College of the Holy Cross, Spring Semester, 2021
Math 241 (Professor Hwang)
Worksheet 4, Due April 21

Work in pairs or groups of three, and to turn in one write-up per group.

Let Ω be a solid object whose density at (x, y, z) is $\delta(x, y, z)$, and write $dV = dx\, dy\, dz$ for the volume element in Cartesian coordinates. We define the *volume* V, *mass* m, and *center of mass* $\bar{\mathbf{x}} = (\bar{x}, \bar{y}, \bar{z})$ of Ω to be

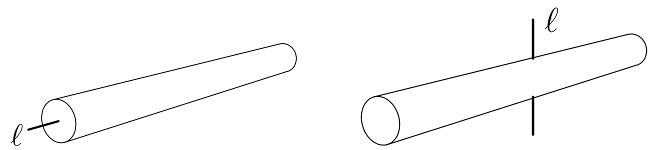
$$V = \iiint_\Omega dV, \qquad m = \iiint_\Omega \delta(x, y, z)\, dV, \qquad \bar{x} = \frac{1}{m} \iiint_\Omega x\delta(x, y, z)\, dV, \quad \text{etc.}$$

Volume and mass have familiar physical meanings. The center of mass is the object's "balance point", and for many purposes an object can be treated as if all its mass is concentrated at $\bar{\mathbf{x}}$. For example, if a stone is tossed into the air (and we neglect air resistance), then $\bar{\mathbf{x}}$ travels in a parabolic arc, even if the stone itself tumbles in a complicated way.

Let ℓ be a line, and $r(x, y, z)$ the distance to ℓ. The *moment of inertia* of Ω about ℓ is

$$I = \iiint_\Omega r(x, y, z)^2 \delta(x, y, z)\, dV.$$

The moment of inertia measures the body's "resistance" to rotation about ℓ. The smaller I is, the more easily the body turns about ℓ. A piece of chalk has small moment about its own axis (left), but relatively large moment about an axis ℓ perpendicular to its own axis (right).



In this worksheet, you'll calculate some standard centers of mass and moments of inertia by setting up and solving integrals in cylindrical and spherical coordinates. Write each integral "conceptually" (as above), then express the integral with respect to an appropriate coordinate system, and finally evaluate the integral.

Example Let Ω be a right circular cone of height h and radius R. In cylindrical coordinates, Ω is defined by

$$0 \le \theta \le 2\pi, \quad 0 \le r \le R, \quad \tfrac{h}{R} r \le z \le h.$$

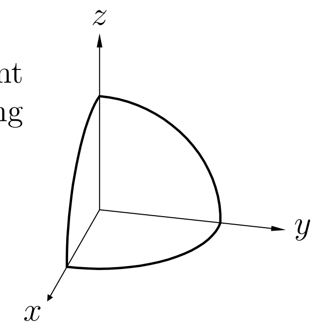
If the density of material is δz, the mass of Ω is

$$\int_0^{2\pi} \int_0^R \int_{hr/R}^h \delta z r\, dz\, dr\, d\theta = \cdots = \tfrac{1}{4}\pi\delta h^2 R^2,$$

and the moment of inertia about the z-axis is

$$\int_0^{2\pi} \int_0^R \int_{hr/R}^h \delta z r^3\, dz\, dr\, d\theta = \cdots = \tfrac{1}{12}\pi\delta h^2 R^4.$$

1. Let Ω be the portion of a solid ball of radius R in the first octant ($x \geq 0$, $y \geq 0$, and $z \geq 0$). Find the center of mass of Ω, assuming the density is



(a) Constant.

(b) Proportional to ρ^2.

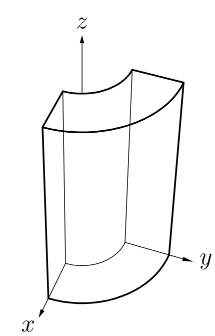
(c) Proportional to r, the distance to the z-axis.

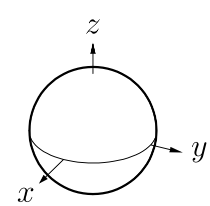
2. Let Ω be the solid defined by $1 \leq \sqrt{x^2+y^2} \leq 2$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 5$. Find the center of mass, assuming the density of Ω is

(a) Constant.

(b) Proportional to r.

(c) Proportional to r^2.





3. (a) Let Ω be a ball of radius R, centered at the origin and having constant density δ. Compute the moment of inertia of Ω about the z-axis. Express your answer in terms of both δ and the total mass $M = \delta \cdot V$.

(b) A solid ball and a solid cylinder have the same radius, mass, and uniform density. (Figure at right.) Which rolls more rapidly down an inclined plane?