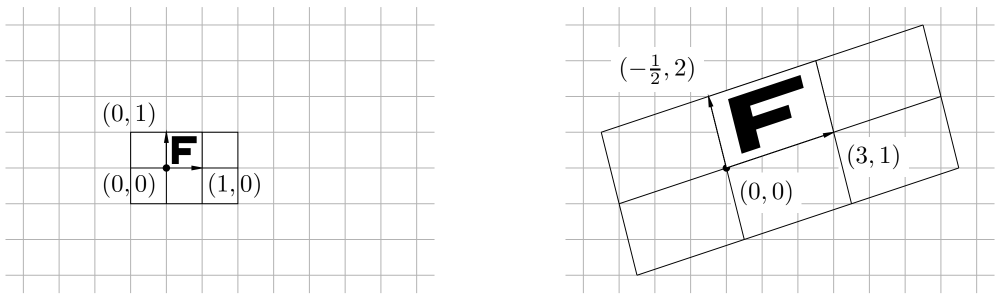
College of the Holy Cross, Spring Semester, 2021
Math 241 (Professor Hwang)
Worksheet 0, Due February 15

Work in groups of three of four; turn in only one write-up per group.

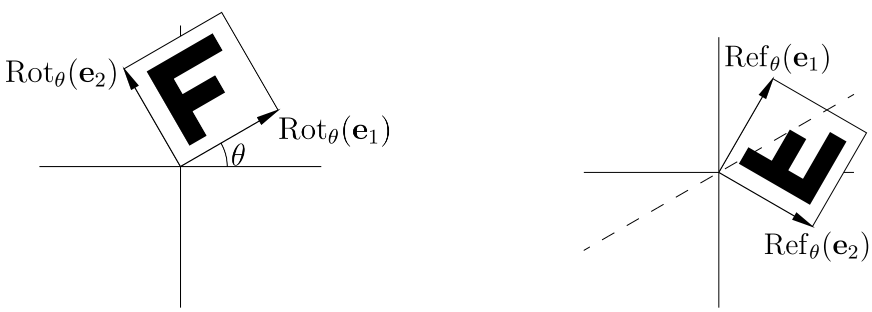
The act of multiplying by a matrix sends the standard basis vectors to the matrix columns. For example, the matrix $A = \begin{bmatrix} 3 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$, which induces $\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} 3x^1 - \frac{1}{2}x^2 \\ x^1 + 2x^2 \end{bmatrix}$, acts on the plane by sending the unit square to the parallelogram spanned by $(3, 1)$ and $(-\frac{1}{2}, 2)$:



Example Rotation about $(0, 0)$ by angle θ maps the standard basis vectors to

$$\mathrm{Rot}_\theta(\mathbf{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \mathrm{Rot}_\theta(\mathbf{e}_2) = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

so the transformation Rot_θ has matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.



Example Let θ be a real number. Reflection across the line through the origin making angle θ with the positive x-axis maps the standard basis vectors to

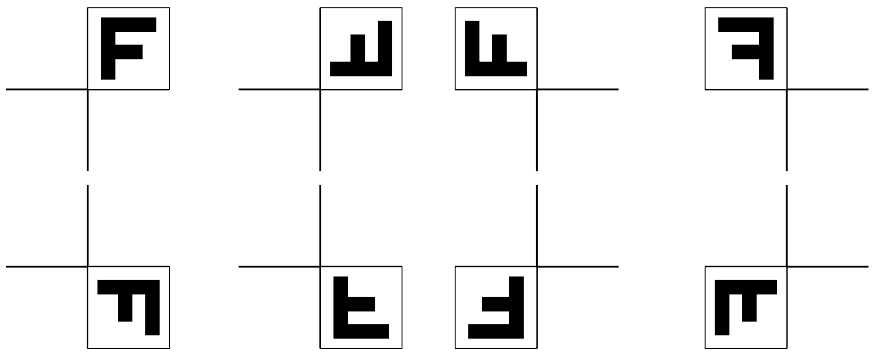
$$\mathrm{Ref}_\theta(\mathbf{e}_1) = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}, \qquad \mathrm{Ref}_\theta(\mathbf{e}_2) = \begin{bmatrix} \cos(2\theta - \frac{\pi}{2}) \\ \sin(2\theta - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix},$$

so the transformation Ref_θ has matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

Exercise 1. Eight matrices and eight transformations are given below. Match each matrix with the corresponding image of the unit F, and compute $T_A(\mathbf{x})$ for each A.

$$\text{(i)}\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\quad \text{(ii)}\ \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\quad \text{(iii)}\ \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\quad \text{(iv)}\ \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

$$\text{(v)}\ \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\quad \text{(vi)}\ \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\quad \text{(vii)}\ \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\quad \text{(viii)}\ \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$



Exercise 2. For each matrix A, compute $T_A(\mathbf{x}) = A\mathbf{x}$, and describe T_A geometrically.

$$S_c : \begin{bmatrix}c & 0\\ 0 & c\end{bmatrix},\quad J : \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad H : \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad V : \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

Exercise 3. Explain why $\mathrm{Rot}_\theta^2 = \mathrm{Rot}_{2\theta}$, and use this together with matrix multiplication to deduce the double-angle formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

Exercise 4. Use the geometric idea of the preceding exercise to deduce formulas for $\cos(\theta+\varphi)$ and $\sin(\theta + \varphi)$.