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A Beginner's Guide to Holomorphic Manifolds

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Preface

This book constitutes notes from a one-semester graduate course called "Complex Manifolds and Hermitian Differential Geometry" given during the Spring Term, 1997, at the University of Toronto. Its aim is not to give a thorough treatment of the algebraic and differential geometry of holomorphic manifolds, but to introduce material of current interest as quickly and concretely as possible with a minimum of prerequisites. There are several excellent references available for the reader who wishes to see subjects in more depth.

The coverage includes standard introductory analytic material on holomorphic manifolds, sheaf cohomology and deformation theory, differential geometry of vector bundles (Hodge theory, and Chern classes via curvature), and some applications to the topology and projective embeddability of Kählerian manifolds. The final chapter is a short survey of extremal Kähler metrics and related topics, emphasizing the geometric and "soft" analytic aspects. There is a large number of exercises, particularly for a book at this level. The exercises introduce several specific but colorful examples scattered through "folklore" and "the literature." Because there are recurrent themes and varying viewpoints in the subject, some of the exercises overlap considerably.

The course attendees were mostly advanced graduate students in mathematics, but it is hoped that these notes will reach a wider audience, including theoretical physicists. The "ideal" reader would be familiar with smooth manifolds (charts, forms, flows, Lie groups, vector bundles), differential geometry (metrics, connections, and curvature), and basic algebraic topology (simplicial and singular cohomology, the long exact sequence, and the fundamental group), but in reality the prerequisites are less strenuous, though a good reference for each subject should be kept at hand.

Apologies are perhaps in order for the departure from standard terminology, see Table 1. A manifold M is “complex” if TM is a complex vector bundle, and is “holomorphic” if TM is a holomorphic vector bundle. This is in accord with the informal usage of “complex/holomorphic category,” as well as the standard usage for vector bundles. Similar considerations apply to other types of structure: “complex” describes algebraic data, while “holomorphic” connotes integrability. In this usage, the real six-sphere S^6 is a complex manifold, but it is not known whether or not S^6 admits a holomorphic structure.

Standard Term	Replacement
Complex	Holomorphic
Almost-complex	Complex

TABLE 1. Non-standard terminology.

Acknowledgements

The influence of the book of Griffiths-Harris is great, and that of several other authors (especially Chern, Kobayashi, Kobayashi-Nomizu, Kodaira, and Wells) is substantial. I am particularly indebted to E. Hironaka for the lecture on Singularities of Algebraic Varieties. In addition, it is a pleasure to thank I. Graham for the opportunity to offer the original course, G. Valent for interest in and support of this project, and S. Kobayashi for mathematical and pedagogical inspiration. This project was supported by an NSERC Canada Individual Research Grant; I am also grateful to the LPTHE, Université de Paris VII, for generous financial support.

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Holomorphic Functions and Atlases

A function $f : D \to \mathbf{C}$ of one complex variable is (complex) differentiable in a domain D if the ordinary Newton quotient

$$f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for every point $z \in D$. For present purposes, there are two other useful characterizations of this condition. The first is to identify the complex line $\mathbf{C}$ with the real plane $\mathbf{R}^2$. The function f is complex differentiable if and only if the associated function $f : D \to \mathbf{R}^2$ has complex-linear derivative at every point, in which case f is said to be *holomorphic*. Concretely, there is a ring homorphism

$$a + bi \in \mathbf{C} \hookrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbf{R}^{2\times 2}, \tag{1.1}$$

so $f' = Df$ is complex-linear if and only if $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy the Cauchy-Riemann equations.

On the other hand, if f is holomorphic in a disk of radius $> r$ centered at z_0, then for all z with $|z - z_0| < r$, the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i} \int_{|w - z_0| = r} \frac{f(w)\, dw}{w - z}.$$

Writing $1/(w - z)$ as a geometric series in $z - z_0$ and integrating term-by-term shows that a holomorphic function may be expressed locally as a convergent power series. In words, a holomorphic function is complex-analytic. Intuitively, the averaging process effected by the contour integral makes the integrand smoother; if f is of class $\mathcal{C}^k$, then the expression on the right is of class $\mathcal{C}^{k+1}$. Since f (times a smooth function) is the integrand, f itself must be smooth. This is the prototypical bootstrap argument, and perhaps the most elementary example of "elliptic regularity."

