) in two ways, first using Eq. (2) and then using Eq. (3).

**SOLUTION** By (2),

\[
M_x = \int_0^1 y \sqrt{1 - y^2} \, dy.
\]

Using the substitution \( u = 1 - y, \, du = -dy \), we find

\[
M_x = \left[ \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_0^1 = \frac{4}{15}.
\]

By (3),

\[
M_x = \frac{1}{2} \int_0^1 (1 - x^2)^2 \, dx = \frac{1}{2} \left( x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right) \bigg|_0^1 = \frac{4}{15}.
\]
9. Let $T$ be the triangular lamina in Figure 21.

(a) Show that the horizontal cut at height $y$ has length $4 - \frac{2}{3}y$ and use Eq. (2) to compute $M_x$ (with $\rho = 1$).

(b) Use the Symmetry Principle to show that $M_y = 0$ and find the center of mass.

\[ \text{FIGURE 21 Isosceles triangle.} \]

**SOLUTION**

(a) The equation of the line from $(2, 0)$ to $(0, 6)$ is $y = -3x + 6$, so

\[ x = \frac{2 - \frac{1}{3}y}{3}. \]

The length of the horizontal cut at height $y$ is then

\[ 2 \left( 2 - \frac{1}{3}y \right) = 4 - \frac{2}{3}y, \]

and

\[ M_x = \int_0^6 y \left( 4 - \frac{2}{3}y \right) \, dy = 24. \]

(b) Because the triangular lamina is symmetric with respect to the $y$-axis, $x_{cm} = 0$, which implies that $M_y = 0$. The total mass of the lamina is

\[ M = 2 \int_0^2 (-3x + 6) \, dx = 12, \]

so $y_{cm} = 24/12$. Finally, the coordinates of the center of mass are $(0, 2)$.

In Exercises 10–17, find the centroid of the region lying underneath the graph of the function over the given interval.

10. $f(x) = 6 - 2x$, \hspace{1em} [0, 3]

**SOLUTION** The moments of the region are

\[ M_x = \frac{1}{2} \int_0^3 (6 - 2x)^2 \, dx = 18 \quad \text{and} \quad M_y = \int_0^3 x(6 - 2x) \, dx = 9. \]

The area of the region is

\[ A = \int_0^3 (6 - 2x) \, dx = 9, \]

so the coordinates of the centroid are

\[ \left( \frac{M_y}{A}, \frac{M_x}{A} \right) = (1, 2). \]

11. $f(x) = \sqrt{x}$, \hspace{1em} [1, 4]

**SOLUTION** The moments of the region are

\[ M_x = \frac{1}{2} \int_1^4 x \, dx = \frac{15}{4} \quad \text{and} \quad M_y = \int_1^4 x \sqrt{x} \, dx = \frac{62}{5}. \]

The area of the region is

\[ A = \int_1^4 \sqrt{x} \, dx = \frac{14}{3}, \]

so the coordinates of the centroid are

\[ \left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{93}{35}, \frac{45}{56} \right). \]
12. \( f(x) = x^3, \quad [0, 1] \)

**SOLUTION** The moments of the region are

\[
M_x = \frac{1}{2} \int_0^1 x^6 \, dx = \frac{1}{14} \quad \text{and} \quad M_y = \int_0^1 x^4 \, dx = \frac{1}{5}.
\]

The area of the region is

\[
A = \int_0^1 x^3 \, dx = \frac{1}{4},
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{4}{5}, \frac{2}{7} \right).
\]

13. \( f(x) = 9 - x^2, \quad [0, 3] \)

**SOLUTION** The moments of the region are

\[
M_x = \frac{1}{2} \int_0^3 (9 - x^2)^2 \, dx = \frac{324}{5} \quad \text{and} \quad M_y = \int_0^3 x(9 - x^2) \, dx = \frac{81}{4}.
\]

The area of the region is

\[
A = \int_0^3 (9 - x^2) \, dx = 18,
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{9}{8}, \frac{18}{7} \right).
\]

14. \( f(x) = (1 + x^2)^{-1/2}, \quad [0, 3] \)

**SOLUTION** The moments of the region are

\[
M_x = \frac{1}{2} \int_0^3 \frac{1}{1 + x^2} \, dx = \frac{\tan^{-1} x}{2} \bigg|_0^3 = \frac{1}{2} \tan^{-1} 3 \quad \text{and} \quad M_y = \int_0^3 \frac{x}{\sqrt{1 + x^2}} \, dx = \sqrt{10} - 1.
\]

The area of the region is

\[
A = \int_0^3 \frac{1}{\sqrt{1 + x^2}} \, dx = \ln |x + \sqrt{1 + x^2}| \bigg|_0^3 = \ln(3 + \sqrt{10}),
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{\sqrt{10} - 1}{\ln(3 + \sqrt{10})}, \frac{\tan^{-1} 3}{2 \ln(3 + \sqrt{10})} \right).
\]

15. \( f(x) = e^{-x}, \quad [0, 4] \)

**SOLUTION** The moments of the region are

\[
M_x = \frac{1}{2} \int_0^4 e^{-2x} \, dx = \frac{1}{4} \left( 1 - e^{-8} \right) \quad \text{and} \quad M_y = \int_0^4 x e^{-x} \, dx = e^{-x} (x + 1) \bigg|_0^4 = 1 - 5e^{-4}.
\]

The area of the region is

\[
A = \int_0^4 e^{-x} \, dx = 1 - e^{-4},
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{1 - 5e^{-4}}{1 - e^{-4}}, \frac{1 - e^{-8}}{4(1 - e^{-4})} \right).
\]
16. \( f(x) = \ln x \), \( [1, 2] \)

**Solution** The moments of the region are

\[
M_x = \frac{1}{2} \int_1^2 (\ln x)^2 \, dx = \frac{1}{2} \left( (\ln x)^2 - 2\ln x + 2x \right) \bigg|_1^2 = (\ln 2)^2 - 2\ln 2 + 1; \text{ and}
\]

\[
M_y = \int_1^2 x \ln x \, dx = \left( \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \bigg|_1^2 = 2\ln 2 - \frac{3}{4}.
\]

The area of the region is

\[
A = \int_1^2 \ln x \, dx = (x \ln x - x) \bigg|_1^2 = 2\ln 2 - 1,
\]

so the coordinates of the centroid are

\[
\left( \frac{M_x}{A}, \frac{M_y}{A} \right) = \left( \frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1}, \frac{(\ln 2)^2 - 2\ln 2 + 1}{2\ln 2 - 1} \right).
\]

17. \( f(x) = \sin x \), \( [0, \pi] \)

**Solution** The moments of the region are

\[
M_x = \frac{1}{2} \int_0^\pi \sin^2 x \, dx = \frac{1}{4} (x - \sin x \cos x) \bigg|_0^\pi = \frac{\pi}{4}; \text{ and}
\]

\[
M_y = \int_0^\pi x \sin x \, dx = (-x \cos x + \sin x) \bigg|_0^\pi = \pi.
\]

The area of the region is

\[
A = \int_0^\pi \sin x \, dx = 2,
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{\pi}{2}, \frac{\pi}{4} \right).
\]

18. Calculate the moments and center of mass of the lamina occupying the region between the curves \( y = x \) and \( y = x^2 \) for \( 0 \leq x \leq 1 \).

**Solution** The moments of the lamina are

\[
M_x = \frac{1}{2} \int_0^1 (x^2 - x^4) \, dx = \frac{1}{15} \quad \text{and} \quad M_y = \int_0^1 x(x - x^2) \, dx = \frac{1}{12}.
\]

The area of the lamina is

\[
A = \int_0^1 (x - x^2) \, dx = \frac{1}{6},
\]

so the coordinates of the centroid are

\[
\left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{1}{2}, \frac{2}{3} \right).
\]

19. Sketch the region between \( y = x + 4 \) and \( y = 2 - x \) for \( 0 \leq x \leq 2 \). Using symmetry, explain why the centroid of the region lies on the line \( y = 3 \). Verify this by computing the moments and the centroid.

**Solution** A sketch of the region is shown below.
The region is clearly symmetric about the line \( y = 3 \), so we expect the centroid of the region to lie along this line. We find
\[
M_x = \frac{1}{2} \int_0^2 \left( (x + 4)^2 - (2 - x)^2 \right) dx = 24;
\]
\[
M_y = \int_0^2 x \left( (x + 4) - (2 - x) \right) dx = \frac{28}{3}; \quad \text{and}
\]
\[
A = \int_0^2 \left( (x + 4) - (2 - x) \right) dx = 8.
\]
Thus, the coordinates of the centroid are \( \left( \frac{7}{6}, 3 \right) \).

In Exercises 20–25, find the centroid of the region lying between the graphs of the functions over the given interval.

20. \( y = x, \quad y = \sqrt{x}, \quad [0, 1] \)

**SOLUTION**

The moments of the region are
\[
M_x = \frac{1}{2} \int_0^1 (x - x^2) dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x) dx = \frac{1}{15}.
\]
The area of the region is
\[
A = \int_0^1 (\sqrt{x} - x) dx = \frac{1}{6},
\]
so the coordinates of the centroid are
\[
\left( \frac{6}{15}, \frac{1}{2} \right).
\]

21. \( y = x^2, \quad y = \sqrt{x}, \quad [0, 1] \)

**SOLUTION**

The moments of the region are
\[
M_x = \frac{1}{2} \int_0^1 (x - x^2) dx = \frac{3}{20} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x^2) dx = \frac{3}{20}.
\]
The area of the region is
\[
A = \int_0^1 (\sqrt{x} - x^2) dx = \frac{1}{3},
\]
so the coordinates of the centroid are
\[
\left( \frac{9}{20}, \frac{9}{20} \right).
\]

Note: This makes sense, since the functions are inverses of each other. This makes the region symmetric with respect to the line \( y = x \). Thus, by the symmetry principle, the center of mass must lie on that line.

22. \( y = x^{-1}, \quad y = 2 - x, \quad [1, 2] \)

**SOLUTION**

The moments of the region are
\[
M_x = \frac{1}{2} \int_1^2 \left[ \left( \frac{1}{x} \right)^2 - (2 - x)^2 \right] dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_1^2 x \left( \frac{1}{x} - (2 - x) \right) dx = \frac{1}{3}.
\]
The area of the region is
\[
A = \int_1^2 \left( \frac{1}{x} - (2 - x) \right) dx = \ln 2 - \frac{1}{2},
\]
so the coordinates of the centroid are
\[
\left( \frac{2}{6 \ln 2 - 3}, \frac{1}{12 \ln 2 - 6} \right).
\]

23. \( y = e^x, \quad y = 1, \quad [0, 1] \)

**SOLUTION**

The moments of the region are
\[
M_x = \frac{1}{2} \int_0^1 (e^{2x} - 1) dx = \frac{e^2 - 3}{4} \quad \text{and} \quad M_y = \int_0^1 x(e^x - 1) dx = \left( x e^x - e^x - \frac{1}{2} x^2 \right)_0^1 = \frac{1}{2}.
\]
The area of the region is
\[ A = \int_0^1 (e^x - 1) \, dx = e - 2, \]
so the coordinates of the centroid are
\[ \left( \frac{1}{2(e - 2)} \frac{e^2 - 3}{4(e - 2)} \right). \]

24. \( y = \ln x, \quad y = x - 1, \quad [1, 3] \)

**SOLUTION**  The moments of the region are
\[ M_x = \frac{1}{2} \int_1^3 (x - 1)^2 - (\ln x)^2 \, dx = \left( \frac{1}{3} x^3 - x^2 - x(\ln x)^2 + 2x \ln x \right) \bigg|_{\ln 3} = 3 \ln 3 - \frac{3}{2}(\ln 3)^2 - \frac{2}{3}; \quad \text{and} \]
\[ M_y = \int_1^3 x((x - 1) - \ln x) \, dx = \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \bigg|_{1} = \frac{20}{3} - \frac{9}{2} \ln 3. \]

The area of the region is
\[ A = \int_1^3 (x - 1 - \ln x) \, dx = \left( \frac{1}{2} x^2 - x \ln x \right) \bigg|_{1} = 4 - 3 \ln 3, \]
so the coordinates of the centroid are
\[ \left( \frac{40 - 27 \ln 3}{24 - 18 \ln 3}, \frac{18 \ln 3 - 9(\ln 3)^2 - 4}{24 - 18 \ln 3} \right). \]

25. \( y = \sin x, \quad y = \cos x, \quad [0, \pi/4] \)

**SOLUTION**  The moments of the region are
\[ M_x = \frac{1}{2} \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx = \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx = \frac{1}{4}; \quad \text{and} \]
\[ M_y = \int_0^{\pi/4} x(\cos x - \sin x) \, dx = [(x - 1) \sin x + (x + 1) \cos x] \bigg|_{0}^{\pi/4} = \frac{\pi \sqrt{2}}{4} - 1. \]

The area of the region is
\[ A = \int_0^{\pi/4} (\cos x - \sin x) \, dx = \sqrt{2} - 1, \]
so the coordinates of the centroid are
\[ \left( \frac{\pi \sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)} \right). \]

26. Sketch the region enclosed by \( y = x + 1 \) and \( y = (x - 1)^2 \) and find its centroid.

**SOLUTION**  A sketch of the region is shown below.
The moments of the region are
\[ M_x = \frac{1}{2} \int_0^3 \left[ (x+1)^2 - (x-1)^2 \right] dx = \frac{1}{2} \left( \frac{1}{3}(x+1)^3 - \frac{1}{5}(x-1)^5 \right) \bigg|_0^3 = \frac{1}{2} \left( \frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5} \right) = \frac{36}{5} \]
\[ M_y = \int_0^3 x (x+1) - (x-1)^2 dx = \int_0^3 \left( 3x^2 - x^3 \right) dx = \left( x^3 - \frac{1}{4}x^4 \right) \bigg|_0^3 = \frac{27}{4} \]

The area of the region is
\[ A = \int_0^3 \left[ (x+1) - (x-1)^3 \right] dx = \int_0^3 \left( 3x^2 - x^3 \right) dx = \left( -\frac{1}{3}x^3 + \frac{3}{2}x^2 \right) \bigg|_0^3 = 9 \]
so that the coordinates of the centroid are
\[ \left( \frac{27}{4} \cdot \frac{2}{9}, \frac{36}{5} \cdot \frac{2}{9} \right) = \left( \frac{3}{5}, \frac{8}{5} \right) \]

27. Sketch the region enclosed by \( y = 0, \ y = (x+1)^3, \) and \( y = (1-x)^3, \) and find its centroid.

**SOLUTION**  A sketch of the region is shown below.

The moments of the region are
\[ M_x = \frac{1}{2} \left( \int_{-1}^0 (x+1)^6 dx + \int_0^1 (1-x)^6 dx \right) = \frac{1}{7}, \text{ and} \]
\[ M_y = 0 \text{ by the Symmetry Principle.} \]

The area of the region is
\[ A = \int_{-1}^0 (x+1)^3 dx + \int_0^1 (1-x)^3 dx = \frac{1}{2}, \]
so the coordinates of the centroid are \( (0, \frac{2}{7}) \).

**In Exercises 28–32, find the centroid of the region.**

28. Top half of the ellipse \( \left( \frac{x}{2} \right)^2 + \left( \frac{y}{4} \right)^2 = 1 \)

**SOLUTION**  The equation of the top half of the ellipse is \( y = \sqrt{16 - 4x^2}. \) Thus,
\[ M_x = \frac{1}{2} \int_{-2}^2 \left( \sqrt{16 - 4x^2} \right)^2 dx = \frac{64}{3}. \]
By the Symmetry Principle, \( M_y = 0. \) The area of the region is one-half the area of an ellipse with major axis 4 and minor axis 2; i.e., \( \frac{1}{2} \pi (4)(2) = 4 \pi. \) Finally, the coordinates of the centroid are
\[ \left( 0, \frac{16}{3\pi} \right). \]

29. Top half of the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) for arbitrary \( a, b > 0 \)

**SOLUTION**  The equation of the top half of the ellipse is
\[ y = \sqrt{b^2 - \frac{b^2x^2}{a^2}} \]
Thus,
\[ M_x = \frac{1}{2} \int_{-a}^a \left( \sqrt{b^2 - \frac{b^2x^2}{a^2}} \right)^2 dx = \frac{2ab^2}{3}. \]
By the Symmetry Principle, \( M_y = 0 \). The area of the region is one-half the area of an ellipse with axes of length \( a \) and \( b \); i.e., \( \frac{1}{2} \pi ab \). Finally, the coordinates of the centroid are

\[
\left(0, \frac{4b}{3\pi}\right).
\]

30. Semicircle of radius \( r \) with center at the origin

**Solution** The equation of the top half of the circle is \( y = \sqrt{r^2 - x^2} \). Thus,

\[
M_x = \frac{1}{2} \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 \, dx = \frac{2r^3}{3}.
\]

By the Symmetry Principle, \( M_y = 0 \). The area of the region is one-half the area of a circle of radius \( r \); i.e., \( \frac{1}{2} \pi r^2 \). Finally, the coordinates of the centroid are

\[
\left(0, \frac{4r}{3\pi}\right).
\]

31. Quarter of the unit circle lying in the first quadrant

**Solution** By the Symmetry Principle, the center of mass must lie on the line \( y = x \) in the first quadrant. Therefore, we need only calculate one of the moments of the region. With \( y = \sqrt{1 - x^2} \), we find

\[
M_y = \int_{0}^{1} x \sqrt{1 - x^2} \, dx = \frac{1}{3}.
\]

The area of the region is one-quarter of the area of a unit circle; i.e., \( \frac{1}{4} \pi \). Thus, the coordinates of the centroid are

\[
\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right).
\]

32. Region between \( y = x(a - x) \) and the \( x \)-axis for \( a > 0 \)

**Solution** By symmetry, \( x_{CM} = \frac{1}{2}a \). On the other hand,

\[
M_x = \frac{1}{2} \int_{0}^{a} [x(a - x)]^2 \, dx = \frac{1}{2} \int_{0}^{a} (a^2 x^2 - 2ax^3 + x^4) \, dx
\]

\[
= \frac{1}{2} \left( \frac{1}{3}a^5 - \frac{1}{2}a^5 + \frac{1}{5}a^5 \right) = \frac{1}{60}a^5
\]

\[
M = \int_{0}^{a} x(a - x) \, dx = \frac{1}{2}a^3 0 \frac{1}{3}a^3 = \frac{1}{6}a^3
\]

Therefore, \( y_{CM} = \frac{M}{M_x} = \frac{1}{10}a^2 \).

33. Find the centroid of the shaded region of the semicircle of radius \( r \) in Figure 22. What is the centroid when \( r = 1 \) and \( h = \frac{1}{2} \)? *Hint:* Use geometry rather than integration to show that the area of the region is \( r^2 \sin^{-1}(\sqrt{1 - h^2/r^2}) - \frac{1}{2}r \sqrt{r^2 - h^2} \).

**Solution** From the symmetry of the region, it is obvious that the centroid lies along the \( y \)-axis. To determine the \( y \)-coordinate of the centroid, we must calculate the moment about the \( x \)-axis and the area of the region. Now, the length of the horizontal cut of the semicircle at height \( y \) is

\[
\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.
\]
Therefore, taking $\rho = 1$, we find

$$M_x = 2 \int_0^r y\sqrt{r^2 - y^2} \, dy = \frac{2}{3} (r^2 - h^2)^{3/2}.$$

Observe that the region is comprised of a sector of the circle with the triangle between the two radii removed. The angle of the sector is $2\theta$, where $\theta = \sin^{-1} \sqrt{1 - h^2/r^2}$, so the area of the sector is $\frac{1}{2} r^2 (2\theta) = r^2 \sin^{-1} \sqrt{1 - h^2/r^2}$. The triangle has base $2\sqrt{r^2 - h^2}$ and height $h$, so the area is $h \sqrt{r^2 - h^2}$. Therefore,

$$Y_{CM} = \frac{M_x}{A} = \frac{\frac{2}{3} (r^2 - h^2)^{3/2}}{r^2 \sin^{-1} \sqrt{1 - h^2/r^2} - h \sqrt{r^2 - h^2}}.$$

When $r = 1$ and $h = 1/2$, we find

$$Y_{CM} = \frac{\frac{2}{3} (3/4)^{3/2}}{\sin^{-1} \sqrt{3/4} - 1/2} = \frac{3\sqrt{3}}{4\pi - 3\sqrt{3}}.$$

34. Sketch the region between $y = x^n$ and $y = x^m$ for $0 \leq x \leq 1$, where $m > n \geq 0$, and find the COM of the region. Find a pair $(n, m)$ such that the COM lies outside the region.

**Solution** A sketch of the region for $x^3$ and $x^4$ is below.

Since $m > n \geq 0$, the graph of $x^n$ lies above that of $x^m$ for $x$ between 0 and 1. Thus the moments are

$$M_x = \frac{1}{2} \left[ \int_0^1 x^{2n} - x^{2m} \, dx \right] = \frac{1}{2} \left( \frac{1}{2n + 1} x^{2n+1} - \frac{1}{2m + 1} x^{2m+1} \right) \bigg|_0^1 = \frac{1}{2} \left( \frac{1}{2n + 1} - \frac{1}{2m + 1} \right) = \frac{m - n}{(2n + 1)(2m + 1)}$$

$$M_y = \int_0^1 x(x^n - x^m) \, dx = \int_0^1 x^{n+1} - x^{m+1} \, dx = \left( \frac{1}{n+2} x^{n+2} - \frac{1}{m+2} x^{m+2} \right) \bigg|_0^1 = \frac{m - n}{(n+2)(m+2)}$$

The area of the region is

$$A = \int_0^1 x^n - x^m \, dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m - n}{(n+1)(m+1)}$$

Thus the center of mass has coordinates

$$\left( \frac{(n+1)(m+1)}{(n+2)(m+2)}, \frac{(n+1)(m+1)}{(2n+1)(2m+1)} \right)$$

In the case graphed above, for $n = 3$, $m = 4$, the center of mass is

$$\left( \frac{20}{30}, \frac{20}{63} \right) = \left( \frac{2}{3}, \frac{20}{63} \right)$$

and

$$\left( \frac{2}{3} \right)^3 = \frac{8}{27} < \frac{20}{63}$$

Thus the point $\left( \frac{2}{3}, \frac{8}{27} \right)$ lies on $y = x^3$ and then the curve $y = x^3$ lies below the center of mass of the region.
35. Find the formula for the volume of a right circular cone of height \( H \) and radius \( R \) using the Theorem of Pappus as applied to the triangle bounded by the \( x \)-axis, the \( y \)-axis, and the line \( y = \frac{-H}{R}x + H \), rotated about the \( y \)-axis.

