College of the Holy Cross, Spring Semester, 2021
Math 241 (Professor Hwang)
Worksheet 5, Due May 7

Let $\mathcal{D}$ be a plane region (such as a disk or rectangle), and let $\mathbf{X} : \mathcal{D} \to S$ be a regular parametrization of a surface, i.e., a continuously-differentiable map for which $\mathbf{X}_S \times \mathbf{X}_t$ is never 0. If f is a continuous, real-valued function on S, we define the *scalar surface integral*

$$\int_S f\, dS = \int_{\mathcal{D}} f\big(\mathbf{X}(s,t)\big) \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt.$$

Particularly, if $f \equiv 1$, the scalar surface integral gives the *area* of S; if $\delta(x,y,z)$ represents the density of material at $\mathbf{X}(s,t)$, the scalar surface integral of δ represents the *mass* of S; if r represents the distance from some line ℓ, then the integral of $r^2\,\delta$ is the *moment of inertia* of S about ℓ; and so forth.

Throughout, $0 < a < R$ are real numbers. If you can do an integral by geometry, symmetry, or algebra, feel free to do so. Watch for and take opportunities to re-use the results of earlier parts!

Exercise 1. Let S be the sphere of radius R centered at the origin. Evaluate the following.

(a) $\displaystyle\int_S R^2\, dS = \int_S (x^2 + y^2 + z^2)\, dS$

(b) $\displaystyle\int_S x^2\, dS = \int_S y^2\, dS = \int_S z^2\, dS$

(c) $\displaystyle\int_S x\, dS = \int_S y\, dS = \int_S z\, dS$

(d) Calculate the moment of inertia of S about an axis, assuming constant density; express your answer in terms of mass.

Exercise 2. Let S be the hemisphere $\{z \geq 0\}$ of radius R parametrized by

$$\mathbf{X}(s,t) = (R\cos s \sin t, R \sin s \sin t, R\cos t), \qquad 0 \leq s \leq 2\pi, \quad 0 \leq t \leq \pi/2.$$

Calculate the following, assuming constant density.

(a) The centroid of S.

(b) The moments of inertia of S about each coordinate axis, in terms of the mass of S.

Exercise 3. The surface T parametrized by

$$\mathbf{X}(s,t) = \big((R + a\cos s)\cos t, (R + a\cos s)\sin t, a \sin s\big), \qquad 0 \leq s \leq 2\pi, \quad 0 \leq t \leq 2\pi,$$

is called a *torus* of *major radius* R and *minor radius* a. Calculate the following:

(a) The area of T.

(b) $\displaystyle\int_T x^2\, dS = \int_T y^2\, dS$ and $\int_T z^2\, dS$

(c) The moment of inertia of T about each coordinate axis (in terms of the mass, assuming constant density).