1.1 Functions of Several Complex Variables

For functions of more than one variable, much of this philosophy carries over by the same reasoning. Let $D \subset \mathbf{C}^n$ be an open set. A function $f : D \to \mathbf{C}$ is holomorphic if the Cauchy-Riemann equations hold on D. More precisely, write $z^\alpha = x^\alpha + iy^\alpha$ and $f = u + iv$ with u and v real-valued. Then u and v may be regarded as functions on a subset of $\mathbf{R}^{2n}$, and f is *holomorphic* if f is of class $\mathcal{C}^1$ and

$$\frac{\partial u}{\partial x^\alpha} = \frac{\partial v}{\partial y^\alpha}, \qquad \frac{\partial u}{\partial y^\alpha} = -\frac{\partial v}{\partial x^\alpha} \tag{1.2}$$

at each point of D. Holomorphicity is related to "separate" holomorphicity (Osgood's Lemma, Proposition 1.1 below), that is, holomorphicity of the functions obtained by fixing $n-1$ of the variables and varying the remaining one; the hypotheses allow differentiation under the integral sign in the Cauchy integral formula. The continuity hypothesis may be dropped (Hartogs' Theorem), though the proof becomes more difficult.

Proposition 1.1 *Let $D \subset \mathbf{C}^n$ be a non-empty open set. If $f : D \to \mathbf{C}$ is continuous and separately holomorphic, then f is holomorphic.*

Let $r = (r_1, \ldots, r_n)$ be a *radius*, that is, an n-tuple of positive real numbers, and let $z_0 = (z_0^1, \ldots, z_0^n) \in \mathbf{C}^n$. If r and r' are radii, then $r' < r$ is taken to mean $r'_\alpha < r_\alpha$ for $\alpha = 1, \ldots, n$. The *polydisk* of radius r centered at z_0 is, by definition,

$$\begin{aligned} \Delta_r(z_0) &= \{z \in \mathbf{C}^n : |z^\alpha - z_0^\alpha| < r_\alpha \quad \text{for } \alpha = 1, \ldots, n\} \\ &= \{z \in \mathbf{C}^n : |z - z_0| < r\}. \end{aligned}$$

Thus a polydisk is exactly a Cartesian product of ordinary disks. While polydisks are not generally domains of convergence for power series of several variables, they are often the most convenient sets to use for local purposes.

Let D be a non-empty open set in $\mathbf{C}^n$. A function $f : D \to \mathbf{C}$ is *complex analytic* if, for every $z_0 \in D$, there is a complex power series centered at z_0 that converges and is equal to f on some polydisk $\Delta_r(z_0)$. Multi-indices simplify notation substantially; if $I = (i_1, \dots, i_n)$ is a multi-index, then set $|I| = i_1 + \cdots + i_n$, $z^I = (z^1)^{i_1} \cdots (z^n)^{i_n}$, and

$$f_I = \frac{\partial f}{\partial z^I} = \frac{\partial^k f}{(\partial z^1)^{i_1} \cdots (\partial z^n)^{i_n}}.$$

Analyticity means there is a polydisk $\Delta_r(z_0)$ such that

$$f(z) = \sum_{k=0}^{\infty} \sum_{|I|=k} \frac{1}{k!} f_I(z_0)(z - z_0)^I \tag{1.3}$$

for all $z \in \Delta_r(z_0)$. As in the case of one variable, holomorphicity and analyticity are equivalent, as is seen by using (an obvious generalization of) the Cauchy integral formula.

The concepts of holomorphicity and analyticity extend in the obvious way to functions with values in $\mathbf{C}^m$, which are usually called "holomorphic maps." A holomorphic map between open subsets of $\mathbf{C}^n$ that possesses a holomorphic inverse is a *biholomorphism*. Remarkably, a one-to-one holomorphic map between open subsets of $\mathbf{C}^n$ is a biholomorphism, see Theorem 5.3; this result has no analogue in the smooth category, even for real polynomial maps, as is shown by $x \mapsto x^3$. The set of biholomorphisms between open subsets of $\mathbf{C}^n$ is a pseudogroup: The composite of two biholomorphisms is a biholomorphism wherever it is defined. It is sometimes useful to consider *anti-holomorphic* maps. These are exactly complex conjugates of holomorphic maps. The set of anti-holomorphic maps is not a pseudogroup, since a composite of two anti-holomorphic maps is holomorphic.

Holomorphic functions of $n > 1$ variables satisfy versions of the identity theorem and maximum principle, see Propositions 1.2 and 1.3 below. However, there are substantial differences from the situation for functions of one variable. The zero set of a holomorphic function of n variables is *never* discrete (see Proposition 1.4 below). Moreover, the zero set must be "properly situated" in $\mathbf{C}^n$. For example, let Δ be a polydisk centered at the origin in $\mathbf{C}^2$. The function z^1 vanishes along the z^2-axis, which is real-linearly isomorphic to $\mathbf{R}^2$. On the other hand, if $f : \Delta \to \mathbf{C}$ vanishes on $\mathbf{R}^2 \cap \Delta$, then $f \equiv 0$ on Δ, as is verified by inspecting the coefficients in the series expansion of f. Intuitively, what matters is whether or not the *complex* span of the real tangent space to the zero set is all of $\mathbf{C}^n$. The statement of Proposition 1.2 is certainly not the strongest possible, but is adequate for present purposes.