**Solution** The line \( y = \frac{-H}{R}x + H \) has an \( x \)-intercept of \( R \) and a \( y \)-intercept of \( H \); accordingly, the triangle bounded by the \( x \)-axis, the \( y \)-axis, and the line \( y = \frac{-H}{R}x + H \) has base \( R \), height \( H \), and area

\[
A = \frac{1}{2}RH.
\]

Because the \( x \)-coordinate of the centroid of the triangle is \( x_{CM} = \frac{1}{3}R \) by Exercise 3, the distance traveled by the centroid when the triangle is rotated about the \( y \)-axis is

\[
2\pi \cdot \frac{1}{3}R = \frac{2\pi R}{3}.
\]

Therefore, by the Theorem of Pappus, the volume of a right circular cone of height \( H \) and radius \( R \) is

\[
V = \frac{1}{2}RH \cdot \frac{2\pi R}{3} = \frac{1}{3}\pi R^2H.
\]

36. Use the Theorem of Pappus to find the centroid of the half-disk bounded by \( y = \sqrt{R^2 - x^2} \) and the \( x \)-axis.

**Solution** Rotating the half-disk bounded by \( y = \sqrt{R^2 - x^2} \) and the \( x \)-axis about the \( x \)-axis produces a sphere of radius \( R \). The volume of this sphere is

\[
V = \frac{4}{3}\pi R^3.
\]

The area of the half-disk is

\[
A = \frac{1}{2}\pi R^2.
\]

By the Theorem of Pappus,

\[
\frac{4}{3}\pi R^2 = \frac{1}{2}\pi R^2 \cdot 2\pi y_{CM} \quad \text{or} \quad y_{CM} = \frac{4R}{3\pi}.
\]

By symmetry, \( x_{CM} = 0 \); thus, the centroid of the half-disk bounded by \( y = \sqrt{R^2 - x^2} \) and the \( x \)-axis is

\[
\left(0, \frac{4R}{3\pi}\right).
\]

In Exercises 37–39, use the additivity of moments to find the COM of the region.

37. Isosceles triangle of height 2 on top of a rectangle of base 4 and height 3 (Figure 23)

**Solution** The region is symmetric with respect to the \( y \)-axis, so \( M_y = 0 \) by the Symmetry Principle. The moment about the \( x \)-axis for the rectangle is

\[
M^\text{rect}_x = \frac{1}{2} \int_{-2}^{2} x^2 \, dx = 18,
\]

whereas the moment about the \( x \)-axis for the triangle is

\[
M^\text{triangle}_x = \int_{3}^{5} y(10 - 2y) \, dy = \frac{44}{3}.
\]

The total moment about the \( x \)-axis is then

\[
M_x = M^\text{rect}_x + M^\text{triangle}_x = 18 + \frac{44}{3} = \frac{98}{3}.
\]
Because the area of the region is $12 + 4 = 16$, the coordinates of the center of mass are

$$ \left( 0, \frac{49}{24} \right). $$

### 38. An ice cream cone consisting of a semicircle on top of an equilateral triangle of side 6 (Figure 24)

**Solution** The region is symmetric with respect to the $y$-axis, so $M_y = 0$ by the Symmetry Principle. The moment about the $x$-axis for the triangle is

$$ M_{\text{triangle}} = \frac{2}{\sqrt{3}} \int_{0}^{3} \sqrt{3} y^2 dy = 54. $$

For the semicircle, first note that the center is $(0, 3\sqrt{3})$, so the equation is $x^2 + (y - 3\sqrt{3})^2 = 9$, and

$$ M_{\text{semi}} = 2 \int_{3\sqrt{3}}^{3+3\sqrt{3}} y \sqrt{9 - (y - 3\sqrt{3})^2} dy. $$

Using the substitution $w = y - 3\sqrt{3}, dw = dy$, we find

$$ M_{\text{semi}} = 2 \int_{0}^{3} (w + 3\sqrt{3}) \sqrt{9 - w^2} dw = 2 \int_{0}^{3} w \sqrt{9 - w^2} dw + 6 \sqrt{3} \int_{0}^{3} \sqrt{9 - w^2} dw = 18 + \frac{27\pi \sqrt{3}}{2}. $$

where we have used the fact that $\int_{0}^{3} \sqrt{9 - w^2} dw$ represents the area of one-quarter of a circle of radius 3. The total moment about the $x$-axis is then

$$ M_x = M_{\text{triangle}} + M_{\text{semi}} = 72 + \frac{27\pi \sqrt{3}}{2}. $$

Because the area of the region is $9\sqrt{3} + \frac{9\pi}{2}$, the coordinates of the center of mass are

$$ \left( 0, \frac{16 + 3\pi \sqrt{3}}{\pi + 2\sqrt{3}} \right). $$

### 39. Three-quarters of the unit circle (remove the part in the fourth quadrant)

**Solution** By the Symmetry Principle, the center of mass must lie on the line $y = -x$. Let region 1 be the semicircle above the $x$-axis and region 2 be the quarter circle in the third quadrant. Because region 1 is symmetric with respect to the $y$-axis, $M_{x1} = 0$ by the Symmetry Principle. Furthermore

$$ M_{x2} = \int_{-1}^{0} x \sqrt{1 - x^2} dx = -\frac{1}{3}. $$

Thus, $M_x = M_{x1} + M_{x2} = 0 + (-\frac{1}{3}) = -\frac{1}{3}$. The area of the region is $3\pi / 4$, so the coordinates of the centroid are

$$ \left( -\frac{4}{9\pi}, \frac{4}{9\pi} \right). $$

### 40. Let $S$ be the lamina of mass density $\rho = 1$ obtained by removing a circle of radius $r$ from the circle of radius $2r$ shown in Figure 25. Let $M_{x1}^S$ and $M_{x2}^S$ denote the moments of $S$. Similarly, let $M_{x1}^{\text{big}}$ and $M_{x1}^{\text{small}}$ be the $y$-moments of the larger and smaller circles.

(a) Use the Symmetry Principle to show that $M_{x1}^S = 0$. 
(b) Show that \( M_y^S = M_y^{\text{big}} - M_y^{\text{small}} \) using the additivity of moments.

(c) Find \( M_y^{\text{big}} \) and \( M_y^{\text{small}} \) using the fact that the COM of a circle is its center. Then compute \( M_y^S \) using (b).

(d) Determine the COM of \( S \).

\[
\text{FIGURE 25}
\]

**SOLUTION**

(a) Because \( S \) is symmetric with respect to the \( x \)-axis, \( M_x^S = 0 \).

(b) Because the small circle together with the region \( S \) comprise the big circle, by the additivity of moments,

\[
M_y^S + M_y^{\text{small}} = M_y^{\text{big}}.
\]

Thus \( M_y^S = M_y^{\text{big}} - M_y^{\text{small}} \).

(c) The center of the big circle is the origin, so \( x_{\text{big}} = 0 \); consequently, \( M_y^{\text{big}} = 0 \). On the other hand, the center of the small circle is \((-r, 0)\), so \( x_{\text{small}} = -r \); consequently

\[
M_y^{\text{small}} = x_{\text{small}} \cdot A_{\text{small}} = -r \cdot \pi r^2 = -\pi r^3.
\]

By the result of part (b), it follows that \( M_y^S = 0 - (-\pi r^3) = \pi r^3 \).

(d) The area of the region \( S \) is \( 4\pi r^2 - \pi r^2 = 3\pi r^2 \). The coordinates of the center of mass of the region \( S \) are then

\[
\left( \frac{\pi r^3}{3\pi r^2}, 0 \right) = \left( \frac{r}{3}, 0 \right).
\]

41. Find the COM of the laminas in Figure 26 obtained by removing squares of side 2 from a square of side 8.

\[
\text{FIGURE 26}
\]

**SOLUTION** Start with the square on the left. Place the square so that the bottom left corner is at \((0, 0)\). By the Symmetry Principle, the center of mass must lie on the lines \( y = x \) and \( y = 8 - x \). The only point in common to these two lines is \((4, 4)\), so the center of mass is \((4, 4)\).

Now consider the square on the right. Place the square as above. By the Symmetry principle, \( x_{\text{cm}} = 4 \). Now, let \( S_1 \) denote the square in the upper left, \( S_2 \) denote the square in the upper right, and \( B \) denote the entire square. Then

\[
M_x^{S_1} = \frac{1}{2} \int_0^2 (8^2 - y^2) \, dy = 28;
\]

\[
M_x^{S_2} = \frac{1}{2} \int_6^8 (8^2 - y^2) \, dy = 28; \quad \text{and}
\]

\[
M_x^B = \frac{1}{2} \int_0^8 y^2 \, dy = 256.
\]

By the additivity of moments, \( M_x = 256 - 28 - 28 = 200 \). Finally, the area of the region is \( A = 64 - 4 - 4 = 56 \), so the coordinates of the center of mass are

\[
\left( 4, \frac{200}{56} \right) = \left( 4, \frac{25}{7} \right).
\]
Further Insights and Challenges

42. A median of a triangle is a segment joining a vertex to the midpoint of the opposite side. Show that the centroid of a triangle lies on each of its medians, at a distance two-thirds down from the vertex. Then use this fact to prove that the three medians intersect at a single point. Hint: Simplify the calculation by assuming that one vertex lies at the origin and another on the x-axis.

**Solution** Orient the triangle by placing one vertex at (0, 0) and the long side of the triangle along the x-axis. Label the vertices (0, 0), (a, 0), (b, c). Thus, the equations of the short sides are \( y = \frac{c}{b}x \) and \( y = \frac{c}{b-a} - \frac{ac}{b-a} x \). Now,

\[
M_x = \frac{1}{2} \int_0^b (cx/b)^2 \, dx + \frac{1}{2} \int_b^a \left( \frac{cx}{b-a} \right)^2 \, dx = \frac{ac^2}{6};
\]
\[
M_y = \int_0^b x(cx/b) \, dx + \int_a^c x \left( \frac{cx}{b-a} \right) \, dx = \frac{ac(a+b)}{6}; \quad \text{and}
\]
\[ M = \frac{ac}{2}. \]

so the center of mass is \( \left( \frac{a+b}{3}, \frac{c}{3} \right) \). To show that the centroid lies on each median, let \( y_1 \) be the median from \( (b, c) \), \( y_2 \) the median from \( (0, 0) \) and \( y_3 \) the median from \( (a, 0) \). We find

\[
y_1(x) = \frac{2c}{2b-a} (x - a/2), \quad \text{so} \quad y_1 \left( \frac{a+b}{3} \right) = \frac{c}{3};
\]
\[
y_2(x) = \frac{c}{a+b} x, \quad \text{so} \quad y_2 \left( \frac{a+b}{3} \right) = \frac{c}{3};
\]
\[
y_3(x) = \frac{c}{b-2a} (x - a), \quad \text{so} \quad y_3 \left( \frac{a+b}{3} \right) = \frac{c}{3}.
\]

This shows that the center of mass lies on each median. We now show that the center of mass is \( \frac{2}{3} \) of the way from each vertex. For \( y_1 \), note that \( x = b \) gives the vertex and \( x = \frac{a}{2} \) gives the midpoint of the opposite side, so two-thirds of this distance is

\[
x = b + \frac{2}{3} \left( \frac{a}{2} - b \right) = \frac{a+b}{3}.
\]

the x-coordinate of the center of mass. Likewise, for \( y_2 \), two-thirds of the distance from \( x = 0 \) to \( x = \frac{a+b}{2} \) is \( \frac{a+b}{3} \), and for \( y_3 \), the two-thirds point is

\[
x = a + \frac{2}{3} \left( \frac{b}{2} - a \right) = \frac{a+b}{3}.
\]

A similar method shows that the y-coordinate is also two-thirds of the way along the median. Thus, since the centroid lies on all three medians, we can conclude that all three medians meet at a single point, namely the centroid.

43. Let \( P \) be the COM of a system of two weights with masses \( m_1 \) and \( m_2 \) separated by a distance \( d \). Prove Archimedes’s Law of the (weightless) Lever: \( P \) is the point on a line between the two weights such that \( m_1 L_1 = m_2 L_2 \), where \( L_j \) is the distance from mass \( j \) to \( P \).

**Solution** Place the lever along the x-axis with mass \( m_1 \) at the origin. Then \( M_y = m_2 d \) and the x-coordinate of the center of mass, \( P \), is

\[
\frac{m_2 d}{m_1 + m_2}.
\]

Thus,

\[
\ell_1 = \frac{m_2 d}{m_1 + m_2}, \quad \ell_2 = d - \frac{m_2 d}{m_1 + m_2} = \frac{m_1 d}{m_1 + m_2},
\]

and

\[
\ell_1 m_1 = m_1 \frac{m_2 d}{m_1 + m_2} = m_2 \frac{m_1 d}{m_1 + m_2} = \ell_2 m_2.
\]
44. Find the COM of a system of two weights of masses \( m_1 \) and \( m_2 \) connected by a lever of length \( d \) whose mass density \( \rho \) is uniform. \textit{Hint:} The moment of the system is the sum of the moments of the weights and the lever.

**Solution**  
Let \( A \) be the cross-sectional area of the rod. Place the rod with \( m_1 \) at the origin and rod lying on the positive \( x \)-axis. The \( y \)-moment of the rod is \( M_y = \frac{1}{2} \rho Ad^2 \), the \( y \)-moment of the mass \( m_2 \) is \( M_y = m_2 d \), and the total mass of the system is \( M = m_1 + m_2 + \rho Ad \). Therefore, the \( x \)-coordinate of the center of mass is

\[
\frac{m_2d + \frac{1}{2} \rho Ad^2}{m_1 + m_2 + \rho Ad}.
\]

45. **Symmetry Principle**  
Let \( R \) be the region under the graph of \( y = f(x) \) over the interval \([-a, a]\), where \( f(x) \geq 0 \). Assume that \( R \) is symmetric with respect to the \( y \)-axis.

(a) Explain why \( y = f(x) \) is even—that is, why \( f(x) = f(-x) \).

(b) Show that \( y = xf(x) \) is an odd function.

(c) Use (b) to prove that \( M_y = 0 \).

(d) Prove that the COM of \( R \) lies on the \( y \)-axis (a similar argument applies to symmetry with respect to the \( x \)-axis).

**Solution**  
(a) By the definition of symmetry with respect to the \( y \)-axis, \( f(x) = f(-x) \), so \( f \) is even.

(b) Let \( g(x) = xf(x) \) where \( f \) is even. Then

\[
g(-x) = -xf(-x) = -xf(x) = -g(x),
\]

and thus \( g \) is odd.

(c) \( M_y = \rho \int_{-a}^{a} xf(x) \, dx = 0 \) since \( xf(x) \) is an odd function.

(d) By part (c), \( x_{cm} = \frac{M_y}{M} = 0 \) and therefore it lies along the \( y \)-axis.

46. Prove directly that Eqs. (2) and (3) are equivalent in the following situation. Let \( f \) be a positive decreasing function on \([0, b]\) such that \( f(b) = 0 \) and \( g(y) = f^{-1}(y) \). Show that

\[
\frac{1}{2} \int_0^b f(x)^2 \, dx = \int_0^d yg(y) \, dy
\]

**Hint:** First apply the substitution \( y = f(x) \) to the integral on the left and observe that \( dx = g'(y) \, dy \). Then apply Integration by Parts.

**Solution**  
\( f(x) \geq 0 \) and \( f'(x) < 0 \) shows that \( f \) has an inverse \( g \) on \([a, b]\). Because \( f(b) = 0 \), \( f(0) = d \), and \( f^{-1}(x) = g(x) \), it follows that \( g(d) = 0 \) and \( g(0) = b \). If we let \( x = g(y) \), then \( dx = g'(y) \, dy \). Thus, with \( y = f(x) \),

\[
\frac{1}{2} \int_0^b f(x)^2 \, dx = \frac{1}{2} \int_0^b y^2 \, dx = \frac{1}{2} \int_d^0 y^2 g'(y) \, dy.
\]

Using Integration by Parts with \( u = y^2 \) and \( v' = g'(y) \, dy \), we find

\[
\frac{1}{2} \int_d^0 y^2 g'(y) \, dy = \frac{1}{2} \left[ y^2 g(y) \bigg|_d^0 - 2 \int_d^0 yg(y) \, dy \right] = \frac{1}{2} \left[ 0 - d^2 g(d) \right] - \int_d^0 yg(y) \, dy = \int_0^d yg(y) \, dy.
\]

47. Let \( R \) be a lamina of uniform density submerged in a fluid of density \( \rho \) (Figure 27). Prove the following law: The fluid force on one side of \( R \) is equal to the area of \( R \) times the fluid pressure on the centroid. **Hint:** Let \( g(y) \) be the horizontal width of \( R \) at depth \( y \). Express both the fluid pressure \([\text{Eq. (2)} \text{ in Section 8.2}]\) and \( y \)-coordinate of the centroid in terms of \( g(y) \).

![Figure 27](https://via.placeholder.com/150)
SOLUTION Let \( \rho \) denote the uniform density of the submerged lamina. Then

\[
M_x = \rho \int_a^b y g(y) \, dy,
\]

and the mass of the lamina is

\[
M = \rho \int_a^b g(y) \, dy = \rho A,
\]

where \( A \) is the area of the lamina. Thus, the \( y \)-coordinate of the centroid is

\[
y_{cm} = \frac{\rho \int_a^b y g(y) \, dy}{\rho A} = \frac{\int_a^b y g(y) \, dy}{A}.
\]

Now, the fluid force on the lamina is

\[
F = w \int_a^b y g(y) \, dy = w \int_a^b y g(y) \, dy A = w y_{cm} A.
\]

In other words, the fluid force on the lamina is equal to the fluid pressure at the centroid of the lamina times the area of the lamina.

8.4 Taylor Polynomials

Preliminary Questions

1. What is \( T_3 \) centered at \( a = 3 \) for a function \( f \) such that \( f(3) = 9 \), \( f'(3) = 8 \), \( f''(3) = 4 \), and \( f'''(3) = 12 \)?

SOLUTION In general, with \( a = 3 \),

\[
T_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{6}(x-3)^3.
\]

Using the information provided, we find

\[
T_3(x) = 9 + 8(x-3) + 2(x-3)^2 + 2(x-3)^3.
\]

2. The dashed graphs in Figure 9 are Taylor polynomials for a function \( f \). Which of the two is a Maclaurin polynomial?

![Figure 9](image)

SOLUTION A Maclaurin polynomial always gives the value of \( f(0) \) exactly. This is true for the Taylor polynomial sketched in (B); thus, this is the Maclaurin polynomial.

3. For which value of \( x \) does the Maclaurin polynomial \( T_n \) satisfy \( T_n(x) = f(x) \), no matter what \( f \) is?

SOLUTION A Maclaurin polynomial always gives the value of \( f(0) \) exactly.

4. Let \( T_n \) be the Maclaurin polynomial of a function \( f \) satisfying \( |f^{(4)}(x)| \leq 1 \) for all \( x \). Which of the following statements follow from the error bound?

(a) \( |T_4(2) - f(2)| \leq \frac{2}{3} \)

(b) \( |T_3(2) - f(2)| \leq \frac{1}{3} \)

(c) \( |T_3(2) - f(2)| \leq \frac{1}{4} \)

SOLUTION For a function \( f(x) \) satisfying \( |f^{(4)}(x)| \leq 1 \) for all \( x \),

\[
|T_3(2) - f(2)| \leq \frac{1}{24} |f^{(4)}(2)| 2^4 \leq \frac{16}{24} = \frac{2}{3}.
\]

Thus, (b) is the correct answer.
**Exercises**

In Exercises 1–14, calculate the Taylor polynomials $T_2$ and $T_3$ centered at $x = a$ for the given function and value of $a$.

1. $f(x) = \sin x, \quad a = 0$

   **SOLUTION** First, we calculate and evaluate the needed derivatives:
   
   $f(x) = \sin x \quad f(a) = 0$
   
   $f'(x) = \cos x \quad f'(a) = 1$
   
   $f''(x) = -\sin x \quad f''(a) = 0$
   
   $f'''(x) = -\cos x \quad f'''(a) = -1$
   
   Now,
   
   $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x$; and
   
   $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$
   
   $= 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-1}{6}(x-0)^3 = x - \frac{1}{6}x^3$.

2. $f(x) = \sin x, \quad a = \frac{\pi}{2}$

   **SOLUTION** First, we calculate and evaluate the needed derivatives:
   
   $f(x) = \sin x \quad f(a) = 1$
   
   $f'(x) = \cos x \quad f'(a) = 0$
   
   $f''(x) = -\sin x \quad f''(a) = -1$
   
   $f'''(x) = -\cos x \quad f'''(a) = 0$
   
   Now,
   
   $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$
   
   $= 1 + 0 \left(x - \frac{\pi}{2}\right) + \frac{-1}{2} \left(x - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2} \left(x - \frac{\pi}{2}\right)^2$; and
   
   $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$
   
   $= 1 + 0 \left(x - \frac{\pi}{2}\right) + \frac{-1}{2} \left(x - \frac{\pi}{2}\right)^2 + \frac{0}{6} \left(x - \frac{\pi}{2}\right)^3 = 1 - \frac{1}{2} \left(x - \frac{\pi}{2}\right)^2$.

