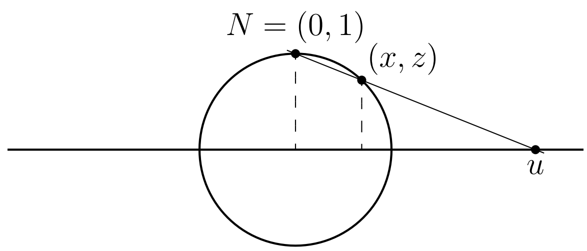
College of the Holy Cross, Spring 2013
Math 392 (Hyperbolic Geometry)
Stereographic Projection

Stereographic Projection in the Plane

Consider the unit circle S^1 defined by $x^2 + z^2 = 1$ in the (x, z)-plane. The point $N = (0, 1)$ is the *north pole* and its antipode $S = (0, -1)$ is the *south pole*. Each non-horizontal line through N intersects the circle in exactly one point $(x, z) \neq N$, and intersects the x-axis in exactly one point, u. We may therefore view the circle S^1 as the number line with a single point added "at infinity".



Definition 1. The mapping $\pi_N : S^1 \setminus \{N\} \to \mathbf{R}$ defined by $\pi_N(x, z) = u$ is called *stereographic projection* (away from N). The inverse map $\pi_N^{-1} : \mathbf{R} \to S^1 \setminus \{N\}$ is called the *rational parameterization of the circle.*

Theorem 2. *For all (x, z) with $z < 1$ on the circle, and for all real u, we have*

$$\pi_N(x, z) = \frac{x}{1 - z}, \qquad \pi_N^{-1}(u) = \left(\frac{2u}{u^2 + 1}, \frac{u^2 - 1}{u^2 + 1}\right).$$

Proof. By similar triangles (dashed lines in the figure), $u/1 = (u - x)/z$. Rearranging to solve for u, we have $uz = u - x$, or $x = u(1 - z)$, or $u = x/(1 - z)$.

For the inverse map, since $x^2 + z^2 = 1$, or $x^2 = 1 - z^2 = (1 - z)(1 + z)$, substitution gives $u^2(1 - z)^2 = (1 - z)(1 + z)$. As long as $z < 1$ (i.e., $(x, z) \neq N$), we have $u^2(1 - z) = 1 + z$, or $u^2 - 1 = z(u^2 + 1)$.

Solving for $x = u(1 - z)$ gives $(x, z) = (2u, u^2 - 1)/(u^2 + 1)$. $\square$

Exercise 1. Show that u is rational if and only if x and z are rational. Conclude that there are infinitely many points on the circle having rational coordinates, and hence infinitely many non-similar right triangles with all sides of rational length.

Exercise 2. Show that if $u \neq 0$, then $\pi_N^{-1}(1/u) = (x, -z)$. That is, the inversion mapping $u \mapsto 1/u$ on the number line corresponds to reflection of the circle across the horizontal axis.

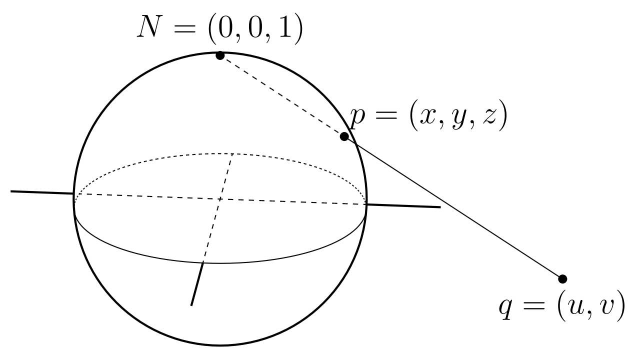
Exercise 3. Let $\phi(x, z) = (-z, x)$ be the counterclockwise quarter-turn of the circle. Calculate the composition $f = \pi_N \circ \phi \circ \pi_N^{-1}$, and check that f cyclically permutes the four points -1, 0, 1, ∞.

Calculate the compositions $f \circ f$, $f \circ f \circ f$, and $f \circ f \circ f \circ f$. (Hint: The last can be found by inspection in two different ways.)

Exercise 4. Let π_S denote stereographic projection from the south pole. Find formulas for $\pi_S(x, z)$, for the inverse $\pi_S^{-1}(u)$, and for the compositions $\pi_S \circ \pi_N^{-1}$ and $\pi_N^{-1} \circ \pi_S$.

Stereographic Projection in Space

Consider the unit sphere S^2 defined by $x^2 + y^2 + z^2 = 1$ in the Cartesian (x, y, z)-space. The point $N = (0, 0, 1)$ is the *north pole* and its antipode $S = (0, 0, -1)$ is the *south pole*. Each non-horizontal line through the north pole intersects the sphere in a unique point $(x, y, z) \neq N$ and intersects the (x, y)-plane in exactly one point, (u, v). We may therefore view the sphere S^2 as the plane with a single point added "at infinity".



Definition 3. The map $\pi_N : S^2 \setminus \{N\} \to \mathbf{R}^2$ defined by $\pi_N(x, y, z) = (u, v)$ is called *stereographic projection* (away from N).

Definition 4. A mapping f from a surface S_1 to a surface S_2 is said to be *conformal* or *angle-preserving* if, whenever two smooth curves γ_1 and γ_2 meet at a point p in S_1 making an angle θ, the image curves $f \circ \gamma_1$ and $f \circ \gamma_2$ meet at $q = f(p)$ in S_2 making an angle θ.

Example 5. Rigid motions and translations are conformal. If S_1 and S_2 are spheres centered at the origin, then the dilatation f mapping S_1 to S_2 is conformal.

Theorem 6. *Stereographic projection $\pi_N : S^2 \setminus \{N\} \to \mathbf{R}^2$ is conformal.*

Proof. Let $p \neq N$ be a point of the sphere, γ_1 and γ_2 smooth curves meeting at p. Consider the plane P_1 in space containing p, N, and the tangent vector of γ_1 at p, and the plane P_2 containing p, N, and the tangent to γ_2 at p.

The plane P_i cuts the sphere in a circle C_i passing through both p and N. Further, C_i is tangent to γ_i at p, so the line $\pi_N(C_i)$ is tangent to $\pi_N \circ \gamma_i$ at $q = \pi_N(p)$; and the tangents to the image curves lie in P_i since π_N is effected by projecting along lines through N.

The angles between the following curves are equal for the indicated reason:

γ_1, γ_2 at p; and C_1, C_2 at p	(curves are respectively tangent);
C_1, C_2 at N	(symmetry);
$f(C_1)$ and $f(C_2)$ at q	(P_1, P_2 cut by parallel planes);
$f \circ \gamma_1$ and $f \circ \gamma_2$ at q	(curves are respectively tangent). $\square$

Exercise 5. Find formulas for π_N and the inverse mapping π_N^{-1}.
Suggestion: In the plane containing the origin, N, and (x, y, z), the mapping π_N is "essentially" projection from the circle to the real numbers. Cylindrical coordinates may be useful.

Give a formula for π_S, stereographic projection from the south pole, and for the composition $\pi_S \circ \pi_N^{-1}$.