Proposition 1.2 *Let $f : \Delta \to \mathbf{C}$ be a holomorphic function on a polydisk, and suppose $f|_U \equiv 0$ for some non-empty open set $U \subset \Delta$. Then $f \equiv 0$.*

Proof The zero set is closed, so it suffices to show the zero set is open. Let z_0 be a point of the closure $\overline{U}$, and let (z_n) be a sequence in U that converges to z_0. The power series coefficients of f (see equation (1.3) above) vanish at z_n for all n, and are continuous on Δ. Consequently, they vanish at z_0, so the function f vanishes identically on a *neighborhood* of z_0. $\Box$

Proposition 1.3 *Let Δ be a polydisk. If $f : \Delta \to \mathbf{C}$ is holomorphic and if $|f|$ has a local maximum at $p \in \Delta$, then f is constant on Δ.*

Proof The restriction of f to every line through p is locally constant by the one-variable maximum principle, so f is locally constant. Now apply Proposition 1.2 to the function $f - f(p)$. $\Box$

A remarkable (and essentially topological) extension result for holomorphic functions of $n > 1$ variables is *Hartogs' Phenomenon.* There is no analogue for holomorphic functions of one variable.

Proposition 1.4 *Let Δ be a polydisk in $\mathbf{C}^n$, $n \geq 2$, and let $K \subset \Delta$ be a compactly contained subset. If $f : \Delta \setminus K \to \mathbf{C}$ is holomorphic, then there exists a holomorphic function $\tilde{f} : \Delta \to \mathbf{C}$ which extends f.*

Proof (Sketch) It suffices to assume Δ is centered at the origin. Fix z^2, $\ldots$, z^n, then choose r_1 so that $z^1 < r_1$ whenever $z \in K$ (possible by the compactness assumption.) The integral

$$\tilde{f}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=r_1} \frac{f(\zeta, z^2, \ldots, z^n)}{\zeta - z^1}\, d\zeta$$

is well-defined, equal to f provided the "slice" misses K (which can happen by the compactness assumption, because $n \geq 2$), and is holomorphic in the sub-polydisk where $|z^1| < r_1$. Enlarging r_1 does not change the integral, so the previous equation defines an extension of f. $\Box$

Let $f : \Delta \to \mathbf{C}$ be holomorphic on a polydisk in $\mathbf{C}^n$, $n \geq 2$. Proposition 1.4 implies, in particular, that f cannot have an isolated singularity, nor can it have an isolated zero since then $1/f$ would have an isolated singularity. More generally, the zero set of f cannot cannot be compact, and cannot be contained in a set of complex codimension greater than one.

1.2 Holomorphic Manifolds

A "holomorphic manifold" is a smooth manifold, locally modelled on the complex Euclidean space $\mathbf{C}^n$ and whose transition functions are holomorphic. More precisely, a *holomorphic manifold* is a pair $(M, \mathfrak{J})$ consisting of a smooth, real manifold of real dimension $2n$ and a maximal atlas whose overlap maps lie in the pseudogroup of biholomorphic maps between open subsets of $\mathbf{C}^n$—briefly, a *holomorphic atlas.* There are various other ways of specifying the same data, discussed below.

Not every $2n$-dimensional manifold admits a holomorphic atlas, and a single smooth manifold may admit many "inequivalent" holomorphic atlases. Generally, determination of the set of holomorphic atlases up to equivalence on a particular smooth manifold is extremely difficult, even if the manifold is compact. The most famous open question along these lines concerns (non-)existence of a holomorphic atlas on the six-dimensional sphere, but there are other open questions of greater interest that are almost as easily stated. Further details are deferred until more tools and terminology are available.

A map between holomorphic manifolds is "holomorphic" if, with respect to arbitrary charts, the induced map is holomorphic. More precisely, $f : M \to M'$ is *holomorphic* at $p \in M$ if there exists a chart (φ, U) near p and a chart (ψ, V) near $f(p) \in M'$ such that $\psi \circ f \circ \varphi^{-1}$ is a holomorphic map between open subsets of complex Euclidean spaces. This condition is independent of the choice of charts because overlap maps are biholomorphic.