3. $f(x) = \frac{1}{1+x}, \quad a = 2$

   **SOLUTION** First, we calculate and evaluate the needed derivatives:
   
   $f(x) = \frac{1}{1+x} \quad f(a) = \frac{1}{3}$
   
   $f'(x) = -\frac{1}{(1+x)^2} \quad f'(a) = -\frac{1}{9}$
   
   $f''(x) = \frac{2}{(1+x)^3} \quad f''(a) = \frac{2}{27}$
   
   $f'''(x) = -\frac{6}{(1+x)^4} \quad f'''(a) = -\frac{2}{27}$
   
   Now,
   
   $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$
   
   $= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2$.
\[ T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \]

\[ = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2/27}{2!}(x-2)^2 - \frac{2/27}{3!}(x-2)^3 = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 \]

4. \( f(x) = \frac{1}{1+x^2}, \quad a = -1 \)

**Solution**  
First, we calculate and evaluate the needed derivatives:

\[
\begin{align*}
  f(x) &= \frac{1}{1+x^2} \quad f(a) = 1/2 \\
  f'(x) &= -\frac{2x}{(x^2+1)^2} \quad f'(a) = 1/2 \\
  f''(x) &= \frac{2(3x^2-1)}{(x^2+1)^3} \quad f''(a) = 1/2 \\
  f'''(x) &= -\frac{24x(x^2-1)}{(x^2+1)^4} \quad f'''(a) = 0 \\
\end{align*}
\]

Now,

\[
\begin{align*}
  T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
  &= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{2}(x+1)^2 = \frac{1}{2} + \frac{1}{2}(x+1)^2; \quad \text{and} \\
  T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\
  &= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{2}(x+1)^2 + \frac{0}{6}(x+1)^3 = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+1)^2. \\
\end{align*}
\]

5. \( f(x) = x^4 - 2x, \quad a = 3 \)

**Solution**  
First calculate and evaluate the needed derivatives:

\[
\begin{align*}
  f(x) &= x^4 - 2x \quad f(a) = 75 \\
  f'(x) &= 4x^3 - 2 \quad f'(a) = 106 \\
  f''(x) &= 12x^2 \quad f''(a) = 108 \\
  f'''(x) &= 24x \quad f'''(a) = 72 \\
\end{align*}
\]

Now,

\[
\begin{align*}
  T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
  &= 75 + 106(x-3) + \frac{108}{2}(x-3)^2 \\
  &= 75 + 106(x-3) + 54(x-3)^2 \\
  T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{6!}(x-a)^3 \\
  &= 75 + 106(x-3) + \frac{108}{2}(x-3)^2 + \frac{72}{6!}(x-3)^3 \\
  &= 75 + 106(x-3) + 54(x-3)^2 + 12(x-3)^3. \\
\end{align*}
\]

6. \( f(x) = \frac{x^2 + 1}{x + 1}, \quad a = -2 \)

**Solution**  
First calculate and evaluate the needed derivatives:

\[
\begin{align*}
  f(x) &= \frac{x^2 + 1}{x + 1} \quad f(a) = -5 \\
  f'(x) &= \frac{x^2 + 2x - 1}{(x+1)^2} \quad f'(a) = -1 \\
\end{align*}
\]
\[ f'''(x) = \frac{4}{(x+1)^3} \quad f''(a) = -4 \]
\[ f'''(x) = -\frac{12}{(x+1)^4} \quad f'''(a) = -12 \]

Now,
\[ T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 = -5 - (x+2) + \frac{-4}{2} (x+2)^2 \]
\[ = -5 - (x+2) - 2(x+2)^2 \]
\[ T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \]
\[ = -5 - (x+2) + \frac{-4}{2} (x+2)^2 + \frac{-12}{3!} (x+2)^3 \]
\[ = -5 - (x+2) - 2(x+2)^2 - 2(x+2)^3 \]

7. \( f(x) = \tan x, \quad a = 0 \)

**Solution** First, we calculate and evaluate the needed derivatives:
\[ f'(x) = \tan x \quad f'(a) = 0 \]
\[ f''(x) = \sec^2 x \quad f''(a) = 1 \]
\[ f'''(x) = 2 \sec^2 x \tan x \quad f'''(a) = 0 \]
\[ f''''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x \quad f''''(a) = 2 \]

Now,
\[ T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 = 0 + 1(x-0) + \frac{0}{2} (x-0)^2 = x; \text{ and} \]
\[ T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 + \frac{f'''(a)}{6} (x-a)^3 \]
\[ = 0 + 1(x-0) + \frac{0}{2} (x-0)^2 + \frac{2}{6} (x-0)^3 = x + \frac{1}{3} x^3. \]

8. \( f(x) = \tan x, \quad a = \frac{\pi}{4} \)

**Solution** First, we calculate and evaluate the needed derivatives:
\[ f'(x) = \sec^2 x \quad f'(a) = 2 \]
\[ f''(x) = 2 \sec^2 x \tan x \quad f''(a) = 4 \]
\[ f'''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x \quad f'''(a) = 16 \]

Now,
\[ T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 = 1 + 2 \left( x - \frac{\pi}{4} \right) + \frac{4}{2} \left( x - \frac{\pi}{4} \right)^2 \]
\[ = 1 + 2 \left( x - \frac{\pi}{4} \right) + 2 \left( x - \frac{\pi}{4} \right)^2; \text{ and} \]
\[ T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 + \frac{f'''(a)}{6} (x-a)^3 \]
\[ = 1 + 2 \left( x - \frac{\pi}{4} \right) + \frac{4}{2} \left( x - \frac{\pi}{4} \right)^2 + \frac{16}{6} \left( x - \frac{\pi}{4} \right)^3 = 1 + 2 \left( x - \frac{\pi}{4} \right) + 2 \left( x - \frac{\pi}{4} \right)^2 + \frac{8}{3} \left( x - \frac{\pi}{4} \right)^3. \]

9. \( f(x) = e^{-x} + e^{-2x}, \quad a = 0 \)

**Solution** First, we calculate and evaluate the needed derivatives:
\[ f(x) = e^{-x} + e^{-2x} \quad f'(a) = 2 \]
Now,
\[
f'(x) = -e^{-x} - 2e^{-2x}
\]
\[
f''(x) = e^{-x} + 4e^{-2x}
\]
\[
f'''(x) = -e^{-x} - 8e^{-2x}
\]

Now,
\[
T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2
\]
\[
= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 = 2 - 3x + \frac{5}{2}x^2; \quad \text{and}
\]
\[
T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3
\]
\[
= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 + \frac{-9}{6}(x-0)^3 = 2 - 3x + \frac{5}{2}x^2 - \frac{3}{2}x^3.
\]

10. \( f(x) = e^{2x}, \quad a = \ln 2 \)

**SOLUTION** First calculate and evaluate the needed derivatives:

\[
f(x) = e^{2x} \quad f(a) = 4
\]
\[
f'(x) = 2e^{2x} \quad f'(a) = 8
\]
\[
f''(x) = 4e^{2x} \quad f''(a) = 16
\]
\[
f'''(x) = 8e^{2x} \quad f'''(a) = 32
\]

Now
\[
T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 4 + 8(x-\ln 2) + \frac{16}{2!}(x-\ln 2)^2
\]
\[
= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2
\]
\[
T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3
\]
\[
= 4 + 8(x - \ln 2) + \frac{16}{2!}(x - \ln 2)^2 + \frac{32}{6}(x - \ln 2)^3
\]
\[
= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 + \frac{16}{3}(x - \ln 2)^3
\]

11. \( f(x) = x^2e^{-x}, \quad a = 1 \)

**SOLUTION** First, we calculate and evaluate the needed derivatives:

\[
f(x) = x^2e^{-x} \quad f(a) = 1/e
\]
\[
f'(x) = (2x - x^2)e^{-x} \quad f'(a) = 1/e
\]
\[
f''(x) = (x^2 - 4x + 2)e^{-x} \quad f''(a) = -1/e
\]
\[
f'''(x) = (-x^2 + 6x - 6)e^{-x} \quad f'''(a) = -1/e
\]

Now,
\[
T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2
\]
\[
= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 = \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2; \quad \text{and}
\]
\[
T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3
\]
\[
= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 + \frac{(-1/e)}{6}(x-1)^3
\]
\[
= \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3.
\]
12. \( f(x) = \cosh 2x, \quad a = 0 \)

**Solution** First calculate and evaluate the needed derivatives:
\[
\begin{align*}
  f(x) &= \cosh 2x \quad f(a) = 1 \\
  f'(x) &= 2 \sinh 2x \quad f'(a) = 0 \\
  f''(x) &= 4 \cosh 2x \quad f''(a) = 4 \\
  f'''(x) &= 8 \sinh 2x \quad f'''(a) = 0
\end{align*}
\]
so that
\[
T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 = 1 + 0(x - 0) + \frac{4}{2!}(x - 0)^2 = 1 + 2x^2; \quad \text{and}
\]
\[
T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 \\
= 1 + 0(x - 0) + 2(x - 0)^2 + \frac{0}{3!}(x - 0)^3 = 1 + 2x^2
\]

13. \( f(x) = \frac{\ln x}{x}, \quad a = 1 \)

**Solution** First calculate and evaluate the needed derivatives:
\[
\begin{align*}
  f(x) &= \frac{\ln x}{x} \quad f(a) = 0 \\
  f'(x) &= \frac{1 - \ln x}{x^2} \quad f'(a) = 1 \\
  f''(x) &= -\frac{3 + 2 \ln x}{x^3} \quad f''(a) = -3 \\
  f'''(x) &= -\frac{11 - 6 \ln x}{x^4} \quad f'''(a) = 11
\end{align*}
\]
so that
\[
T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 = 0 + 1(x - 1) + \frac{-3}{2!}(x - 1)^2 \\
= (x - 1) - \frac{3}{2}(x - 1)^2; \quad \text{and}
\]
\[
T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 \\
= 0 + 1(x - 1) + \frac{-3}{2!}(x - 1)^2 + \frac{11}{3!}(x - 1)^3 \\
= (x - 1) - \frac{3}{2}(x - 1)^2 + \frac{11}{6}(x - 1)^3
\]

14. \( f(x) = \ln(x + 1), \quad a = 0 \)

**Solution** First, we calculate and evaluate the needed derivatives:
\[
\begin{align*}
  f(x) &= \ln(x + 1) \quad f(a) = 0 \\
  f'(x) &= \frac{1}{x + 1} \quad f'(a) = 1 \\
  f''(x) &= -\frac{1}{(x + 1)^2} \quad f''(a) = -1 \\
  f'''(x) &= \frac{2}{(x + 1)^3} \quad f'''(a) = 2
\end{align*}
\]
Now,
\[
T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 = 0 + 1(x - 0) + \frac{-1}{2}(x - 0)^2 = x - \frac{1}{2}x^2; \quad \text{and}
\]
\[
T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{6}(x - a)^3 \\
= 0 + 1(x - 0) + \frac{-1}{2}(x - 0)^2 + \frac{2}{6}(x - 0)^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3.
\]
15. Show that the $n$th Maclaurin polynomial for $f(x) = e^x$ is

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

**SOLUTION** With $f(x) = e^x$, it follows that $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all $n$. Thus,

$$T_n(x) = 1 + 1(x - 0) + \frac{1}{2!}(x - 0)^2 + \cdots + \frac{1}{n!}(x - 0)^n = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

16. Show that the $n$th Taylor polynomial for $f(x) = \frac{1}{x + 1}$ at $a = 1$ is

$$T_n(x) = \frac{1}{2} \left( \frac{x - 1}{x} \right)^2 + \frac{1}{8} \left( \frac{x - 1}{x} \right)^3 + \cdots + \frac{(-1)^n (x - 1)^n}{2n+1}.$$

**SOLUTION** Let $f(x) = \frac{1}{x + 1}$. Then

$$f(x) = \frac{1}{x + 1}, \quad f(1) = \frac{1}{2}, \quad f'(x) = \frac{1}{(x + 1)^2}, \quad f'(1) = \frac{1}{4}, \quad f''(x) = \frac{2}{(x + 1)^3}, \quad f''(1) = \frac{1}{4}.$$

Therefore,

$$T_n(x) = \frac{1}{2} + \left[ \frac{1}{4} \right] (x - 1) + \frac{1}{4} \frac{(x - 1)^2}{2!} + \cdots + \frac{(-1)^n (x - 1)^n}{2n+1}.$$

17. Show that the Maclaurin polynomials for $f(x) = \sin x$ are

$$T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

**SOLUTION** Let $f(x) = \sin x$. Then

$$f(x) = \sin x, \quad f(0) = 0,
\quad f'(x) = \cos x, \quad f'(0) = 1,
\quad f''(x) = -\sin x, \quad f''(0) = 0,
\quad f'''(x) = -\cos x, \quad f'''(0) = -1,
\quad f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0,
\quad f^{(5)}(x) = \cos x, \quad f^{(5)}(0) = 1,
\quad \vdots, \quad \vdots$$

Consequently,

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

and

$$T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + 0 = T_{2n+1}(x).$$
18. Show that the Maclaurin polynomials for \( f(x) = \ln(1 + x) \) are
\[
T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}
\]

**SOLUTION** Let \( f(x) = \ln(1 + x) \). Then
\[

g(x) = (1 + x)^{-1} \quad \Rightarrow \quad f(x) = \ln g(x) = \ln(1 + x)
\]
\[
\Rightarrow \quad f'(x) = \frac{1}{1 + x} = g'(x) = -\frac{1}{1 + x}
\]
\[
f''(x) = -\frac{1}{(1 + x)^2} = -2 \frac{1}{(1 + x)^2} = -2 \frac{1}{1 + x} = -2 g''(x)
\]
\[
f'''(x) = 2 \frac{1}{(1 + x)^3} = 2 \frac{1}{1 + x} = 2 g'''(x)
\]
\[
\vdots
\]
\[
f^{(n)}(x) = (-1)^{n-1} \frac{n!}{x^n} = (-1)^{n-1} \frac{n!}{x^n} = (-1)^{n-1} \frac{n!}{x^n}
\]
\[
\Rightarrow \quad f^{(n)}(0) = (-1)^{n-1} \frac{n!}{0^n} = (-1)^{n-1} \frac{n!}{0^n}
\]
so that in general
\[
f^{(n)}(x) = (-1)^{n-1} (n-1)! (1 + x)^{-n} \quad \Rightarrow \quad f^{(n)}(0) = (-1)^{n-1} n!
\]

Thus
\[
T_n(x) = x - \frac{1}{2!} x^2 + \frac{1}{3!} x^3 - \cdots + (-1)^{n-1} \frac{1}{n!} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}
\]

**In Exercises 19–24, find \( T_n \) centered at \( a \) for all \( n. \)**

19. \( f(x) = \frac{1}{1 + x} \), \( a = 0 \)

**SOLUTION** We have
\[
\frac{1}{1 + x} = (\ln(1 + x))^\prime
\]
so that from Exercise 18, letting \( g(x) = \ln(1 + x) \),
\[
f^{(n)}(x) = g^{(n+1)}(x) = (-1)^n n! (x + 1)^{-1-n} \quad \Rightarrow \quad f^{(n)}(0) = (-1)^n n!
\]

Then
\[
T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots + (-1)^{n-1} \frac{1}{n!} x^n
\]
\[
= 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1} \frac{x^n}{n!}
\]

20. \( f(x) = \frac{1}{x - 1} \), \( a = 4 \)

**SOLUTION** Let \( f(x) = \frac{1}{x - 1} \). Then
\[
f(x) = \frac{1}{x - 1} \quad \Rightarrow \quad f(4) = \frac{1}{3} = \frac{(-1)^0 n!}{30 + 1}
\]
\[
f'(x) = -\frac{1}{(x - 1)^2} \quad \Rightarrow \quad f'(4) = -\frac{1}{9} = \frac{(-1)^1 1!}{31 + 1}
\]
\[
f''(x) = \frac{2}{(x - 1)^3} \quad \Rightarrow \quad f''(4) = \frac{2}{27} = \frac{(-1)^2 2!}{32 + 1}
\]
\[
\vdots
\]
\[
f^{(n)}(x) = \frac{(-1)^n n!}{(x - 1)^{n+1}} \quad \Rightarrow \quad f^{(n)}(4) = \frac{(-1)^n n!}{3n + 1}
\]
Therefore,
\[
T_n(x) = \frac{1}{3} + \left(-\frac{1}{9}\right) (x - 4) + \frac{2/27}{2} (x - 4)^2 + \cdots + \frac{(-1)^n n! (x - 4)^n}{3n + 1}
\]
\[
= \frac{1}{3} - \frac{1}{9} (x - 4) + \frac{1}{27} (x - 4)^2 + \cdots + \frac{(-1)^n}{3n + 1} (x - 4)^n.
\]
21. \(f(x) = e^x, \quad a = 1\)

**SOLUTION**  Let \(f(x) = e^x\). Then \(f^{(n)}(x) = e^x\) and \(f^{(n)}(1) = e\) for all \(n\). Therefore,

\[T_n(x) = e + e(x - 1) + \frac{e}{2!}(x - 1)^2 + \cdots + \frac{e}{n!}(x - 1)^n.\]

22. \(f(x) = x^{-2}, \quad a = 2\)

**SOLUTION**  We have

\[
\begin{align*}
f(x) &= x^{-2} & f(2) &= \frac{1}{4} \\
f'(x) &= -2x^{-3} & f'(2) &= -\frac{1}{4} \\
f''(x) &= 6x^{-4} & f''(2) &= \frac{3}{8} \\
f'''(x) &= -24x^{-5} & f'''(2) &= -\frac{3}{4} \\
&\vdots & & \vdots \\
f^{(n)}(x) &= (-1)^n (n+1)!x^{-n-2} & f^{(n)}(2) &= (-1)^n \frac{(n+1)!}{2^{n+2}} (x - 2)^n
\end{align*}
\]

so that

\[
T_n(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x - 2)^n
\]

\[
= \frac{1}{4} - \frac{1}{4}(x - 2) + \frac{3}{16}(x - 2)^2 + \cdots + (-1)^n \frac{n + 1}{2^{n+2}} (x - 2)^n
\]

23. \(f(x) = \cos x, \quad a = \frac{\pi}{4}\)

**SOLUTION**  Let \(f(x) = \cos x\). Then

\[
\begin{align*}
f(x) &= \cos x & f(\pi/4) &= \frac{1}{\sqrt{2}} \\
f'(x) &= -\sin x & f'(\pi/4) &= -\frac{1}{\sqrt{2}} \\
f''(x) &= -\cos x & f''(\pi/4) &= -\frac{1}{\sqrt{2}} \\
f'''(x) &= \sin x & f'''(\pi/4) &= \frac{1}{\sqrt{2}}
\end{align*}
\]

This pattern of four values repeats indefinitely. Thus,

\[
f^{(n)}(\pi/4) = \begin{cases} 
(-1)^{(n+1)/2} \frac{1}{\sqrt{2}} & n \text{ odd} \\
(-1)^{n/2} \frac{1}{\sqrt{2}} & n \text{ even}
\end{cases}
\]

and

\[
T_n(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} (x - \pi/4) - \frac{1}{2\sqrt{2}} (x - \pi/4)^2 + \frac{1}{6\sqrt{2}} (x - \pi/4)^3 \cdots.
\]

In general, the coefficient of \((x - \pi/4)^n\) is

\[
\pm \frac{1}{(\sqrt{2})^n n!}
\]

with the pattern of signs +, −, −, +, −, −, ….
24. \( f(\theta) = \sin 3\theta, \ a = 0 \)

**SOLUTION** We have

\[
\begin{align*}
  f(\theta) &= \sin 3\theta & f(0) &= 0 \\
  f'(\theta) &= 3\cos 3\theta & f'(0) &= 3 \\
  f''(\theta) &= -9\sin 3\theta & f''(0) &= 0 \\
  f'''(\theta) &= -27\cos 3\theta & f'''(0) &= -27 \\
  f^{(4)}(\theta) &= 81\sin 3\theta & f^{(4)}(0) &= 0
\end{align*}
\]

and in general

\[
\begin{align*}
  f^{(2n)}(\theta) &= (-1)^n 3^{2n} \sin 3\theta & f^{(2n)}(0) &= 0 \\
  f^{(2n+1)}(\theta) &= (-1)^n 3^{2n+1} \cos 3\theta & f^{(2n+1)}(0) &= (-1)^n 3^{2n+1}
\end{align*}
\]

Thus

\[
T_n(x) = 3\theta - \frac{27}{3!} \theta^3 + \frac{243}{5!} \theta^5 - \ldots
\]

where the coefficient of \( \theta^{2n+1} \) is \((-1)^n \frac{3^{2n+1}}{(2n+1)!} \).

In Exercises 25–28, find \( T_2 \) and use a calculator to compute the error \( |f(x) - T_2(x)| \) for the given values of \( a \) and \( x \).

25. \( y = e^x, \ a = 0, \ x = -0.5 \)

**SOLUTION** Let \( f(x) = e^x \). Then \( f'(x) = e^x, \ f''(x) = e^x, \ f(a) = 1, \ f'(a) = 1 \) and \( f''(a) = 1 \). Therefore

\[
T_2(x) = 1 + 1(x - 0) + \frac{1}{2} (x - 0)^2 = 1 + x + \frac{1}{2} x^2,
\]

and

\[
T_2(-0.5) = 1 + (-0.5) + \frac{1}{2} (-0.5)^2 = 0.625.
\]

Using a calculator, we find

\[
f(-0.5) = \frac{1}{\sqrt{e}} = 0.606531,
\]

so

\[
|T_2(-0.5) - f(-0.5)| = 0.0185.
\]

26. \( y = \cos x, \ a = 0, \ x = \frac{\pi}{12} \)

**SOLUTION** Let \( f(x) = \cos x \). Then \( f'(x) = -\sin x, \ f''(x) = -\cos x, \ f(a) = 1, \ f'(a) = 0 \) and \( f''(a) = -1 \). Therefore

\[
T_2(x) = 1 + 0(x - 0) + \frac{1}{2} (x - 0)^2 = 1 - \frac{1}{2} x^2,
\]

and

\[
T_2\left(\frac{\pi}{12}\right) = 1 - \frac{1}{2} \left(\frac{\pi}{12}\right)^2 \approx 0.965731.
\]

Using a calculator, we find

\[
f\left(\frac{\pi}{12}\right) = 0.965926,
\]

so

\[
|T_2\left(\frac{\pi}{12}\right) - f\left(\frac{\pi}{12}\right)| = 0.000195.
\]

27. \( y = x^{-2/3}, \ a = 1, \ x = 1.2 \)

**SOLUTION** Let \( f(x) = x^{-2/3} \). Then \( f'(x) = -\frac{2}{3} x^{-5/3}, \ f''(x) = \frac{10}{9} x^{-8/3}, \ f(1) = 1, \ f'(1) = -\frac{2}{3}, \) and \( f''(1) = \frac{10}{9} \).

Thus

\[
T_2(x) = 1 - \frac{2}{3} (x - 1) + \frac{10}{2 \cdot 9} (x - 1)^2 = 1 - \frac{2}{3} (x - 1) + \frac{5}{9} (x - 1)^2
\]
and
\[ T_2(1.2) = 1 - \frac{2}{3}(0.2) + \frac{5}{9}(0.2)^2 = \frac{8}{9} \approx 0.88889 \]

Using a calculator, \( f(1.2) = (1.2)^{-2/3} \approx 0.88555 \) so that
\[ |T_2(1.2) - f(1.2)| \approx 0.00334 \]

28. \( y = e^{\sin x}, \quad a = \frac{\pi}{2}, \quad x = 1.5 \)

**SOLUTION** Let \( f(x) = e^{\sin x} \). Then \( f'(x) = \cos x e^{\sin x}, \quad f''(x) = \cos^2 x e^{\sin x} - \sin x e^{\sin x}, \quad f(a) = e, \quad f'(a) = 0 \) and \( f''(a) = -e \). Therefore
\[
T_2(x) = e + 0 \left(x - \frac{\pi}{2}\right) + -\frac{e}{2} \left(x - \frac{\pi}{2}\right)^2 = e - \frac{e}{2} \left(x - \frac{\pi}{2}\right)^2,
\]
and
\[
T_2(1.5) = e - \frac{e}{2} \left(1.5 - \frac{\pi}{2}\right)^2 \approx 2.711469651.
\]

Using a calculator, we find \( f(1.5) = 2.711481018 \), so
\[ |T_2(1.5) - f(1.5)| = 1.14 \times 10^{-5}. \]

29. **CAS** Compute \( T_3 \) for \( f(x) = \sqrt{x} \) centered at \( a = 1 \). Then use a plot of the error \( |f(x) - T_3(x)| \) to find a value \( c > 1 \) such that the error on the interval \([1, c]\) is at most 0.25.

**SOLUTION** We have
\[
\begin{align*}
 f(x) &= x^{1/2} \\
 f(1) &= 1 \\
 f'(x) &= \frac{1}{2} x^{-1/2} \\
 f'(1) &= \frac{1}{2} \\
 f''(x) &= -\frac{1}{4} x^{-3/2} \\
 f''(1) &= -\frac{1}{4} \\
 f'''(x) &= \frac{3}{8} x^{-5/2} \\
 f'''(1) &= \frac{3}{8}
\end{align*}
\]

Therefore
\[ T_3(x) = 1 + \frac{1}{2} (x-1) - \frac{1}{4 \cdot 2!} (x-1)^2 + \frac{3}{8 \cdot 3!} (x-1)^3 = 1 + \frac{1}{2} (x-1) - \frac{1}{8} (x-1)^2 + \frac{1}{16} (x-1)^3 \]

A plot of \( |f(x) - T_3(x)| \) is below.

It appears that for \( x \in [1, 2.9] \) that the error does not exceed 0.25. The error at \( x = 3 \) appears to just exceed 0.25.

30. **CAS** Plot \( f(x) = 1/(1 + x) \) together with the Taylor polynomials \( T_n \) at \( a = 1 \) for \( 1 \leq n \leq 4 \) on the interval \([-2, 8]\) (be sure to limit the upper plot range).

(a) Over which interval does \( T_4 \) appear to approximate \( f \) closely?

(b) What happens for \( x < -1 \)?

(c) Use your computer algebra system to produce and plot \( T_{30} \) together with \( f \) on \([-2, 8]\). Over which interval does \( T_{30} \) appear to give a close approximation?

**SOLUTION** Let \( f(x) = \frac{1}{1+x} \). Then
\[
\begin{align*}
 f(x) &= \frac{1}{1+x} \\
 f(1) &= \frac{1}{2}
\end{align*}
\]
CHAPTER 8 FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

\[ f'(x) = -\frac{1}{(1 + x)^2} \quad f'(1) = -\frac{1}{4} \]
\[ f''(x) = \frac{2}{(1 + x)^3} \quad f''(1) = \frac{1}{4} \]
\[ f'''(x) = -\frac{6}{(1 + x)^4} \quad f'''(1) = -\frac{3}{8} \]
\[ f^{(4)}(x) = \frac{24}{(1 + x)^5} \quad f^{(4)}(1) = \frac{3}{4} \]

and

\[ T_1(x) = \frac{1}{2} - \frac{1}{4}(x - 1); \]
\[ T_2(x) = \frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{8}(x - 1)^2; \]
\[ T_3(x) = \frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{8}(x - 1)^2 - \frac{1}{16}(x - 1)^3; \text{ and} \]
\[ T_4(x) = \frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{8}(x - 1)^2 - \frac{1}{16}(x - 1)^3 + \frac{1}{32}(x - 1)^4. \]

A plot of \( f(x) \), \( T_1(x) \), \( T_2(x) \), \( T_3(x) \) and \( T_4(x) \) is shown below.

(a) The graph below displays \( f(x) \) and \( T_4(x) \) over the interval \([−0.5, 2.5]\). It appears that \( T_4(x) \) gives a close approximation to \( f(x) \) over the interval \((0.1, 2)\).

(b) For \( x < -1 \), \( f(x) \) is negative, but the Taylor polynomials are positive; thus, the Taylor polynomials are poor approximations for \( x < -1 \).

(c) The graph below displays \( f(x) \) and \( T_{30}(x) \) over the interval \([-2, 8]\). It appears that \( T_{30}(x) \) gives a close approximation to \( f(x) \) over the interval \((-1, 3)\).

31. Let \( T_3 \) be the Maclaurin polynomial of \( f(x) = e^x \). Use the error bound to find the maximum possible value of \( |f(1.1) - T_3(1.1)| \). Show that we can take \( K = e^{1.1} \).
**SOLUTION** Since \( f(x) = e^x \), we have \( f^{(n)}(x) = e^x \) for all \( n \); since \( e^x \) is increasing, the maximum value of \( e^x \) on the interval \([0, 1]\) is \( K = e^{1.1} \). Then by the error bound,

\[
|e^{1.1} - T_3(1.1)| \leq K \frac{(1.1 - 0)^4}{4!} = e^{1.1} \frac{1.1^4}{24} \approx 0.183
\]

**32.** Let \( T_2 \) be the Taylor polynomial of \( f(x) = \sqrt{x} \) at \( a = 4 \). Apply the error bound to find the maximum possible value of the error \( |f(3.9) - T_2(3.9)| \).

**SOLUTION** We have \( f(x) = x^{1/2}, f'(x) = \frac{1}{2}x^{-1/2}, f''(x) = -\frac{1}{4}x^{-3/2}, \) and \( f'''(x) = \frac{3}{8}x^{-5/2} \). This is a decreasing function of \( x \), so its maximum value on \([3.9, 4]\) is achieved at \( x = 3.9 \); that value is \( \frac{3}{8 \cdot 3.9^{1/2}} \approx 0.0125 \), so we can take \( K = 0.0125 \). Then

\[
|f(x) - T_2(x)| \leq K \frac{|3.9 - 4|^3}{3!} = 0.0125 \times 0.001 \frac{6}{6} \approx 2.08 \times 10^{-6}
\]

**In Exercises 33–36, compute the Taylor polynomial indicated and use the error bound to find the maximum possible size of the error. Verify your result with a calculator.**

**33.** \( f(x) = \cos x, \quad a = 0; \quad |\cos 0.25 - T_3(0.25)| \)

**SOLUTION** The Maclaurin series for \( \cos x \) is

\[
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

so that

\[
T_3(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}
\]

\([T_3(0.25)] = 0.9689127604
\]

In addition, \( f^{(6)}(x) = -\cos x \) so that \(|f^{(6)}(x)| \leq 1\) and we may take \( K = 1 \) in the error bound formula. Then

\[
|\cos 0.25 - T_3(0.25)| \leq K \frac{0.25^6}{6!} = \frac{1}{2^{12} \cdot 6!} \approx 3.390842014 \times 10^{-7}
\]

(The true value is \( \cos 0.25 \approx 0.9689124217 \) and the difference is in fact \( \approx 3.387 \times 10^{-7} \).)

**34.** \( f(x) = x^{11/2}, \quad a = 1; \quad |f(1.2) - T_4(1.2)| \)

**SOLUTION** Let \( f(x) = x^{11/2} \). Then

\[
f(x) = x^{11/2} \quad \quad f(1) = 1
\]

\[
f'(x) = \frac{11}{2} x^{9/2} \quad \quad f'(1) = \frac{11}{2}
\]

\[
f''(x) = \frac{99}{4} x^{7/2} \quad \quad f''(1) = \frac{99}{4}
\]

\[
f'''(x) = \frac{693}{8} x^{5/2} \quad \quad f'''(1) = \frac{693}{8}
\]

\[
f^{(4)}(x) = \frac{3465}{16} x^{3/2} \quad \quad f^{(4)}(1) = \frac{3465}{16}
\]

and

\[
T_4(x) = 1 + \frac{11}{2} (x - 1) + \frac{99}{8} (x - 1)^2 + \frac{231}{16} (x - 1)^3 + \frac{1155}{128} (x - 1)^4.
\]

Using the Error Bound,

\[
|f(1.2) - T_4(1.2)| \leq K \frac{|1.2 - 1|^5}{5!} = \frac{K}{375000}
\]

where \( K \) is a number such that \(|f^{(5)}(x)| \leq K\) for \( x \) between 1 and 1.2. Now,

\[
f^{(5)}(x) = \frac{10395}{32} x^{1/2},
\]
which is increasing for $x > 1$. Consequently, on the interval $[1, 1.2]$, $f^{(4)}(x)$ is maximized at $x = 1.2$. We can therefore take $K = \frac{10395}{32\cdot \sqrt{1.2}}$, and then
\[
|f(1.2) - T_4(1.2)| \leq \frac{10395}{(32)(375000)} \sqrt{1.2} \approx 9.489 \times 10^{-4}.
\]

35. $f(x) = x^{-1/2}, \ a = 4; \ |f(4.3) - T_3(4.3)|$

**Solution** We have
\[
\begin{align*}
f(x) &= x^{-1/2} \quad f(4) = \frac{1}{2} \\
f'(x) &= -\frac{1}{2}x^{-3/2} \quad f'(4) = -\frac{1}{16} \\
f''(x) &= \frac{3}{4}x^{-5/2} \quad f''(4) = \frac{3}{128} \\
f'''(x) &= -\frac{15}{8}x^{-7/2} \quad f'''(4) = -\frac{15}{1024} \\
f^{(4)}(x) &= \frac{105}{16}x^{-9/2}
\end{align*}
\]
so that
\[
T_3(x) = \frac{1}{2} - \frac{1}{16}(x - 4) + \frac{3}{256}(x - 4)^2 - \frac{5}{2048}(x - 4)^3
\]
Using the error bound formula,
\[
|f(4.3) - T_3(4.3)| \leq K \frac{|4.3 - 4|^4}{4!} = \frac{27K}{80,000}
\]
where $K$ is a number such that $|f^{(4)}(x)| \leq K$ for $x$ between 4 and 4.3. Now, $f^{(4)}(x)$ is a decreasing function for $x > 1$, so it takes its maximum value on $[4, 4.3]$ at $x = 4$; there, its value is
\[
K = \frac{105}{16} = \frac{105}{8192}
\]
so that
\[
|f(4.3) - T_3(4.3)| \leq \frac{27 \cdot 105}{80,000} = \frac{27 \cdot 105}{8192 \cdot 80,000} \approx 4.3258667 \times 10^{-6}
\]

36. $f(x) = \sqrt{1+x}, \ a = 8; \ |\sqrt{9.02} - T_3(8.02)|$

**Solution** Let $f(x) = \sqrt{1+x}$. Then
\[
\begin{align*}
f(x) &= \sqrt{1+x} \quad f(8) = 3 \\
f'(x) &= \frac{1}{2}(x + 1)^{-1/2} \quad f'(8) = \frac{1}{6} \\
f''(x) &= -\frac{1}{4}(x + 1)^{-3/2} \quad f''(8) = -\frac{1}{108} \\
f'''(x) &= \frac{3}{8}(x + 1)^{-5/2} \quad f'''(8) = \frac{1}{648}
\end{align*}
\]
and
\[
T_3(x) = 3 + \frac{1}{6}(x - 8) - \frac{1}{108 \cdot 2!}(x - 8)^2 + \frac{1}{648 \cdot 3!}(x - 8)^3 = 3 + \frac{1}{6}(x - 8) - \frac{1}{216}(x - 8)^2 + \frac{1}{3888}(x - 8)^3.
\]
Therefore
\[
T_3(8.02) = 3 + \frac{1}{6}(0.02) - \frac{1}{216}(0.02)^2 + \frac{1}{3888}(0.02)^3 = 3.003331484.
\]
Using the Error Bound, we have
\[
|\sqrt{9.02} - T_3(8.02)| \leq K \frac{|8.02 - 8|^4}{4!} = \frac{K}{150,000,000}
\]
where $K$ is a number such that $|f^{(4)}(x)| \leq K$ for $x$ between 8 and 8.02. Now
\[
f^{(4)}(x) = -\frac{15}{16}(1 + x)^{-7/2},
\]
which is a decreasing function for $8 \leq x \leq 8.02$, so we may take
\[
K = \frac{15}{16} \cdot 9^{-7/2} = \frac{15}{34992}.
\]
Thus,
\[
|\sqrt{9.02} - T_3(8.02)| \leq \frac{15/34992}{150,000,000} \approx 2.858 \times 10^{-12}.
\]

37. Calculate the Maclaurin polynomial $T_3$ for $f(x) = \tan^{-1} x$. Compute $T_3\left(\frac{1}{2}\right)$ and use the error bound to find a bound for the error $|\tan^{-1} \frac{1}{2} - T_3\left(\frac{1}{2}\right)|$. Refer to the graph in Figure 10 to find an acceptable value of $K$. Verify your result by computing $|\tan^{-1} \frac{1}{2} - T_3\left(\frac{1}{2}\right)|$ using a calculator.

SOLUTION Let $f(x) = \tan^{-1} x$. Then
\[
\begin{align*}
f(x) &= \tan^{-1} x \\
f'(x) &= \frac{1}{1 + x^2} \quad f'(0) = 1 \\
f''(x) &= -\frac{2x}{(1 + x^2)^2} \quad f''(0) = 0 \\
f'''(x) &= \frac{(1 + x^2)^2(-2) - (-2x)(2)(1 + x^2)(2x)}{(1 + x^2)^4} \quad f'''(0) = -2
\end{align*}
\]
and
\[
T_3(x) = 0 + 1(x - 0) + \frac{0}{2}(x - 0)^2 + \frac{-2}{6}(x - 0)^3 = x - \frac{x^3}{3}.
\]
Since $f^{(4)}(x) \leq 5$ for $x \geq 0$, we may take $K = 5$ in the error bound; then,
\[
\left|\tan^{-1} \left(\frac{1}{2}\right) - T_3\left(\frac{1}{2}\right)\right| \leq \frac{5(1/2)^4}{4!} = \frac{5}{384}.
\]

38. Let $f(x) = \ln(x^3 - x + 1)$. The third Taylor polynomial at $a = 1$ is
\[
T_3(x) = 2(x - 1) + (x - 1)^2 - \frac{7}{3}(x - 1)^3
\]
Find the maximum possible value of $|f(1.1) - T_3(1.1)|$, using the graph in Figure 11 to find an acceptable value of $K$. Verify your result by computing $|f(1.1) - T_3(1.1)|$ using a calculator.
Let $y = f(x) = \ln(x^3 - x + 1)$. We have

$$f(x) = \ln(x^3 - x + 1)$$

which is in accordance with the error bound above.

39. Let $T_2$ be the Taylor polynomial at $a = 0.5$ for $f(x) = \cos(x^2)$. Use the error bound to find the maximum possible value of $|f(0.6) - T_2(0.6)|$. Plot $f^{(3)}$ to find an acceptable value of $K$.

**SOLUTION** We have

$$f(x) = \cos(x^2) \quad f(0.5) = \cos(0.25) \approx 0.9689$$

$$f'(x) = -2x \sin(x^2) \quad f'(0.5) = -\sin(0.25) \approx -0.247403953$$

$$f''(x) = -4x^2 \cos(x^2) - 2 \sin(x^2) \quad f''(0.5) = -\cos(0.25) - 2 \sin(0.25) \approx -1.463720340$$

$$f^{(3)}(x) = 8x^3 \sin(x^2) - 12x \cos(x^2)$$

so that

$$T_2(x) = 0.9689 - 0.247203953(x - 0.5) - 0.73186017(x - 0.5)^2$$

and $T_2(0.6) \approx 0.9368534237$. A graph of $f^{(3)}(x)$ for $x$ near 0.5 is below.

Clearly the maximum value of $|f^{(3)}(x)|$ on $[0.5, 0.6]$ is bounded by 7 (near $x = 0.5$), so we may take $K = 7$; then

$$|f(0.6) - T_2(0.6)| \leq K \frac{|0.6 - 0.5|^3}{3!} = \frac{7}{6000} \approx 0.0011666667$$

40. Calculate the Maclaurin polynomial $T_2$ for $f(x) = \sech x$ and use the error bound to find the maximum possible value of $|f^{(4)}(\frac{1}{2}) - T_2^{(4)}(\frac{1}{2})|$. Plot $f^{(4)}$ to find an acceptable value of $K$.

**SOLUTION** To compute $T_2(x)$ for $f(x) = \sech x$, we take the first two derivatives:

$$f(x) = \sech x \quad f(0) = 1$$

$$f'(x) = -\sech x \tanh x \quad f'(0) = 0$$

$$f''(x) = \sech x \tanh^2 x - \sech^3 x \quad f''(0) = -1$$
From this,
\[ T_2(x) = 1 - \frac{1}{2}x^2. \]

and
\[ T_2\left(\frac{1}{2}\right) = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{8} = \frac{7}{8}. \]

Using the Error Bound, we have
\[ \left| f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right| \leq K \frac{\left|1/2\right|^3}{6} = \frac{K}{48}. \]

where \( K \) is a number such that \( |f'''(x)| \leq K \) for \( x \) between 0 and \( \frac{1}{2} \). Here,
\[ f'''(x) = -\text{sech} \, x \tanh^3 x + 2 \text{sech}^3 x \tanh x + 3 \text{sech}^2 x \text{tanh} x \mbox{ sech} x \tanh x \]
\[ = 5 \text{sech}^2 x \text{tanh} x - \text{sech} x \tanh^3 x. \]

A plot of \( f'''(x) \) is given below. From the plot, we see that \( |f'''(x)| \leq 2 \) for all \( x \) between 0 and \( 1/2 \). Thus,
\[ \left| f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right| \leq \frac{2}{48} = \frac{1}{24}. \]

In Exercises 41–44, use the error bound to find a value of \( n \) for which the given inequality is satisfied. Then verify your result using a calculator.

**41.** \( |\cos 0.1 - T_n(0.1)| \leq 10^{-7}, \quad a = 0 \)

**SOLUTION** Using the error bound with \( K = 1 \) (every derivative of \( f(x) = \cos x \) is \( \pm \sin x \) or \( \pm \cos x \), so \( |f^(n)(x)| \leq 1 \) for all \( n \)), we have
\[ |T_n(0.1) - \cos 0.1| \leq \frac{(0.1)^{n+1}}{(n + 1)!}. \]

With \( n = 3, \)
\[ \frac{(0.1)^4}{4!} \approx 4.17 \times 10^{-6} > 10^{-7}, \]

but with \( n = 4, \)
\[ \frac{(0.1)^5}{5!} \approx 8.33 \times 10^{-8} < 10^{-7}, \]

so we choose \( n = 4 \). Now,
\[ T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4, \]

so
\[ T_4(0.1) = 1 - \frac{1}{2}(0.1)^2 + \frac{1}{24}(0.1)^4 = 0.995004166. \]

Using a calculator, \( \cos 0.1 = 0.995004165 \), so
\[ |T_4(0.1) - \cos 0.1| = 1.387 \times 10^{-8} < 10^{-7}. \]
42. \(|\ln 1.3 - T_n(1.3)| \leq 10^{-4}\), \(a = 1\)

**Solution** Let \(f(x) = \ln x\). Then \(f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = 2x^{-3}, f^{(4)}(x) = -6x^{-4}\), etc. In general, 
\[f^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}.\]

Now, \(|f^{(n+1)}(x)|\) is decreasing on the interval \([1, 1.3]\), so \(|f^{(n+1)}(x)| \leq |f^{(n+1)}(1)| = n!\) for all \(x \in [1, 1.3]\). We can therefore take \(K = n!\) in the error bound, and 
\[|\ln 1.3 - T_n(1.3)| \leq n! \frac{|1.3 - 1|^{n+1}}{(n+1)!} = \frac{(0.3)^{n+1}}{n+1}.\]

With \(n = 5\), 
\[\frac{(0.3)^6}{6} = 1.215 \times 10^{-4} > 10^{-4},\]
but with \(n = 6\), 
\[\frac{(0.3)^7}{7} = 3.124 \times 10^{-5} < 10^{-4}.\]

Therefore, the error is guaranteed to be below \(10^{-4}\) for \(n = 6\). Now, 
\[T_6(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \frac{1}{5}(x - 1)^5 - \frac{1}{6}(x - 1)^6\]
and \(T_6(1.3) \approx 0.2623395000\). Using a calculator, \(\ln(1.3) \approx 0.2623642645\); the difference is 
\[\ln(1.3) - T_6(1.3) \approx 0.0000247645 < 10^{-4}\]

43. \(|\sqrt{1.3} - T_n(1.3)| \leq 10^{-6}\), \(a = 1\)

**Solution** Using the Error Bound, we have 
\[|\sqrt{1.3} - T_n(1.3)| \leq K \frac{|1.3 - 1|^{n+1}}{(n+1)!} = K \frac{(0.3)^{n+1}}{(n+1)!},\]
where \(K\) is a number such that \(|f^{(n+1)}(x)| \leq K\) for \(x\) between 1 and 1.3. For \(f(x) = \sqrt{x}\), \(|f^{(n)}(x)|\) is decreasing for \(x > 1\), hence the maximum value of \(|f^{(n+1)}(x)|\) occurs at \(x = 1\). We may therefore take 
\[K = |f^{(n+1)}(1)| = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{n+1}} \frac{(2n+2)!}{(n+1)!} = \frac{2n+2)!}{(n+1)!} 2^{2n+2}.
\]

Then 
\[|\sqrt{1.3} - T_n(1.3)| \leq \frac{(2n+2)!}{(n+1)!} 2^{2n+2} \frac{(0.3)^{n+1}}{(n+1)!} = \frac{(2n+2)!}{(n+1)!} 2^{2n+2} (0.075)^{n+1}.
\]

With \(n = 9\), 
\[\frac{(20)!}{(10)!}^2 (0.075)^{10} = 1.040 \times 10^{-6} > 10^{-6},\]
but with \(n = 10\) 
\[\frac{(22)!}{(11)!}^2 (0.075)^{11} = 2.979 \times 10^{-7} < 10^{-6}.\]

Hence, \(n = 10\) will guarantee the desired accuracy. Using technology to compute and evaluate \(T_{10}(1.3)\) gives 
\[T_{10}(1.3) \approx 1.140175414, \quad \sqrt{1.3} \approx 1.140175425\]
and 
\[|\sqrt{1.3} - T_{10}(1.3)| \approx 1.1 \times 10^{-8} < 10^{-6}\]
44. \( |e^{-0.1} - T_n(-0.1)| \leq 10^{-6}, \quad a = 0 \)

**SOLUTION** Using the Error Bound, we have

\[
|e^{-0.1} - T_n(-0.1)| \leq K \frac{|-0.1 - 0|^{n+1}}{(n+1)!} = K \frac{1}{10^{n+1}(n+1)!}
\]

where \( K \) is a number such that \( |f^{(n+1)}(x)| \leq K \) for \( x \) between \(-0.1\) and 0. Since \( f(x) = e^x, f^{(n)}(x) = e^x \) for all \( n \); this is an increasing function, so it takes its maximum value at \( x = 0 \); this value is 1. So we may take \( K = 1 \) and then

\[
|e^{-0.1} - T_n(-0.1)| \leq \frac{1}{10^{n+1}(n+1)!}
\]

With \( n = 3 \)

\[
\frac{1}{10^4 \cdot 24} = \frac{1}{240,000} \approx 4.16666667 \times 10^{-6} > 10^{-6}
\]

but with \( n = 4 \)

\[
\frac{1}{10^5 \cdot 120} = \frac{1}{12,000,000} \approx 8.33333333 \times 10^{-8} < 10^{-6}
\]

Thus \( n = 4 \) will guarantee the desired accuracy. Using technology to compute \( T_4(x) \) and evaluate,

\[
T_4(-0.1) \approx 0.9048375000, \quad e^{-0.1} \approx 0.9048374180
\]

and

\[
|e^{-0.1} - T_4(-0.1)| \approx 8.2 \times 10^{-8} < 10^{-6}
\]

45. Let \( f(x) = e^{-x} \) and \( T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} \). Use the error bound to show that for all \( x \geq 0 \),

\[
|f(x) - T_3(x)| \leq \frac{x^4}{24}
\]

If you have a GU, illustrate this inequality by plotting \( y = f(x) - T_3(x) \) and \( y = \frac{x^4}{24} \) together over \([0, 1]\).

**SOLUTION** Note that all derivatives of \( \sin x \) are either \( \pm \cos x \) or \( \pm \sin x \) so are bounded in absolute value by 1. Thus we may take \( K = 1 \) in the Error Bound. Thus for \( x \),

\[
|f(x) - T_3(x)| \leq K \frac{|x - 0|^4}{4!} = \frac{x^4}{24}
\]

A plot of \( f(x) - T_3(x) \) and \( \frac{x^4}{24} \) is shown below.

46. Use the error bound with \( n = 4 \) to show that

\[
\left| \sin x - \left( x - \frac{x^3}{3!} \right) \right| \leq \frac{|x|^5}{120} \quad \text{(for all \( x \))}
\]

**SOLUTION** Note that all derivatives of \( \sin x \) are either \( \pm \cos x \) or \( \pm \sin x \) so are bounded in absolute value by 1. Thus we may take \( K = 1 \) in the Error Bound. Now,

\[
T_4(x) = x - \frac{x^3}{3!}
\]
so that

\[ |\sin x - T_4(x)| = |\sin x - \left(x - \frac{x^3}{6}\right)| \leq K \frac{|x - 0|^5}{5!} = \frac{|x|^5}{120}. \]

47. Let \( T_n \) be the Taylor polynomial for \( f(x) = \ln x \) at \( x = 1 \), and let \( c > 1 \). Show that

\[ |\ln x - T_n(x)| \leq \frac{|c - 1|^{n+1}}{n+1} \]

Then find a value of \( n \) such that \(|\ln 1.5 - T_n(1.5)| \leq 10^{-2} \).

**SOLUTION** With \( f(x) = \ln x \), we have

\[ f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -6x^{-4}, \]

and, in general,

\[ f^{(k+1)}(x) = (-1)^k k! x^{-k-1}. \]

Notice that \(|f^{(k+1)}(x)| = k! |x|^{-k-1}\) is a decreasing function for \( x > 0 \). Therefore, the maximum value of \(|f^{(k+1)}(x)|\) on \([1, c]\) is \(|f^{(k+1)}(1)|\). Using the Error Bound, we have

\[ |\ln x - T_n(x)| \leq \frac{|c - 1|^{n+1}}{(n+1)!}, \]

where \( K \) is a number such that \( |f^{(n+1)}(x)| \leq K \) for \( x \) between 1 and \( c \). We have just determined that we may take \( K = |f^{(n+1)}(1)| = n! \). Then

\[ |\ln c - T_n(c)| \leq n! \frac{|c - 1|^{n+1}}{(n+1)!} = \frac{|c - 1|^{n+1}}{n + 1}. \]

Evaluating at \( c = 1.5 \) gives

\[ |\ln 1.5 - T_n(1.5)| \leq \frac{1.5 - 1|^{n+1}}{n + 1} = \frac{(0.5)^{n+1}}{n + 1}. \]

With \( n = 3 \),

\[ \frac{(0.5)^4}{4} = 0.015625 > 10^{-2}. \]

but with \( n = 4 \),

\[ \frac{(0.5)^5}{5} = 0.00625 < 10^{-2}. \]

Hence, \( n = 4 \) will guarantee the desired accuracy.

48. Let \( n \geq 1 \). Show that if \(|x|\) is small, then

\[ (x + 1)^{1/n} \approx 1 + \frac{x}{n} + \frac{1 - n}{2n^2} x^2 \]

Use this approximation with \( n = 6 \) to estimate \(1.5^{1/6} \).

**SOLUTION** Let \( f(x) = (x + 1)^{1/n} \). Then

\[ f(x) = (x + 1)^{1/n} \quad f(0) = 1 \]

\[ f'(x) = \frac{1}{n} (x + 1)^{1/n-1} \quad f'(0) = \frac{1}{n} \]

\[ f''(x) = \frac{1}{n} \left( \frac{1}{n} - 1 \right) (x + 1)^{1/n-2} \quad f''(0) = \frac{1}{n} \left( \frac{1}{n} - 1 \right) \]

and

\[ T_2(x) = 1 + \frac{1}{n} x + \left( \frac{1}{n^2} - \frac{1}{n} \right) \frac{x^2}{2} = 1 + \frac{x}{n} + \left( \frac{1 - n}{2n^2} \right) x^2. \]

With \( n = 6 \) and \( x = 0.5 \),

\[ 1.5^{1/6} \approx T_2(0.5) = \frac{307}{288} \approx 1.065972. \]
49. Verify that the third Maclaurin polynomial for \( f(x) = e^x \sin x \) is equal to the product of the third Maclaurin polynomials of \( f(x) = e^x \) and \( f(x) = \sin x \) (after discarding terms of degree greater than 3 in the product).

**Solution** Let \( f(x) = e^x \sin x \). Then

\[
\begin{align*}
  f(x) &= e^x \sin x \\
  f(0) &= 0 \\
  f'(x) &= e^x (\cos x + \sin x) \\
  f'(0) &= 1 \\
  f''(x) &= 2e^x \cos x \\
  f''(0) &= 2 \\
  f'''(x) &= 2e^x (\cos x - \sin x) \\
  f'''(0) &= 2
\end{align*}
\]

and

\[
T_3(x) = 0 + (1)x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 = x + x^2 + \frac{x^3}{3}.
\]

Now, the third Maclaurin polynomial for \( e^x \) is \( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \), and the third Maclaurin polynomial for \( \sin x \) is \( x - \frac{x^3}{3!} \).

Multiplying these two polynomials, and then discarding terms of degree greater than 3, yields

\[
e^x \sin x \approx x + x^2 + \frac{x^3}{3},
\]

which agrees with the Maclaurin polynomial obtained from the definition.

50. Find the fourth Maclaurin polynomial for \( f(x) = \sin x \cos x \) by multiplying the fourth Maclaurin polynomials for \( f(x) = \sin x \) and \( f(x) = \cos x \).

**Solution** The fourth Maclaurin polynomial for \( \sin x \) is \( x - \frac{x^3}{3!} \), and the fourth Maclaurin polynomial for \( \cos x \) is \( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \). Multiplying these two polynomials, and then discarding terms of degree greater than 4, we find that the fourth Maclaurin polynomial for \( f(x) = \sin x \cos x \) is

\[
T_4(x) = x - \frac{2x^3}{3}.
\]

51. Find the Maclaurin polynomials \( T_n \) for \( f(x) = \cos(x^2) \). You may use the fact that \( T_n(x) \) is equal to the sum of the terms up to degree \( n \) obtained by substituting \( x^2 \) for \( x \) in the \( n \)th Maclaurin polynomial of \( \cos x \).

**Solution** The Maclaurin polynomials for \( \cos x \) are of the form

\[
T_{2n}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.
\]

Accordingly, the Maclaurin polynomials for \( \cos(x^2) \) are of the form

\[
T_{4n}(x) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}.
\]

52. Find the Maclaurin polynomials of \( 1/(1 + x^2) \) by substituting \(-x^2\) for \( x \) in the Maclaurin polynomials of \( 1/(1-x) \).

**Solution** The Maclaurin polynomials for \( \frac{1}{1-x} \) are of the form

\[
T_n(x) = 1 + x + x^2 + \cdots + x^n.
\]

Accordingly, the Maclaurin polynomials for \( \frac{1}{1+x^2} \) are of the form

\[
T_{2n}(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n.
\]

53. Let \( f(x) = 3x^3 + 2x^2 - x - 4 \). Calculate \( T_j \) for \( j = 1, 2, 3, 4, 5 \) at both \( a = 0 \) and \( a = 1 \). Show that \( T_3(x) = f(x) \) in both cases.

**Solution** Let \( f(x) = 3x^3 + 2x^2 - x - 4 \). Then

\[
\begin{align*}
f(x) &= 3x^3 + 2x^2 - x - 4 \\
f(0) &= -4 \\
f(1) &= 0 \\
f'(x) &= 9x^2 + 4x - 1 \\
f'(0) &= -1 \\
f'(1) &= 12 \\
f''(x) &= 18x + 4 \\
f''(0) &= 4 \\
f''(1) &= 22 \\
f'''(x) &= 18 \\
f'''(0) &= 18 \\
f'''(1) &= 18 \\
f^{(4)}(x) &= 0 \\
f^{(4)}(0) &= 0 \\
f^{(4)}(1) &= 0 \\
f^{(5)}(x) &= 0 \\
f^{(5)}(0) &= 0 \\
f^{(5)}(1) &= 0
\end{align*}
\]
At $a = 0$,
\begin{align*}
T_1(x) &= -4 - x; \\
T_2(x) &= -4 - x + 2x^2; \\
T_3(x) &= -4 - x + 2x^2 + 3x^3 = f(x); \\
T_4(x) &= T_3(x); \quad \text{and} \\
T_5(x) &= T_3(x).
\end{align*}

At $a = 1$,
\begin{align*}
T_1(x) &= 12(x - 1); \\
T_2(x) &= 12(x - 1) + 11(x - 1)^2; \\
T_3(x) &= 12(x - 1) + 11(x - 1)^2 + 3(x - 1)^3 = -4 - x + 2x^2 + 3x^3 = f(x); \\
T_4(x) &= T_3(x); \quad \text{and} \\
T_5(x) &= T_3(x).
\end{align*}

54. Let $T_n$ be the $n$th Taylor polynomial at $x = a$ for a polynomial $f$ of degree $n$. Based on the result of Exercise 53, guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

**Solution** Based on Exercise 53, we expect $|f(x) - T_n(x)| = 0$. From the Error Bound,
\[ |f(x) - T_n(x)| \leq K \frac{|x - a|^{n+1}}{(n+1)!}, \]
where $K$ is a number such that $|f^{(n+1)}(u)| \leq K$ for $u$ between $a$ and $x$. Since $f^{(n+1)}(x) = 0$ for an $n$th degree polynomial, we may take $K = 0$; the Error Bound then becomes $|f(x) - T_n(x)| = 0$.

55. Let $s(t)$ be the distance of a truck to an intersection. At time $t = 0$, the truck is 60 m from the intersection, travels away from it with a velocity of 24 m/s, and begins to slow down with an acceleration of $a = -3 \text{ m/s}^2$. Determine the second Maclaurin polynomial of $s$, and use it to estimate the truck’s distance from the intersection after 4 s.

**Solution** Place the origin at the intersection, so that $s(0) = 60$ (the truck is traveling away from the intersection). The second Maclaurin polynomial of $s(t)$ is
\[ T_2(t) = s(0) + s'(0)t + \frac{s''(0)}{2}t^2. \]
The conditions of the problem tell us that $s(0) = 60$, $s'(0) = 24$, and $s''(0) = -3$. Thus
\[ T_2(t) = 60 + 24t - \frac{3}{2}t^2, \]
so that after 4 seconds,
\[ T_2(4) = 60 + 24 \cdot 4 - \frac{3}{2} \cdot 4^2 = 132 \text{ m} \]
The truck is 132 m past the intersection.

56. A bank owns a portfolio of bonds whose value $P(r)$ depends on the interest rate $r$ (measured in percent; e.g., $r = 5$ means a 5% interest rate). The bank’s quantitative analyst determines that
\[ P(5) = 100,000, \quad \frac{dP}{dr} \bigg|_{r=5} = -40,000, \quad \frac{d^2P}{dr^2} \bigg|_{r=5} = 50,000. \]
In finance, this second derivative is called bond convexity. Find the second Taylor polynomial of $P(r)$ centered at $r = 5$ and use it to estimate the value of the portfolio if the interest rate moves to $r = 5.5%$.

**Solution** The second Taylor polynomial of $P(r)$ at $r = 5$ is
\[ T_2(r) = P(5) + P'(5)(r - 5) + \frac{P''(5)}{2}(r - 5)^2, \]
From the conditions of the problem, $P(5) = 100,000$, $P'(5) = -40,000$, and $P''(5) = 50,000$, so that
\[ T_2(r) = 100,000 - 40,000(r - 5) + 25,000(r - 5)^2. \]
If the interest rate moves to 5.5%, then the value of the portfolio can be estimated by
\[ T_2(5.5) = 100,000 - 40,000(0.5) + 25,000(0.5)^2 = 86,250. \]
57. A narrow, negatively charged ring of radius $R$ exerts a force on a positively charged particle $P$ located at distance $x$ above the center of the ring of magnitude

$$F(x) = -\frac{kx}{(x^2 + R^2)^{3/2}}$$

where $k > 0$ is a constant (Figure 12).

(a) Compute the third-degree Maclaurin polynomial for $F$.

(b) Show that $F \approx -(k/R^3)x$ to second order. This shows that when $x$ is small, $F(x)$ behaves like a restoring force similar to the force exerted by a spring.

(c) Show that $F(x) \approx -k/x^2$ when $x$ is large by showing that

$$\lim_{x \to \infty} \frac{F(x)}{-k/x^2} = 1$$

Thus, $F(x)$ behaves like an inverse square law, and the charged ring looks like a point charge from far away.

![Figure 12](image_url)

**SOLUTION**

(a) Start by computing and evaluating the necessary derivatives:

$$F(x) = -\frac{kx}{(x^2 + R^2)^{3/2}} \quad F(0) = 0$$

$$F'(x) = \frac{k(2x^2 - R^2)}{(x^2 + R^2)^{5/2}} \quad F'(0) = -\frac{k}{R^3}$$

$$F''(x) = \frac{3kx(3R^2 - 2x^2)}{(x^2 + R^2)^{7/2}} \quad F''(0) = 0$$

$$F'''(x) = \frac{3k(8x^4 - 24x^2R^2 + 3R^4)}{(x^2 + R^2)^{9/2}} \quad F'''(0) = \frac{9k}{R^5}$$

so that

$$T_3(x) = F(0) + F'(0)x + \frac{F''(0)}{2!}x^2 + \frac{F'''(0)}{3!}x^3 = -\frac{k}{R^3}x + \frac{3k}{2R^5}x^3$$

(b) To degree 2, $F(x) \approx T_3(x) \approx -\frac{k}{R^3}x$ as we may ignore the $x^3$ term of $T_3(x)$.

(c) We have

$$\lim_{x \to \infty} \frac{F(x)}{-k/x^2} = \lim_{x \to \infty} \left( \frac{x^2}{k} \cdot \frac{-kx}{(x^2 + R^2)^{3/2}} \right) = \lim_{x \to \infty} \frac{x^3}{(x^2 + R^2)^{3/2}}$$

$$= \lim_{x \to \infty} x^{-3}(x^2 + R^2)^{3/2} = \lim_{x \to \infty} \frac{1}{(1 + R^2/x^2)^{3/2}} = 1$$

Thus as $x$ grows large, $F(x)$ looks like an inverse square function.

58. A light wave of wavelength $\lambda$ travels from $A$ to $B$ by passing through an aperture (circular region) located in a plane that is perpendicular to $\overrightarrow{AB}$ (see Figure 13 for the notation). Let $f(r) = d' + h'$; that is, $f(r)$ is the distance $AC + CB$ as a function of $r$.

(a) Show that $f(r) = \sqrt{d^2 + r^2} + \sqrt{h^2 + r^2}$, and use the Maclaurin polynomial of order 2 to show that

$$f(r) \approx d + h + \frac{1}{2} \left( \frac{1}{d} + \frac{1}{h} \right) r^2$$
(b) The Fresnel zones, used to determine the optical disturbance at B, are the concentric bands bounded by the circles of radius $R_n$ such that $f(R_n) = d + h + n\lambda /2$. Show that $R_n \approx \sqrt{n\lambda L}$, where $L = (d^{-1} + h^{-1})^{-1}$.

(c) Estimate the radii $R_1$ and $R_{100}$ for blue light ($\lambda = 475 \times 10^{-7}$ cm) if $d = h = 100$ cm.

\[ \text{FIGURE 13 The Fresnel zones are the regions between the circles of radius } R_n. \]

**SOLUTION**

(a) From the diagram, we see that $AC = \sqrt{d^2 + r^2}$ and $CB = \sqrt{h^2 + r^2}$. Therefore, $f(r) = \sqrt{d^2 + r^2} + \sqrt{h^2 + r^2}$. Moreover,

\[ f'(r) = \frac{r}{\sqrt{d^2 + r^2}} + \frac{r}{\sqrt{h^2 + r^2}} \quad f''(r) = \frac{d^2}{(d^2 + r^2)^{3/2}} + \frac{h^2}{(h^2 + r^2)^{3/2}}, \]

\[ f(0) = d + h, \quad f'(0) = 0 \text{ and } f''(0) = d^{-1} + h^{-1}. \]

Thus,

\[ f(r) \approx T_2(r) = d + h + \frac{1}{2} \left( \frac{1}{d} + \frac{1}{h} \right) r^2. \]

(b) Solving

\[ f(R_n) \approx d + h + \frac{1}{2} \left( \frac{1}{d} + \frac{1}{h} \right) R_n^2 = d + h + \frac{n\lambda}{2} \]

yields

\[ R_n = \sqrt{n\lambda(d^{-1} + h^{-1})^{-1}} = \sqrt{n\lambda L}, \]

where $L = (d^{-1} + h^{-1})^{-1}$.

(c) With $d = h = 100$ cm, $L = 50$ cm. Taking $\lambda = 475 \times 10^{-7}$ cm, it follows that

\[ R_1 \approx \sqrt{\lambda L} = 0.04873 \text{ cm; and} \]

\[ R_{100} \approx \sqrt{100\lambda L} = 0.4873 \text{ cm}. \]

59. Referring to Figure 14, let $a$ be the length of the chord $AC$ of angle $\theta$ of the unit circle. Derive the following approximation for the excess of the arc over the chord:

\[ \theta - a \approx \frac{\theta^3}{24} \]

**Hint:** Show that $\theta - a = \theta - 2 \sin(\theta/2)$ and use the third Maclaurin polynomial as an approximation.

\[ \text{FIGURE 14 Unit circle.} \]
SOLUTION Draw a line from the center \( O \) of the circle to \( B \), and label the point of intersection of this line with \( AC \) as \( D \). Then \( CD = \frac{a}{2} \), and the angle \( COB \) is \( \frac{\theta}{2} \). Since \( CO = 1 \), we have

\[
\sin \left( \frac{\theta}{2} \right) = \frac{a}{2}
\]

so that \( a = 2 \sin (\theta/2) \). Thus \( \theta - a = \theta - 2 \sin (\theta/2) \). Now, the third Maclaurin polynomial for \( f(\theta) = \sin (\theta/2) \) can be computed as follows: \( f(0) = 0 \), \( f'(0) = \frac{\pi}{2} \cos (\theta/2) \) so that \( f''(0) = -\frac{\pi}{4} \), \( f'''(0) = -\frac{\pi^3}{8} \), and \( f''''(0) = 0 \). Finally, \( f^{(m)}(x) = -\frac{\pi}{8} \cos (\theta/2) \) and \( f''''(0) = -\frac{1}{8} \). Thus

\[
T_3(\theta) = f(0) + f'(0)\theta + \frac{f''(0)}{2!} \theta^2 + \frac{f'''(0)}{3!} \theta^3 = \frac{1}{2} \theta - \frac{1}{24} \theta^3
\]

Finally,

\[
\theta - a = \theta - 2 \sin \left( \frac{\theta}{2} \right) \approx \theta - 2T_3(\theta) = \theta - \left( \theta - \frac{1}{24} \theta^3 \right) = \frac{\theta^3}{24}
\]

60. To estimate the length \( \theta \) of a circular arc of the unit circle, the seventeenth-century Dutch scientist Christian Huygens used the approximation \( \theta \approx (8b - a)/3 \), where \( a \) is the length of the chord \( AC \) of angle \( \theta \) and \( b \) is the length of the chord \( AB \) of angle \( \theta/2 \) (Figure 14).

(a) Prove that \( a = 2 \sin (\theta/2) \) and \( b = 2 \sin (\theta/4) \), and show that the Huygens approximation amounts to the approximation

\[
\theta \approx 16 \sin \left( \frac{\theta}{3} \right) - \frac{2}{3} \sin \left( \frac{\theta}{2} \right)
\]

(b) Compute the fifth Maclaurin polynomial of the function on the right.

(c) Use the error bound to show that the error in the Huygens approximation is less than \( 0.00022\|\theta\|^5 \).

SOLUTION

(a) By the Law of Cosines and the identity \( \sin^2 (\theta/2) = (1 - \cos \theta)/2 \):

\[
a^2 = 1^2 + 1^2 - 2 \cos \theta = 2(1 - \cos \theta) = 4 \sin^2 \left( \frac{\theta}{2} \right)
\]

and so \( a = 2 \sin (\theta/2) \). Similarly, \( b = 2 \sin (\theta/4) \). Substituting these expressions for \( a \) and \( b \) into the Huygens approximation yields

\[
\theta \approx \frac{8}{3} \cdot 2 \sin \left( \frac{\theta}{4} \right) - \frac{2}{3} \cdot 2 \sin \left( \frac{\theta}{2} \right) = \frac{16}{3} \sin \left( \frac{\theta}{4} \right) - \frac{2}{3} \sin \left( \frac{\theta}{2} \right)
\]

(b) The fifth Maclaurin polynomial for \( \sin x \) is \( x - \frac{x^3}{3!} + \frac{x^5}{5!} \); therefore, the fifth Maclaurin polynomial for \( \sin (\theta/2) \) is

\[
\frac{\theta}{2} - \frac{(\theta/2)^3}{6} + \frac{(\theta/2)^5}{120} = \frac{\theta}{2} - \frac{\theta^3}{48} + \frac{\theta^5}{3840}
\]

and the fifth Maclaurin polynomial for \( \sin (\theta/4) \) is

\[
\frac{\theta}{4} - \frac{(\theta/4)^3}{6} + \frac{(\theta/4)^5}{120} = \frac{\theta}{4} - \frac{\theta^3}{384} + \frac{\theta^5}{122880}
\]

Thus, the fifth Maclaurin polynomial for \( f(\theta) = 16 \sin \left( \frac{\theta}{3} \right) - \frac{2}{3} \sin \left( \frac{\theta}{2} \right) \) is

\[
\theta - \frac{1}{7680} \theta^5
\]

(c) Based on the result from part (b), the Huygens approximation for \( \theta \) is equal to the fourth Maclaurin polynomial \( T_4(\theta) \) for \( f(\theta) \), and the error is at most \( K \|\theta\|^5/5! \), where \( K \) is the maximum value of the absolute value of the fifth derivative \( f^{(5)}(\theta) \). Because

\[
f^{(5)}(\theta) = \frac{1}{192} \cos \left( \frac{\theta}{4} \right) - \frac{1}{48} \cos \left( \frac{\theta}{2} \right)^3
\]

we may take \( K = 1/48 + 1/192 = 0.0260417 \), so the error is at most \( \|\theta\|^5 \) times the constant

\[
\frac{0.0261}{5!} = 0.00022.
\]
**Further Insights and Challenges**

61. Show that the $n$th Maclaurin polynomial of $f(x) = \arcsin x$ for $n$ odd is

$$T_n(x) = x + \frac{1}{2} x^3 + \frac{1}{2 \cdot 4} x^5 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} x^n$$

**SOLUTION** Let $f(x) = \sin^{-1}x$. Then

$$f(x) = \sin^{-1}x, \quad f(0) = 0$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f'(0) = 1$$
$$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) \quad f''(0) = 0$$
$$f'''(x) = \frac{2x^2+1}{(1-x^2)^{5/2}} \quad f'''(0) = 1$$
$$f^{(4)}(x) = \frac{-3x(2x^2+3)}{(1-x^2)^{7/2}} \quad f^{(4)}(0) = 0$$
$$f^{(5)}(x) = \frac{24x^4+72x^2+9}{(1-x^2)^{9/2}} = f^{(5)}(0) = 9$$

and

$$T_7(x) = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \frac{225x^7}{7!} = x + \frac{x^3}{2} + \frac{13x^5}{24} + \frac{135x^7}{246}$$

Thus, we can infer that

$$T_n(x) = x + \frac{1}{2} x^3 + \frac{1}{2 \cdot 4} x^5 + \frac{13}{2 \cdot 4 \cdot 6} x^7 + \cdots + \frac{13}{2} \cdots \frac{n-2}{n-1} x^n$$

62. Let $x \geq 0$ and assume that $f^{(n+1)}(t) \geq 0$ for $0 \leq t \leq x$. Use Taylor’s Theorem to show that the $n$th Maclaurin polynomial $T_n$ satisfies

$$T_n(x) \leq f(x), \quad \text{for all } x \geq 0$$

**SOLUTION** From Taylor’s Theorem,

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!} \int_0^x (x-u)^n f^{(n+1)}(u) \, du.$$ 

If $f^{(n+1)}(t) \geq 0$ for all $t$ then

$$\frac{1}{n!} \int_0^x (x-u)^n f^{(n+1)}(u) \, du \geq 0$$

since $(x-u)^n \geq 0$ for $0 \leq u \leq x$. Thus, $f(x) - T_n(x) \geq 0$, or $f(x) \geq T_n(x)$.

63. Use Exercise 62 to show that for $x \geq 0$ and all $n$,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Sketch the graphs of $y = e^x$, $y = T_1(x)$, and $y = T_2(x)$ on the same coordinate axes. Does this inequality remain true for $x < 0$?

**SOLUTION** Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for all $n$. Because $e^x > 0$ for all $x$, it follows from Exercise 62 that $f(x) \geq T_n(x)$ for all $x \geq 0$ and for all $n$. For $f(x) = e^x$,

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

thus,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

From the figure below, we see that the inequality does not remain true for $x < 0$, as $T_2(x) \geq e^x$ for $x < 0$. 

...
This exercise is intended to reinforce the proof of Taylor’s Theorem.

(a) Show that 
\[ f(x) = T_0(x) + \int_a^x f'(u) \, du. \]

(b) Use Integration by Parts to prove the formula
\[ \int_a^x (x-u) f''(u) \, du = f'(x)(x-a) - f'(a)(x-a) + \int_a^x f'(u) \, du. \]

(c) Prove the case \( n = 2 \) of Taylor’s Theorem:
\[ f(x) = T_1(x) + \int_a^x (x-u) f''(u) \, du. \]

SOLUTION

(a)

\[ T_0(x) + \int_a^x f'(u) \, du = T_0(x) + f(x) - f(a) \quad \text{(from FTC2)} \]
\[ = f(x) - f(a) = f(x). \]

(b) Using Integration by Parts with \( w = x-u \) and \( v' = f''(u) \, du \),
\[ \int_a^x (x-u) f''(u) \, du = f'(u)(x-u) \bigg|_a^x + \int_a^x f'(u) \, du \]
\[ = f'(x)(x-x) - f'(a)(x-a) + \int_a^x f'(u) \, du \]
\[ = -f'(a)(x-a) + \int_a^x f'(u) \, du. \]

(c)

\[ T_1(x) + \int_a^x (x-u) f''(u) \, du = f(a) + f'(a)(x-a) + (-f'(a)(x-a)) + \int_a^x f'(u) \, du \]
\[ = f(a) + f(x) - f(a) = f(x). \]

In Exercises 65–69, we estimate integrals using Taylor polynomials. Exercise 66 is used to estimate the error.

65. Find the fourth Maclaurin polynomial \( T_4 \) for \( f(x) = e^{-x^2} \), and calculate \( I = \int_0^{1/2} T_4(x) \, dx \) as an estimate for \( \int_0^{1/2} e^{-x^2} \, dx \). A CAS yields the value \( I \approx 0.461281 \). How large is the error in your approximation? Hint: \( T_4 \) is obtained by substituting \(-x^2\) in the second Maclaurin polynomial for \( e^x \).

SOLUTION

Following the hint, since the second Maclaurin polynomial for \( e^x \) is
\[ 1 + x + \frac{x^2}{2}, \]
we substitute \(-x^2\) for \( x \) to get the fourth Maclaurin polynomial for \( e^{-x^2} \):
\[ T_4(x) = 1 - x^2 + \frac{x^4}{2}. \]

Then
\[ \int_0^{1/2} e^{-x^2} \, dx \approx \int_0^{1/2} T_4(x) \, dx = \left( x - \frac{1}{3} x^3 + \frac{1}{10} x^5 \right) \bigg|_0^{1/2} = \frac{443}{960} \approx 0.461458333. \]
CHAPTER 8  FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

Using a CAS, we have \( \int_{0}^{1/2} e^{-x^2} \, dx \approx 0.4612810064 \), so the error is about \( 1.77 \times 10^{-4} \).

66. Approximating Integrals  Let \( L > 0 \). Show that if two functions \( f \) and \( g \) satisfy \( |f(x) - g(x)| < L \) for all \( x \in [a, b] \), then

\[
\left| \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \right| < L(b-a).
\]

**Solution**  Because \( f(x) - g(x) \leq |f(x) - g(x)| \), it follows that

\[
\left| \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \right| = \left| \int_{a}^{b} (f(x) - g(x)) \, dx \right| \leq \int_{a}^{b} |f(x) - g(x)| \, dx < L(b-a).
\]

67. Let \( T_4 \) be the fourth Maclaurin polynomial for \( f(x) = \cos x \).

(a) Show that

\[
|\cos x - T_4(x)| \leq \frac{(1/2)^6}{6!} \text{ for all } x \in \left[ 0, \frac{1}{2} \right]
\]

**Hint:** \( T_4(x) = T_5(x) \).

(b) Evaluate \( \int_0^{1/2} T_4(x) \, dx \) as an approximation to \( \int_0^{1/2} \cos x \, dx \). Use Exercise 66 to find a bound for the size of the error.

**Solution**

(a) Let \( f(x) = \cos x \). Then

\[
T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.
\]

Moreover, with \( a = 0 \), \( T_4(x) = T_5(x) \) and

\[
|\cos x - T_4(x)| \leq K \frac{|x|^6}{6!},
\]

where \( K \) is a number such that \( |f^{(6)}(u)| \leq K \) for \( u \) between 0 and \( x \). Now \( |f^{(6)}(u)| = |\cos u| \leq 1 \), so we may take \( K = 1 \). Finally, with the restriction \( x \in \left[ 0, \frac{1}{2} \right] \),

\[
|\cos x - T_4(x)| \leq \frac{(1/2)^6}{6!} \approx 0.000022.
\]

(b) \( \int_0^{1/2} \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} \right) \, dx = \frac{1841}{3840} \approx 0.479427. \)

By (a) and Exercise 66, the error associated with this approximation is less than or equal to

\[
\frac{(1/2)^6}{6!} \left( \frac{1}{2} - 0 \right) = \frac{1}{92,160} \approx 1.1 \times 10^{-5}.
\]

Note that \( \int_0^{1/2} \cos x \, dx \approx 0.4794255 \), so the actual error is roughly \( 1.5 \times 10^{-6} \).

68. Let \( Q(x) = 1 - x^2/6 \). Use the error bound for \( f(x) = \sin x \) to show that

\[
\left| \frac{\sin x}{x} - Q(x) \right| \leq \frac{|x|^4}{3!};
\]

Then calculate \( \int_0^1 Q(x) \, dx \) as an approximation to \( \int_0^1 (\sin x/x) \, dx \) and find a bound for the error.
For an additional question, consider the following:

Additionally, this is also true for any function $T(x)$ since $\sin(x)$ is either $\pm \sin x$ or $\pm \cos x$, which are bounded in absolute value by 1. Thus, we may take $K = 1$ in the Error Bound, so

$$|\sin x - Q(x)| = |\sin x - T_3(x)| = |\sin x - Q_3(x)| \leq K \frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Divide both sides of this inequality by $|x|$ to get

$$\left| \frac{\sin x}{x} - Q(x) \right| \leq \frac{|x|^4}{3!}$$

We can thus estimate $\int_0^1 (\sin x/x) \, dx$ by

$$\int_0^1 Q(x) \, dx = \int_0^1 \left(1 - \frac{x^2}{6}\right) \, dx = \left(x - \frac{x^3}{18}\right) \bigg|_0^1 = \frac{17}{18} \approx 0.9444444444$$

The error in this approximation is at most

$$\frac{|x|^4}{5!} = \frac{1}{120} \approx 0.00833333333$$

The true value of the integral is approximately 0.9460830704, which is consistent with the error bound.

69. (a) Compute the sixth Maclaurin polynomial $T_6$ for $f(x) = \sin(x^2)$ by substituting $x^2$ in $P(x) = x - x^3/6$, the third Maclaurin polynomial for $f(x) = \sin x$.

(b) Show that $|\sin(x^2) - T_6(x)| \leq \frac{|x|^{10}}{5!}$.

*Hint:* Substitute $x^2$ for $x$ in the error bound for $|\sin x - P(x)|$, noting that $P$ is also the fourth Maclaurin polynomial for $f(x) = \sin x$.

(c) Use $T_6$ to approximate $\int_0^{1/2} \sin(x^2) \, dx$ and find a bound for the error.

**Solution**

Let $s(x) = \sin x$ and $f(x) = \sin(x^2)$. Then

(a) The third Maclaurin polynomial for $s(x)$ is

$$S_3(x) = x - \frac{x^3}{6}$$

so, substituting $x^2$ for $x$, we see that the sixth Maclaurin polynomial for $\sin(x^2)$ is

$$T_6(x) = x^2 - \frac{x^6}{6}$$

(b) Since all derivatives of $s(x)$ are either $\pm \cos x$ or $\pm \sin x$, they are bounded in magnitude by 1, so we may take $K = 1$ in the Error Bound for $s(x)$. Since the third Maclaurin polynomial $S_3(x)$ for $s(x)$ is also the fourth Maclaurin polynomial $S_4(x)$, we have

$$|\sin x - S_3(x)| = |\sin x - S_4(x)| \leq K \frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Now substitute $x^2$ for $x$ in the above inequality and note from part (a) that $S_3(x^2) = T_6(x)$ to get

$$|\sin(x^2) - S_3(x^2)| = |\sin(x^2) - T_6(x)| \leq \frac{|x|^5}{5!} = \frac{|x|^{10}}{5!}$$

(c) $\int_0^{1/2} \sin(x^2) \, dx \approx \int_0^{1/2} T_6(x) \, dx = \left(\frac{1}{3} x^3 - \frac{1}{42} x^7\right) \bigg|_0^{1/2} \approx 0.04148065476$

From part (b) and Exercise 66, the error is bounded by

$$\frac{x^{10}}{5!} \left(\frac{1}{2} - 0\right) = \frac{(1/2)^{11}}{120} = \frac{1}{2048 \cdot 120} \approx 4.069010417 \times 10^{-6}$$

The true value of the integral is approximately 0.04148102427, which is consistent with the computed error bound.
70. Prove by induction that for all \(k\),
\[
\frac{d^j}{dx^j} \left( \frac{(x-a)^k}{k!} \right) = \frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!}
\]
Use this to prove that \(T_n\) agrees with \(f\) at \(x = a\) to order \(n\).

**Solution**

The first formula is clearly true for \(j = 0\). Suppose the formula is true for an arbitrary \(j\). Then
\[
\frac{d^{j+1}}{dx^{j+1}} \left( \frac{(x-a)^k}{k!} \right) = \frac{d}{dx} \left( \frac{d^j}{dx^j} \left( \frac{(x-a)^k}{k!} \right) \right) = \frac{d}{dx} \left( \frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!} \right)
\]
\[
= \frac{k(k-1)\cdots(k-j+1)(k-(j+1)+1)(x-a)^{k-(j+1)}}{k!}
\]
as desired. Note that if \(k = j\), then the numerator is \(k!\), the denominator is \(k!\) and the value of the derivative is 1; otherwise, the value of the derivative is 0 at \(x = a\). In other words,
\[
\frac{d^j}{dx^j} \left( \frac{(x-a)^k}{k!} \right) \bigg|_{x=a} = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}
\]

Applying this latter formula, it follows that
\[
\frac{d^j}{dx^j} T_n(a) = \sum_{k=0}^{n} \frac{d^j}{dx^j} \left( \frac{f^{(k)}(a)}{k!} (x-a)^k \right) \bigg|_{x=a} = f^{(j)}(a)
\]
as required.

71. Let \(a\) be any number and let
\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]
be a polynomial of degree \(n\) or less.

(a) Show that if \(P^{(j)}(a) = 0\) for \(j = 0, 1, \ldots, n\), then \(P(x) = 0\), that is, \(a_j = 0\) for all \(j\). **Hint:** Use induction, noting that if the statement is true for degree \(n-1\), then \(P'(x) = 0\).

(b) Prove that \(T_n\) is the only polynomial of degree \(n\) or less that agrees with \(f\) at \(x = a\) to order \(n\). **Hint:** If \(Q\) is another such polynomial, apply (a) to \(P(x) = T_n(x) - Q(x)\).

**Solution**

(a) Note first that if \(n = 0\), i.e., if \(P(x) = a_0\) is a constant, then the statement holds; if \(P^{(0)}(a) = P(a) = 0\), then \(a_0 = 0\), so that \(P(x) = 0\). Next, assume the statement holds for all polynomials of degree \(n-1\) or less, and let \(P(x)\) be a polynomial of degree at most \(n\) with \(P^{(j)}(a) = 0\) for \(j = 0, 1, \ldots, n\). If \(P(x)\) has degree less than \(n\), then we know \(P(x) = 0\) by induction, so assume the degree of \(P(x)\) is exactly \(n\). Then
\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]
where \(a_n \neq 0\); also,
\[
P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1
\]
Note that \(P^{(j+1)}(a) = (P^{(j)}(a))'\) for \(j = 0, 1, \ldots, n-1\). But then
\[
0 = P^{(j+1)}(a) = (P^{(j)}(a))' \text{ for all } j = 0, 1, \ldots, n-1
\]
Since \(P'(x)\) has degree at most \(n-1\), it follows by induction that \(P'(x) = 0\). Thus \(a_n = a_{n-1} = \cdots = a_1 = 0\) so that \(P(x) = a_0\). But \(P(a) = 0\) so that \(a_0 = 0\) as well and thus \(P(x) = 0\).

(b) Suppose \(Q(x)\) is a polynomial of degree at most \(n\) that agrees with \(f(x)\) at \(x = a\) up to order \(n\). Let \(P(x) = T_n(x) - Q(x)\). Note that \(P(x)\) is a polynomial of degree at most \(n\) since both \(T_n(x)\) and \(Q(x)\) are. Since both \(T_n(x)\) and \(Q(x)\) agree with \(f(x)\) at \(x = a\) to order \(n\), we have
\[
T_n^{(j)}(a) = f^{(j)}(a), \quad j = 0, 1, 2, \ldots, n
\]
Thus
\[
P^{(j)}(a) = T_n^{(j)}(a) - Q^{(j)}(a) = 0 \quad \text{for } j = 0, 1, 2, \ldots, n
\]
But then by part (a), \(P(x) = 0\) so that \(T_n(x) = Q(x)\).
In Exercises 1–4, calculate the arc length over the given interval.

1. \( y = \frac{x^5}{10} + \frac{x^{-3}}{6}, \ [1, 2] \)

**Solution** Let \( y = \frac{x^5}{10} + \frac{x^{-3}}{6} \). Then

\[
1 + (y')^2 = 1 + \left( \frac{x^4}{2} - \frac{x^{-4}}{2} \right)^2 = 1 + \frac{x^8}{4} - \frac{1}{2} + \frac{x^{-8}}{4} = \frac{x^8}{4} + \frac{1}{2} + \frac{x^{-8}}{4} = \left( \frac{x^4}{2} + \frac{x^{-4}}{2} \right)^2.
\]

Because \( \frac{1}{2}(x^4 + x^{-4}) > 0 \) on \([1, 2]\), the arc length is

\[
s = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \left( \frac{x^4}{2} + \frac{x^{-4}}{2} \right) \, dx = \left[ \frac{x^5}{10} - \frac{x^{-3}}{6} \right]_1^2 = \frac{779}{240}.
\]

2. \( y = e^{x/2} + e^{-x/2}, \ [0, 2] \)

**Solution** Let \( y = e^{x/2} + e^{-x/2} \). Then, \( y' = \frac{1}{2}(e^{x/2} - e^{-x/2}) \), so

\[
1 + (y')^2 = \frac{1}{4}(e^x - 2 + e^{-x}) + 1 = \frac{1}{4}(e^x + 2 + e^{-x}) = \left[ \frac{1}{2}(e^{x/2} + e^{-x/2}) \right]^2.
\]

Because \( \frac{1}{2}(e^{x/2} + e^{-x/2}) > 0 \) on \([0, 2]\),

\[
s = \int_0^2 \frac{1}{2}(e^{x/2} + e^{-x/2}) \, dx = (e^{x/2} - e^{-x/2}) \bigg|_0^2 = e - e^{-1} = 2 \sin(1).
\]

3. \( y = 4x - 2, \ [-2, 2] \)

**Solution** Let \( y = 4x - 2 \). Then

\[
\sqrt{1 + (y')^2} = \sqrt{1 + 4^2} = \sqrt{17}.
\]

Hence,

\[
s = \int_{-2}^2 \sqrt{17} \, dx = 4\sqrt{17}.
\]

4. \( y = x^{2/3}, \ [0, 8] \)

**Solution** Let \( y = x^{2/3} \). Then \( y' = \frac{2}{3}x^{-1/3} \), and

\[
\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4}{9}x^{-2/3}} = \sqrt{\frac{4}{9}x^{-2/3}(\frac{9}{4}x^{2/3} + 1)} = \frac{2}{3}x^{-1/3} \sqrt{1 + \frac{9}{4}x^{2/3}}.
\]

The arc length is

\[
s = \int_0^8 \sqrt{1 + (y')^2} \, dx = \int_0^8 \frac{2}{3}x^{-1/3} \sqrt{1 + \frac{9}{4}x^{2/3}} \, dx.
\]
Compute the trapezoidal approximation

Now, we make the substitution \( u = 1 + \tan^2 x \), \( du = 2 \sec^2 x \, dx \). Then

\[
S = \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{1 + \sec^4 x}} \, dx \approx \frac{\pi}{40} \left( \sqrt{1 + \sec^4 0} + 2 \sqrt{1 + \sec^4 \frac{\pi}{20}} + 2 \sqrt{1 + \sec^4 \frac{\pi}{10}} + 2 \sqrt{1 + \sec^4 \frac{3\pi}{20}} + 2 \sqrt{1 + \sec^4 \frac{\pi}{5}} \right) \\
\approx \frac{\pi}{40} (1.41421356 + 2 \cdot 1.43206164 + 2 \cdot 1.49073513 + 2 \cdot 1.60830125 + 2 \cdot 1.82602534 + 2.23606797) \\
\approx 1.285267058
\]

In Exercises 7–10, calculate the surface area of the solid obtained by rotating the curve over the given interval about the \( x \)-axis.

7. \( y = x + 1 \), \([0, 4]\)

**Solution** Let \( y = x + 1 \). Then \( y' = 1 \), and

\[
y\sqrt{1 + y^2} = (x + 1)\sqrt{1 + 1} = \sqrt{2}(x + 1).
\]

Thus,

\[
SA = 2\pi \int_0^4 \sqrt{2}(x + 1) \, dx = 2\sqrt{2}\pi \left( \frac{x^2}{2} + x \right) \bigg|_0^4 = 24\sqrt{2}\pi.
\]
8. \( y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}, \quad [0, 1] \)

**Solution** Let \( y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4} \). Then

\[
y' = \frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2},
\]

and

\[
1 + (y')^2 = 1 + \left( \frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2} \right)^2 = \frac{x^{-1/2}}{4} + \frac{1}{2} + \frac{x^{1/2}}{4} = \left( \frac{x^{-1/4}}{2} + \frac{x^{1/4}}{2} \right)^2.
\]

Because \( \frac{1}{2}(x^{-1/4} + x^{1/4}) \geq 0 \), the surface area is

\[
2\pi \int_0^1 y\sqrt{1 + (y')^2} dy = 2\pi \int_0^1 \left( \frac{2x^{3/4}}{3} - \frac{2x^{5/4}}{5} \right) \left( \frac{x^{1/4}}{2} + \frac{x^{-1/4}}{2} \right) dx = 2\pi \int_0^1 \left( -\frac{x^{3/2}}{5} - \frac{x}{5} + \frac{x}{3} + \frac{\sqrt{x}}{3} \right) dx
\]

\[
= 2\pi \left( -\frac{2x^{5/2}}{25} + \frac{x^2}{15} + \frac{2x^{3/2}}{9} \right) \bigg|_0^1 = \frac{94}{225}\pi.
\]

9. \( y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}, \quad [1, 2] \)

**Solution** Let \( y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \). Then

\[
y' = \sqrt{x} - \frac{1}{4\sqrt{x}},
\]

and

\[
1 + (y')^2 = 1 + \left( \sqrt{x} - \frac{1}{4\sqrt{x}} \right)^2 = 1 + \left( x - \frac{1}{2} + \frac{1}{16x} \right) = x + \frac{1}{2} + \frac{1}{16x} = \left( \sqrt{x} + \frac{1}{4\sqrt{x}} \right)^2.
\]

Because \( \sqrt{x} + \frac{1}{4\sqrt{x}} \geq 0 \), the surface area is

\[
2\pi \int_a^b y\sqrt{1 + (y')^2} dx = 2\pi \int_1^2 \left( \frac{2}{3}x^{3/2} - \frac{\sqrt{x}}{2} \right) \left( \sqrt{x} + \frac{1}{4\sqrt{x}} \right) dx
\]

\[
= 2\pi \int_1^2 \left( \frac{2}{3}x^2 + \frac{1}{6} - \frac{1}{2} - \frac{1}{8} \right) dx = 2\pi \left( \frac{2x^3}{9} - \frac{x^2}{6} - \frac{1}{8} \right) \bigg|_1 = \frac{67}{36}\pi.
\]

10. \( y = \frac{1}{2}x^2, \quad [0, 2] \)

**Solution** Let \( y = \frac{1}{2}x^2 \). Then \( y' = x \) and

\[
SA = 2\pi \int_0^2 \frac{1}{2} x^2 \sqrt{1 + x^2} dx = \pi \int_0^2 x^2 \sqrt{1 + x^2} dx.
\]

Using the substitution \( x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta \), we find that

\[
\int x^2 \sqrt{1 + x^2} dx = \int \sec^3 \theta \tan^2 \theta \, d\theta = \int (\sec^5 \theta - \sec^3 \theta) \, d\theta
\]

\[
= \left( \frac{3}{4} \sec^3 \theta \tan \theta + \frac{3}{8} \sec \theta \tan^3 \theta + \frac{3}{8} \ln | \sec \theta + \tan \theta | - \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln | \sec \theta + \tan \theta | \right) + C
\]

\[
= \frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8} \sqrt{1 + x^2} - \frac{1}{8} \ln \sqrt{1 + x^2} + x + C.
\]

Finally,

\[
SA = \pi \left( \frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8} \sqrt{1 + x^2} - \frac{1}{8} \ln \sqrt{1 + x^2} + x \right) \bigg|_0^2
\]

\[
= \pi \left( \frac{5\sqrt{5}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8} \ln (2 + \sqrt{5}) \right) = \frac{9\sqrt{5}}{4} \pi - \frac{\pi}{8} \ln (2 + \sqrt{5}).
\]
11. Compute the total surface area of the coin obtained by rotating the region in Figure 1 about the \( x \)-axis. The top and bottom parts of the region are semicircles with a radius of 1 mm.

![Figure 1](image)

**Solution**

The generating half circle of the edge is \( y = 2 + \sqrt{1 - x^2} \). Then,

\[
y' = \frac{-2x}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}}.
\]

and

\[
1 + (y')^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}.
\]

The surface area of the edge of the coin is

\[
2\pi \int_{-1}^{1} y \sqrt{1 + (y')^2} \, dx = 2\pi \int_{-1}^{1} \left(2 + \sqrt{1 - x^2}\right) \frac{1}{\sqrt{1 - x^2}} \, dx
\]

\[
= 2\pi \left(2 \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} + \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx\right)
\]

\[
= 2\pi \left(2 \arcsin|x| \Big|_{-1}^{1} + \int_{-1}^{1} dx\right)
\]

\[
= 2\pi (2\pi + 2) = 4\pi^2 + 4\pi.
\]

We now add the surface area of the two sides of the disk, which are circles of radius 2. Hence the surface area of the coin is:

\[
4\pi^2 + 4\pi + 2\pi \cdot 2^2 = 4\pi^2 + 12\pi.
\]

12. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex at the surface of the water.

**Solution**

To find the fluid force, we must find an expression for the horizontal width \( f(y) \) of the triangle at depth \( y \).

![Triangle](image)

By similar triangles we have:

\[
\frac{y}{f(y)} = \frac{3}{2} \quad \text{so} \quad f(y) = \frac{2y}{3}.
\]

Therefore, the fluid force on the side of the triangle is

\[
F = \rho g \int_{0}^{3} yf(y) \, dy = \rho g \int_{0}^{3} \frac{2y^2}{3} \, dy = \rho g \cdot \frac{2y^3}{9} \bigg|_{0}^{3} = 6\rho g.
\]

For water, \( \rho = 10^3 \); \( g = 9.8 \), so \( F = 6 \cdot 9800 = 58,800 \) N.
13. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex located at a depth of 4 m.

**SOLUTION** We need to find an expression for the horizontal width \( f(y) \) at depth \( y \).

By similar triangles we have:

\[
\frac{f(y)}{y-4} = \frac{2}{3} \quad \text{so} \quad f(y) = \frac{2(y-4)}{3}.
\]

Hence, the force on the side of the triangle is

\[
F = \rho g \int_{4}^{7} y f(y) \, dy = \frac{2\rho g}{3} \int_{4}^{7} (y^2 - 4y) \, dy = \frac{2\rho g}{3} \left( \frac{y^3}{3} - 2y^2 \right) \bigg|_{4}^{7} = 18\rho g.
\]

For water, \( \rho = 10^3; \ g = 9.8, \) so \( F = 18 \cdot 9800 = 176,400 \) N.

14. A plate in the shape of the shaded region in Figure 2 is submerged in water. Calculate the fluid force on a side of the plate if the water surface is \( y = 1 \).

**SOLUTION** Here, we can proceed as follows: Calculate the force that would be exerted on the entire semicircle and then subtract the force that would be exerted on the “missing” portion of the ellipse. The force on the semicircle is

\[
2w \int_{0}^{1} (1-y)\sqrt{1-y^2} \, dy = 2w \int_{0}^{1} \sqrt{1-y^2} \, dy - 2w \int_{0}^{1} y\sqrt{1-y^2} \, dy.
\]

The first integral can be interpreted as the area of one-quarter of a circle of radius 1. Hence,

\[
\int_{0}^{1} \sqrt{1-y^2} \, dy = \frac{\pi}{4}.
\]

On the other hand,

\[
\int_{0}^{1} y\sqrt{1-y^2} \, dy = -\frac{1}{3}(1-y^2)^{3/2} \bigg|_{0}^{1} = \frac{1}{3}.
\]

Thus, the force on the semicircle is

\[
2w \left( \frac{\pi}{4} - \frac{1}{3} \right).
\]

Now for the ellipse. The force that would be exerted on the upper half of the ellipse is

\[
2w \int_{0}^{1/2} (1-y)\sqrt{1-4y^2} \, dy = 2w \int_{0}^{1/2} \sqrt{1-4y^2} \, dy - 2w \int_{0}^{1/2} y\sqrt{1-4y^2} \, dy.
\]
Using the substitution \( w = 2y, dw = 2\, dy \), it follows that
\[
\int_{0}^{1\frac{1}{3}} \sqrt{1 - 4y^2} \, dy = \frac{1}{2} \int_{0}^{1} \sqrt{1 - w^2} \, dw = \frac{\pi}{8},
\]
and
\[
\int_{0}^{1\frac{1}{3}} \sqrt{1 - 4y^2} \, y \, dy = \frac{1}{4} \int_{0}^{1} w\sqrt{1 - w^2} \, dw = \frac{1}{12}.
\]
Thus, the force on the “missing” ellipse is
\[
2w \left( \frac{\pi}{8} - \frac{1}{12} \right).
\]
Finally, the force exerted on the plate shown in Figure 2 is
\[
F = 2w \left( \frac{\pi}{4} - \frac{1}{3} \right) - 2w \left( \frac{\pi}{8} - \frac{1}{12} \right) = \frac{\pi}{4} - \frac{2}{4}w.
\]

15. Figure 3 shows an object whose face is an equilateral triangle with 5-m sides. The object is 2 m thick and is submerged in water with its vertex 3 m below the water surface. Calculate the fluid force on both a triangular face and a slanted rectangular edge of the object.

**SOLUTION** Start with each triangular face of the object. Place the origin at the upper vertex of the triangle, with the positive \( y \)-axis pointing downward. Note that because the equilateral triangle has sides of length 5 feet, the height of the triangle is \( \frac{5\sqrt{3}}{2} \) feet. Moreover, the width of the triangle at location \( y \) is \( \frac{2y}{\sqrt{3}} \). Thus,
\[
F = \frac{2\rho g}{\sqrt{3}} \int_{0}^{5\sqrt{3}/2} (y + 3)y \, dy = \frac{2\rho g}{\sqrt{3}} \left( \frac{1}{3}y^3 + \frac{3}{2}y^2 \right)_{0}^{5\sqrt{3}/2} = \frac{\rho g}{4} \left( 125 + 75\sqrt{3} \right) \approx 624.514 \text{ N}.
\]

Now, consider the slanted rectangular edges of the object. Each edge is a constant 2 feet wide and makes an angle of 60° with the horizontal. Therefore,
\[
F = \frac{\rho g}{\sin 60\degree} \int_{0}^{5\sqrt{3}/2} 2(y + 3) \, dy = \frac{2\rho g}{\sqrt{3}} \left( y^2 + 6y \right)_{0}^{5\sqrt{3}/2} = \rho g \left( \frac{25\sqrt{3}}{2} + 30 \right) \approx 506.176 \text{ N}.
\]
The force on the bottom face can be computed without calculus:
\[
F = \left( 3 + \frac{5\sqrt{3}}{2} \right) (2)(5)\rho g \approx 718.352 \text{ N}.
\]

16. The end of a horizontal oil tank is an ellipse (Figure 4) with equation \( \left( \frac{x}{4} \right)^2 + \left( \frac{y}{3} \right)^2 = 1 \) (length in meters). Assume that the tank is filled with oil of density 900 kg/m³.
(a) Calculate the total force \( F \) on the end of the tank when the tank is full.
(b) Would you expect the total force on the lower half of the tank to be greater than, less than, or equal to \( \frac{1}{2}F \)? Explain. Then compute the force on the lower half exactly and confirm (or refute) your expectation.

**FIGURE 3**

**FIGURE 4**
SOLUTION
(a) Solving the equation of the ellipse for $x$ yields

$$x = \frac{4}{3}\sqrt{9 - y^2}.$$ 

Therefore, a horizontal strip of the ellipse at height $y$ has width $\frac{4}{3}\sqrt{9 - y^2}$. This strip is at a depth of $3 - y$, so the total force on the end of the tank is

$$F = \rho g \int_{-3}^{3} (3 - y) \frac{8}{3}\sqrt{9 - y^2} dy = 8\rho g \int_{-3}^{3} \sqrt{9 - y^2} dy - \frac{8}{3}\rho g \int_{-3}^{3} y \sqrt{9 - y^2} dy.$$ 

The first integral can be interpreted as the area of one-half of a circle of radius 3, so the value of this integral is $9\pi$. The second integral is zero, since the integrand is an odd function and the interval of integration is symmetric about zero. Hence,

$$F = 8\rho g \frac{9\pi}{2} - \frac{8}{3}\rho g(0) = 8 \cdot 900 \cdot 9.8 \cdot \frac{9\pi}{2} \approx 997,518 \text{ N.}$$

(b) The oil in the lower half of the tank is at a greater depth than the oil in the upper half, therefore we expect the total force $F_l$ on the lower half of the tank to be greater than the total force $F_u$ on the upper half. We compute the two forces to verify our expectation. Now,

$$F_l = \rho g \int_{-3}^{0} (3 - y) \frac{8}{3}\sqrt{9 - y^2} dy = 8\rho g \int_{-3}^{0} \sqrt{9 - y^2} dy - \frac{8}{3}\rho g \int_{-3}^{0} y \sqrt{9 - y^2} dy.$$ 

Similarly,

$$F_u = 8\rho g \int_{0}^{3} \sqrt{9 - y^2} dy - \frac{8}{3}\rho g \int_{0}^{3} y \sqrt{9 - y^2} dy.$$ 

The first integral in each expression,

$$\int_{-3}^{0} \sqrt{9 - y^2} dy \quad \text{and} \quad \int_{0}^{3} \sqrt{9 - y^2} dy,$$

can be interpreted as the area of one-quarter of a circle of radius 3, so both integrals have the value $\frac{9\pi}{4}$. Using the substitution $u = 9 - y^2$, $du = -2y dy$ we find

$$\int_{-3}^{0} y \sqrt{9 - y^2} dy = \int_{0}^{9} \sqrt{u}(-\frac{1}{2}) du = \frac{1}{3}u^{3/2} \bigg|_{0}^{9} = -9.$$ 

Moreover, since the integrand is an odd function, we have

$$\int_{0}^{3} y \sqrt{9 - y^2} dy = -\int_{-3}^{0} y \sqrt{9 - y^2} dy = 9.$$ 

Thus,

$$F_l = 8\rho g \frac{9\pi}{4} - \frac{8}{3}\rho g(-9) = (18\pi + 24)\rho g; \quad \text{and} \quad F_u = 8\rho g \frac{9\pi}{4} - \frac{8}{3}\rho g(9) = (18\pi - 24)\rho g.$$ 

We see that $F_l > F_u$. That is, the total force on the lower half of the tank is greater than the total force on the upper half, as expected.

17. Calculate the moments and COM of the lamina occupying the region under $y = x(4 - x)$ for $0 \leq x \leq 4$, assuming a density of $\rho = 1200 \text{ kg/m}^3$.

**SOLUTION** Because the lamina is symmetric with respect to the vertical line $x = 2$, by the symmetry principle, we know that $x_{cm} = 2$. Now,

$$M_x = \rho \int_{0}^{4} f(x)^2 \, dx = \frac{1200}{2} \int_{0}^{4} x^2(4 - x)^2 \, dx = \frac{1200}{2} \left( \frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5 \right) \bigg|_{0}^{4} = 20,480.$$ 

Moreover, the mass of the lamina is

$$M = \rho \int_{0}^{4} f(x) \, dx = 1200 \int_{0}^{4} x(4 - x) \, dx = 1200 \left( 2x^2 - \frac{1}{3}x^3 \right) \bigg|_{0}^{4} = 12,800.$$
Thus, the coordinates of the center of mass are
\[ \left( \frac{2 \cdot 20,480}{12,800}, \frac{8}{5} \right) = \left( 2, \frac{8}{5} \right). \]

18. Sketch the region between \( y = 4(x + 1)^{-1} \) and \( y = 1 \) for \( 0 \leq x \leq 3 \), and find its centroid.

**SOLUTION**

\[
\begin{align*}
M_x &= \frac{1}{2} \int_0^3 \left( \frac{16}{(x + 1)^2} - 1 \right) dx = \frac{1}{2} \left[ \frac{16}{x + 1} - x \right]_0^3 = \frac{9}{2}, \\
M_y &= \int_0^3 x \left( 4(x + 1)^{-1} - 1 \right) dx = \int_0^3 \left( \frac{4x}{x + 1} - x \right) dx \\
&= \int_0^3 \left( \frac{4x - 4}{x + 1} - x \right) dx = \int_0^3 \left( \frac{4 - 4x}{x + 1} - x \right) dx \\
&= \left( 4x - \frac{x^2}{2} - 4 \ln(x + 1) \right)_0^3 = \frac{15}{2} - 4 \ln 4.
\end{align*}
\]

The area of the region is
\[ A = \int_0^3 \left( \frac{4}{x + 1} - 1 \right) dx = (4 \ln(x + 1) - x)_0^3 = 4 \ln 4 - 3, \]
so the coordinates of the centroid are:
\[ \left( \frac{15 - 8 \ln 4}{8 \ln 4 - 6}, \frac{9 - 6 \ln 4}{8 \ln 4 - 6} \right). \]

19. Find the centroid of the region between the semicircle \( y = \sqrt{1 - x^2} \) and the top half of the ellipse \( y = \frac{1}{2} \sqrt{1 - x^2} \) (Figure 2).

**SOLUTION** Since the region is symmetric with respect to the y-axis, the centroid lies on the y-axis. To find \( y_{cm} \) we calculate
\[
M_x = \frac{1}{2} \int_{-1}^{1} \left[ \left( \sqrt{1 - x^2} \right) - \left( \frac{\sqrt{1 - x^2}}{2} \right) \right]^2 dx = \frac{1}{2} \int_{-1}^{1} \frac{3}{4} (1 - x^2) dx = \frac{3}{8} \left( x - \frac{1}{3} x^3 \right)_{-1}^{1} = \frac{1}{2}.
\]
The area of the lamina is \( \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4} \), so the coordinates of the centroid are
\[ \left( 0, \frac{1/2}{\pi/4} \right) = \left( 0, \frac{2}{\pi} \right). \]

20. Find the centroid of the shaded region in Figure 5 bounded on the left by \( x = 2y^2 - 2 \) and on the right by a semicircle of radius 1. **Hint:** Use symmetry and additivity of moments.
SOLUTION  The region is symmetric with respect to the $x$-axis, hence the centroid lies on the $x$-axis; that is, $y_{cm} = 0$. To compute the area and the moment with respect to the $y$-axis, we treat the left side and the right side of the region separately. Starting with the left side, we find

$$M_y^{\text{left}} = 2 \int_{-2}^{0} x \sqrt{x^2 + 1} \, dx \quad \text{and} \quad A^{\text{left}} = 2 \int_{-2}^{0} \sqrt{x^2 + 1} \, dx.$$ 

In each integral we make the substitution $u = \frac{x^2}{2} + 1$, $du = \frac{1}{2} \, dx$, and find

$$M_y^{\text{left}} = 8 \int_{1}^{0} (u - 1)u^{1/2} \, du = 8 \int_{0}^{1} \left( u^{3/2} - u^{1/2} \right) \, du = 8 \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right)|_{0}^{1} = -\frac{32}{15}$$

and

$$A^{\text{left}} = 4 \int_{0}^{1} u^{1/2} \, du = \frac{8}{3} u^{3/2}|_{0}^{1} = \frac{8}{3}.$$ 

On the right side of the region

$$M_y^{\text{right}} = 2 \int_{0}^{1} x \sqrt{1 - x^2} \, dx = -\frac{2}{3} (1 - x^2)^{3/2}|_{0}^{1} = \frac{2}{3},$$

and $A^{\text{right}} = \frac{\pi}{2}$ (because the right side of the region is one-half of a circle of radius 1). Thus,

$$M_y = M_y^{\text{left}} + M_y^{\text{right}} = -\frac{32}{15} + \frac{2}{3} = -\frac{22}{15},$$

$$A = A^{\text{left}} + A^{\text{right}} = \frac{8}{3} + \frac{\pi}{2} = \frac{16 + 3\pi}{6};$$

and the coordinates of the centroid are

$$\left( \frac{-22/15}{(16 + 3\pi)/6}, 0 \right) = \left( \frac{-44}{80 + 15\pi}, 0 \right).$$

21. Use the Theorem of Pappus to find the volume of the solid of revolution obtained by rotating the region in the first quadrant bounded by $y = x^2$ and $y = \sqrt{x}$ about the $y$-axis.

SOLUTION  The moments of the region are

$$M_x = \frac{1}{2} \int_{0}^{1} (x - x^4) \, dx = \frac{3}{20} \quad \text{and} \quad M_y = \int_{0}^{1} x (\sqrt{x} - x^2) \, dx = \frac{3}{20};$$

the area of the region is

$$A = \int_{0}^{1} (\sqrt{x} - x^2) \, dx = \frac{1}{3}.$$

Thus, the coordinates of the centroid are

$$\left( \frac{9}{20}, \frac{9}{20} \right).$$

When the region in the first quadrant bounded by $y = x^2$ and $y = \sqrt{x}$ is rotated about the $y$-axis, the centroid travels a distance of

$$2\pi \left( \frac{9}{20} \right) = \frac{9\pi}{10}.$$
Therefore, by the Theorem of Pappus, the volume of the solid is
\[ V = \frac{1}{5} \cdot \frac{9\pi}{10} = \frac{3\pi}{10}. \]

22. Use the Theorem of Pappus to find a formula for the volume of the solid obtained by rotating the triangle with vertices \((1, 0), (3, 0),\) and \((2, 2)\) about the \(y\)-axis.

**Solution** The triangle has a base of length 2 and a height of 2, so its area is
\[ A = \frac{1}{2}(2)(2) = 2. \]

By symmetry, the centroid of the triangle has an \(x\)-coordinate of 2, so when the triangle is rotated about the \(y\)-axis, the centroid travels a distance of \(2\pi(2) = 4\pi.\) Therefore, by the Theorem of Pappus, the volume of the solid is
\[ V = 2 \cdot 4\pi = 8\pi. \]

In Exercises 23–28, find the Taylor polynomial at \(x = a\) for the given function.

23. \(f(x) = x^3, \quad T_3, \quad a = 1\)

**Solution** We start by computing the first three derivatives of \(f(x) = x^3:\)
\[ f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6. \]

Evaluating the function and its derivatives at \(x = 1,\) we find
\[ f(1) = 1, \quad f'(1) = 3, \quad f''(1) = 6, \quad f'''(1) = 6. \]

Therefore,
\[ T_3(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 \]
\[ = 1 + 3(x - 1) + \frac{6}{2}(x - 1)^2 + \frac{6}{3!}(x - 1)^3 \]
\[ = 1 + 3(x - 1) + 3(x - 1)^2 + 6(x - 1)^3. \]

24. \(f(x) = 3(x + 2)^3 - 5(x + 2), \quad T_3, \quad a = -2\)

**Solution** \(T_3(x)\) is the Taylor polynomial of \(f\) consisting of powers of \((x + 2)\) up to three. Since \(f(x)\) is already in this form we conclude that \(T_3(x) = f(x).\)

25. \(f(x) = x \ln(x), \quad T_4, \quad a = 1\)

**Solution** We start by computing the first four derivatives of \(f(x) = x \ln x:\)
\[ f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1 \]
\[ f''(x) = \frac{1}{x} \]
\[ f'''(x) = -\frac{1}{x^2} \]
\[ f^{(4)}(x) = \frac{2}{x^3} \]

Evaluating the function and its derivatives at \(x = 1,\) we find
\[ f(1) = 0, \quad f'(1) = 1, \quad f''(1) = 1, \quad f'''(1) = -1, \quad f^{(4)}(1) = 2. \]

Therefore,
\[ T_4(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 \]
\[ = 0 + 1(x - 1) + \frac{1}{2!}(x - 1)^2 - \frac{1}{3!}(x - 1)^3 + \frac{2}{4!}(x - 1)^4 \]
\[ = (x - 1) + \frac{1}{2}(x - 1)^2 - \frac{1}{6}(x - 1)^3 + \frac{1}{12}(x - 1)^4. \]
26. \( f(x) = (3x + 2)^{1/3}, \quad T_3, \quad a = 2 \)

**Solution** We start by computing the first three derivatives of \( f(x) = (3x + 2)^{1/3} \):

\[
\begin{align*}
    f'(x) & = \frac{1}{3} (3x + 2)^{-2/3} \cdot 3 = (3x + 2)^{-2/3} \\
    f''(x) & = -\frac{2}{3} (3x + 2)^{-5/3} \cdot 3 = -2(3x + 2)^{-5/3} \\
    f'''(x) & = \frac{10}{9} (3x + 2)^{-8/3} \cdot 3 = 10(3x + 2)^{-8/3}
\end{align*}
\]

Evaluating the function and its derivatives at \( x = 2 \), we find

\[
\begin{align*}
    f(2) & = 2, \quad f'(2) = \frac{1}{4}, \quad f''(2) = -\frac{1}{16}, \quad f'''(2) = \frac{5}{128}
\end{align*}
\]

Therefore,

\[
T_3(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!} (x - 2)^2 + \frac{f'''(2)}{3!} (x - 2)^3 \\
= 2 + \frac{1}{4}(x - 2) - \frac{1}{32}(x - 2)^2 - \frac{5}{768}(x - 2)^3.
\]

27. \( f(x) = xe^{-x^2}, \quad T_4, \quad a = 0 \)

**Solution** We start by computing the first four derivatives of \( f(x) = xe^{-x^2} \):

\[
\begin{align*}
    f'(x) & = e^{-x^2} + x \cdot (-2x)e^{-x^2} = (1 - 2x^2)e^{-x^2} \\
    f''(x) & = -4xe^{-x^2} + (1 - 2x^2) \cdot (-2x)e^{-x^2} = (4x^3 - 6x)e^{-x^2} \\
    f'''(x) & = (12x^2 - 6)e^{-x^2} + (4x^3 - 6x) \cdot (-2x)e^{-x^2} = (-8x^4 + 24x^2 - 6)e^{-x^2} \\
    f^{(4)}(x) & = (-32x^3 + 48x)e^{-x^2} + (-8x^4 + 24x^2 - 6) \cdot (-2x)e^{-x^2} = (16x^5 - 80x^3 + 60x)e^{-x^2}
\end{align*}
\]

Evaluating the function and its derivatives at \( x = 0 \), we find

\[
\begin{align*}
    f(0) & = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -6, \quad f^{(4)}(0) = 0.
\end{align*}
\]

Therefore,

\[
T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 \\
= 0 + x + 0 \cdot x^2 - \frac{6}{3!} x^3 + 0 \cdot x^4 = x - x^3.
\]

28. \( f(x) = \ln(\cos x), \quad T_3, \quad a = 0 \)

**Solution** We start by computing the first three derivatives of \( f(x) = \ln(\cos x) \):

\[
\begin{align*}
    f'(x) & = \frac{-\sin x}{\cos x} = -\tan x \\
    f''(x) & = -\sec^2 x \\
    f'''(x) & = -2 \sec^2 x \tan x
\end{align*}
\]

Evaluating the function and its derivatives at \( x = 0 \), we find

\[
\begin{align*}
    f(0) & = 0, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0.
\end{align*}
\]

Therefore,

\[
T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 = 0 + 0 \cdot x - \frac{1}{2!} x^2 + 0 \cdot \frac{0}{3!} x^3 = -\frac{x^2}{2}.
\]
29. Find the $n$th Maclaurin polynomial for $f(x) = e^{3x}$.

**Solution** We differentiate the function $f(x) = e^{3x}$ repeatedly, looking for a pattern:

- $f'(x) = 3e^{3x}$
- $f''(x) = 3 \cdot 3e^{3x} = 9e^{3x}$
- $f'''(x) = 3 \cdot 3^2e^{3x} = 27e^{3x}$
- \[ f^{(n)}(x) = 3^n e^{3x} \]

Thus, for general $n$, $f^{(n)}(x) = 3^n e^{3x}$ and $f^{(1)}(0) = 1$. Substituting into the formula for the $n$th Taylor polynomial, we obtain:

\[
T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

\[
= 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \cdots + \frac{1}{n!}(3x)^n.
\]

30. Use the fifth Maclaurin polynomial of $f(x) = e^x$ to approximate $\sqrt{e}$. Use a calculator to determine the error.

**Solution** Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all $n$. Hence,

\[
T_5(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!}
\]

\[
= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.
\]

For $x = \frac{1}{2}$ we have

\[
T_5\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \frac{\left(\frac{1}{2}\right)^5}{5!}
\]

\[
= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 1.648697917
\]

Using a calculator, we find that $\sqrt{e} = 1.648721271$. The error in the Taylor polynomial approximation is

\[
|1.648697917 - 1.648721271| = 2.335 \times 10^{-5}.
\]

31. Use the third Taylor polynomial of $f(x) = \tan^{-1}x$ at $a = 1$ to approximate $f(1.1)$. Use a calculator to determine the error.

**Solution** We start by computing the first three derivatives of $f(x) = \tan^{-1}x$:

- $f'(x) = \frac{1}{1+x^2}$
- $f''(x) = -\frac{2x}{(1+x^2)^2}$
- $f'''(x) = \frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{2(3x^2-1)}{(1+x^2)^3}$

Evaluating the function and its derivatives at $x = 1$, we find

- $f(1) = \frac{\pi}{4}$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2}$, $f'''(1) = \frac{1}{2}$.

Therefore,

\[
T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3
\]

\[
= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3.
\]

Setting $x = 1.1$ yields:

\[
T_3(1.1) = \frac{\pi}{4} + \frac{1}{2}(0.1) - \frac{1}{4}(0.1)^2 + \frac{1}{12}(0.1)^3 = 0.832981496.
\]
Using a calculator, we find \( \tan^{-1}1.1 = 0.832981266 \). The error in the Taylor polynomial approximation is

\[
|T_3(1.1) - \tan^{-1}1.1| = |0.832981496 - 0.832981266| = 2.301 \times 10^{-7}.
\]

32. Let \( T_4 \) be the Taylor polynomial for \( f(x) = \sqrt{x} \) at \( a = 16 \). Use the error bound to find the maximum possible size of \( |f(17) - T_4(17)| \).

**SOLUTION** Using the Error Bound, we have

\[
|f(17) - T_4(17)| \leq K \frac{(17 - 16)^5}{5!} = \frac{K}{5!},
\]

where \( K \) is a number such that \( |f^{(5)}(x)| \leq K \) for all \( 16 \leq x \leq 17 \). Starting from \( f(x) = \sqrt{x} \) we find

\[
f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}, \quad f''''(x) = -\frac{15}{16}x^{-7/2}, \quad f'''''(x) = \frac{105}{32}x^{-9/2}.
\]

For \( 16 \leq x \leq 17 \),

\[
|f'''''(x)| \leq 105 \frac{1}{32} x^{-9/2}.
\]

Therefore, we may take

\[
K = \frac{105}{8388608}.
\]

Finally,

\[
|f(17) - T_4(17)| \leq \frac{105}{8388608} \cdot \frac{1}{5!} \approx 1.044 \cdot 10^{-7}.
\]

33. Find \( n \) such that \( |e - T_n(1)| < 10^{-8} \), where \( T_n \) is the \( n \)th Maclaurin polynomial for \( f(x) = e^x \).

**SOLUTION** Using the Error Bound, we have

\[
|e - T_n(1)| \leq K \frac{|1 - 0|^{n+1}}{(n+1)!} = \frac{K}{(n+1)!},
\]

where \( K \) is a number such that \( |f^{(n+1)}(x)| = e^x \leq K \) for all \( 0 \leq x \leq 1 \). Since \( e^x \) is increasing, the maximum value on the interval \( 0 \leq x \leq 1 \) is attained at the endpoint \( x = 1 \). Thus, for \( 0 \leq u \leq 1 \), \( e^u \leq e^1 < 2.8 \). Hence we may take \( K = 2.8 \) to obtain:

\[
|e - T_n(1)| \leq \frac{2.8}{(n+1)!}.
\]

We now choose \( n \) such that

\[
\frac{2.8}{(n+1)!} < 10^{-8}, \quad \frac{(n+1)!}{2.8} > 10^8, \quad (n+1)! > 2.8 \times 10^8
\]

For \( n = 10 \), \( (n+1)! = 3.99 \times 10^7 < 2.8 \times 10^8 \) and for \( n = 11 \), \( (n+1)! = 4.79 \times 10^8 > 2.8 \times 10^8 \). Hence, to make the error less than \( 10^{-8} \), \( n = 11 \) is sufficient; that is,

\[
|e - T_{11}(1)| < 10^{-8}.
\]

34. Let \( T_4 \) be the Taylor polynomial for \( f(x) = x \ln x \) at \( a = 1 \) computed in Exercise 25. Use the error bound to find a bound for \( |f(1.2) - T_4(1.2)| \).

**SOLUTION** Using the Error Bound, we have

\[
|f(1.2) - T_4(1.2)| \leq K \frac{(1.2 - 1)^5}{5!} = \frac{(0.2)^5}{120} K.
\]
where $K$ is a number such that $|f^{(5)}(x)| \leq K$ for all $1 \leq x \leq 1.2$. Starting from $f(x) = x \ln x$, we find
\[
f'(x) = \ln x + x \frac{1}{x} = \ln x + 1, \quad f''(x) = \frac{1}{x}, \quad f'''(x) = -\frac{1}{x^2}, \quad f^{(4)}(x) = -\frac{2}{x^3}, \quad f^{(5)}(x) = -\frac{6}{x^4}.
\]
and
\[
|f^{(5)}(x)| = \frac{6}{x^4} \leq 6 \frac{1}{14^4} = 6.
\]
Hence we may take $K = 6$ to obtain:
\[
|f(1.2) - T_n(1.2)| \leq \frac{(0.2)^5}{120} \times 6 = 1.6 \times 10^{-5}.
\]

35. Verify that $T_n(x) = 1 + x + x^2 + \cdots + x^n$ is the $n$th Maclaurin polynomial of $f(x) = 1/(1 - x)$. Show using substitution that the $n$th Maclaurin polynomial for $f(x) = 1/(1 - x/4)$ is
\[
T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n.
\]
What is the $n$th Maclaurin polynomial for $g(x) = \frac{1}{1 + x}$?

**Solution** Let $f(x) = (1 - x)^{-1}$. Then, $f'(x) = (1 - x)^{-2}$, $f''(x) = 2(1 - x)^{-3}$, $f'''(x) = 3!(1 - x)^{-4}$, and, in general, $f^{(n)}(x) = n!/(1 - x)^{-(n+1)}$. Therefore, $f^{(n)}(0) = n!$ and
\[
T_n(x) = 1 + \frac{1}{1!}x + \frac{2!}{2!}x^2 + \cdots + \frac{n!}{n!}x^n = 1 + x^2 + \cdots + x^n.
\]
Upon substituting $x/4$ for $x$, we find that the $n$th Maclaurin polynomial for $f(x) = \frac{1}{1 - x/4}$ is
\[
T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n.
\]
Substituting $-x$ for $x$, the $n$th Maclaurin polynomial for $g(x) = \frac{1}{1 + x}$ is
\[
T_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n.
\]

36. Let $f(x) = \frac{5}{4 + 3x - x^2}$ and let $a_k$ be the coefficient of $x^k$ in the Maclaurin polynomial $T_n$ for $k \leq n$.

(a) Show that $f(x) = \left( \frac{1/4}{1 - x/4} + \frac{1}{1 + x} \right)$.

(b) Use Exercise 35 to show that $a_k = \frac{1}{4^k+1} + (-1)^k$.

(c) Compute $T_3$.

**Solution**

(a) Start by factoring the denominator and writing the form of the partial fraction decomposition:
\[
f(x) = \frac{5}{4 + 3x - x^2} = \frac{5}{(x + 1)(4 - x)} = \frac{A}{x + 1} + \frac{B}{4 - x}.
\]
Multiplying through by $(x + 1)(4 - x)$, we obtain:
\[
5 = A(4 - x) + B(x + 1).
\]
Substituting $x = 4$ yields $5 = A(0) + B(5)$, so $B = 1$; substituting $x = -1$ yields $5 = A(5) + B(0)$, so $A = 1$. Thus,
\[
f(x) = \frac{1}{x + 1} + \frac{1}{4 - x} = \frac{1}{x + 1} + \frac{1}{4 - x}.
\]
(b) The \( n \)th Maclaurin polynomial for \( f(x) = \frac{1}{1 + x} \) is the sum of the \( n \)th Maclaurin polynomials for the functions \( g(x) = \frac{1}{q} \cdot \frac{1}{1 - q} \) and \( h(x) = \frac{1}{1 + x} \). In Exercise 35, we found that the \( n \)th Maclaurin polynomials \( P_n(x) \) and \( Q_n(x) \) for \( g \) and \( h \) are

\[
P_n(x) = \frac{1}{4} \left( 1 + \frac{1}{4} x + \frac{1}{4^2} x^2 + \cdots + \frac{1}{4^n} x^n \right) = \frac{1}{4} + \frac{1}{4^2} x + \frac{1}{4^3} x^2 + \cdots + \frac{1}{4^n} x^n = \sum_{k=0}^{n} \frac{x^k}{4^k+1}
\]

\[
Q_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k
\]

Therefore,

\[
T_n(x) = P_n(x) + Q_n(x) = \sum_{k=0}^{n} \frac{x^k}{4^k+1} + \sum_{k=0}^{n} (-1)^k x^k = \sum_{k=0}^{n} \left[ \frac{1}{4^k+1} + (-1)^k \right] x^k;
\]

that is, the coefficient of \( x^k \) in \( T_n \) for \( k \leq n \) is

\[
a_k = \frac{1}{4^k+1} + (-1)^k.
\]

(c) From part (b),

\[
a_0 = \frac{1}{4} + 1, \quad a_1 = \frac{1}{4^2} - 1, \quad a_2 = \frac{1}{4^3} + 1, \quad a_3 = \frac{1}{4^4} - 1
\]

so that

\[
T_3(x) = \frac{5}{4} - \frac{15}{16} x + \frac{65}{64} x^2 - \frac{255}{256} x^3
\]

37. Let \( T_n \) be the \( n \)th Maclaurin polynomial for the function \( f(x) = \sin x + \sinh x \).

(a) Show that \( T_5(x) = T_6(x) = T_7(x) = T_8(x) \).

(b) Show that \( |f^{(n)}(x)| \leq 1 + \cosh x \) for all \( n \). \textit{Hint:} Note that \( |\sin x| \leq |\cosh x| \) for all \( x \).

(c) Show that \( |T_n(x) - f(x)| \leq \frac{2.6}{9^n} |x|^9 \) for \(-1 \leq x \leq 1 \).

\textbf{SOLUTION}

(a) Let \( f(x) = \sin x + \sinh x \). Then

\[
f'(x) = \cos x + \cosh x
\]

\[
f''(x) = - \sin x + \sinh x
\]

\[
f'''(x) = - \cos x + \cosh x
\]

\[
f^{(4)}(x) = \sin x + \sinh x
\]

From this point onward, the pattern of derivatives repeats indefinitely. Thus

\[
f(0) = f^{(4)}(0) = f^{(8)}(0) = 0
\]

\[
f'(0) = f^{(5)}(0) = \sin 0 + \sinh 0 = 0
\]

\[
f''(0) = f^{(6)}(0) = - \sin 0 + \sinh 0 = 0
\]

\[
f'''(0) = f^{(7)}(0) = - \cos 0 + \cosh 0 = 0
\]

Consequently,

\[
T_5(x) = f'(0)x + \frac{f^{(5)}(0)}{5!} x^5 = 2x + \frac{1}{60} x^5,
\]

and, because \( f^{(6)}(0) = f^{(7)}(0) = f^{(8)}(0) = 0 \), it follows that

\[
T_6(x) = T_7(x) = T_8(x) = 2x + \frac{1}{60} x^5
\]

(b) First note that \( |\sin x| \leq 1 \) and \( |\cos x| \leq 1 \) for all \( x \). Moreover,

\[
|\sinh x| = \left| \frac{e^x - e^{-x}}{2} \right| \leq \frac{e^x + e^{-x}}{2} = \cosh x.
\]
Now, recall from part (a), that all derivatives of \( f(x) \) contain two terms: the first is \( \pm \sin x \) or \( \pm \cos x \), while the second is either \( \sinh x \) or \( \cosh x \). In absolute value, the trigonometric term is always less than or equal to 1, while the hyperbolic term is always less than or equal to \( \cosh x \). Thus, for all \( n \),

\[
|f^{(n)}(x)| \leq 1 + \cosh x.
\]

(c) Using the Error Bound, we have

\[
|T_8(x) - f(x)| \leq K |x - 0|^{9} \frac{|x|^{9}}{9!} = K |x|^{9} \frac{|x|^{9}}{9!},
\]

where \( K \) is a number such that \( |f^{(9)}(u)| \leq K \) for all \( u \) between 0 and \( x \). By part (b), we know that

\[
f^{(9)}(u) \leq 1 + \cosh u.
\]

Now, \( \cosh u \) is an even function that is increasing on \((0, \infty)\). The maximum value for \( u \) between 0 and \( x \) is therefore \( \cosh x \). Moreover, for \(-1 \leq x \leq 1\), \( \cosh x \leq \cosh 1 \approx 1.543 < 1.6 \). Hence, we may take \( K = 1 + 1.6 = 2.6 \), and

\[
|T_8(x) - f(x)| \leq 2.6 |x|^{9} \frac{|x|^{9}}{9!}.
\]