The maximum principle (Proposition 1.3) implies that *every holomorphic function on a connected, compact holomorphic manifold is constant*; the absolute value must have a maximum value by compactness, so the function is locally constant by the maximum principle, hence globally constant since the manifold is connected. If $i : M \hookrightarrow \mathbf{C}^N$ is a holomorphic map, then each coordinate function on $\mathbf{C}^N$ restricts to a global holomorphic function on the image. In particular, there is no holomorphic analogue of the Whitney embedding theorem; the only connected, compact holomorphic manifold that embeds holomorphically in $\mathbf{C}^N$ is a point.

A holomorphic manifold that embeds as a closed submanifold in a complex Euclidean space is called a *Stein manifold*. The study of Stein manifolds falls most naturally into the realm of several complex variables, though "affine varieties" are of interest in algebraic geometry as well.

There are three commonly considered equivalence relations, each of which is strictly weaker than the previous one. Let $M - 0$ and M be holomorphic manifolds. Then:

- M_0 and M are *biholomorphic* (or "equivalent," or "the same") if there exists a holomorphic map $f : M_0 \to M$ with holomorphic inverse. Assertions regarding uniqueness of holomorphic structure on a fixed manifold M are always meant *up to biholomorphism* unless otherwise specified. As noted above, a single smooth manifold may admit many non-equivalent holomorphic structures, and a topic of intense current research is the study of "moduli spaces" of holomorphic structures on fixed smooth manifolds.

- M_0 and M are *deformation equivalent* if there is a holomorphic family, parametrized by the unit disk $\Delta \subset \mathbf{C}$, that contains both M_0 and M; precisely, if there exists a holomorphic manifold X and a holomorphic submersion $\pi : X \to \Delta$, with $\pi^{-1}(0) = M_0$ and $\pi^{-1}(t) = M$ for some

$t \in \Delta$.

- M_0 and M are *diffeomorphic* if the underlying smooth manifolds are diffeomorphic and the induced orientations coincide. It is shown in Chapter 2 that a holomorphic manifold has a natural orientation.

Biholomorphic manifolds are obviously deformation equivalent; take $X = M \times \Delta$. It is not difficult to see that deformation equivalent manifolds are diffeomorphic, but the proof is deferred to the introduction to deformation theory. Examples below show that neither of these implications is reversible in general.

Examples

Example 1.5 Euclidean space $\mathbf{C}^n$ is a holomorphic manifold. More interesting examples are gotten by dividing by a lattice (i.e. a finitely generated discrete subgroup) $\Lambda \subset \mathbf{C}^n$. Since Λ acts on $\mathbf{C}^n$ by translation and this action is properly discontinuous and holomorphic, the quotient space $\mathbf{C}^n/\Lambda$ inherits the structure of a holomorphic manifold from the standard atlas on $\mathbf{C}^n$. If Λ is generated by an $\mathbf{R}$-basis of $\mathbf{C}^n$, then the quotient is a compact manifold, called a *compact complex n-torus*. Although all compact n-tori are diffeomorphic to a real $2n$-torus, their complex-analytic properties (such as the number of non-constant meromorphic functions, or whether they can be "projectively embedded") depend on arithmetic properties of the lattice.

Generally, if a *discrete* group Γ acts properly discontinuously by biholomorphisms on a manifold M, then the quotient M/Γ inherits a holomorphic structure from M. Another class of examples is the family of *Hopf manifolds*: Let $n > 1$, and let α be a complex number with $|\alpha| > 1$. Consider the action of $\Gamma \simeq \mathbf{Z}$ on $\mathbf{C}^n \setminus \mathbf{0}$ generated by the map $z \mapsto \alpha z$. The quotient is a compact holomorphic manifold diffeomorphic to $S^1 \times S^{2n-1}$. The complex analytic properties of general Hopf *surfaces* are investigated in Exercise 2.3. □

Example 1.6 Open subsets of $\mathbf{C}^n$ are of course holomorphic manifolds, and some of them are important or otherwise remarkable. An *algebraic torus* is a manifold biholomorphic to $(\mathbf{C}^\times)^n$ (cf. Examples 1.5 and 1.8; an algebraic torus must not be confused with an "Abelian variety"). An algebraic torus has the structure of a complex Lie group. Equivariant compactifications form the intensively-studied class of *toric* manifolds.

The general linear group $GL(n, \mathbf{C}) \subset \mathbf{C}^{n \times n}$ is a complex Lie group under matrix multiplication. This manifold has various closed complex subgroups, such as $SL(n, \mathbf{C})$ (matrices of unit determininant) and $O(n, \mathbf{C})$ (complex orthogonal matrices). Compact groups such as $U(n)$ and $SU(n)$ are not complex Lie groups, nor are they holomorphic submanifolds of $GL(n, \mathbf{C})$.

In fact, a compact, connected, complex Lie group is a compact torus. This is not trivial, though it is easy to see that such a group is Abelian: the adjoint representation must be trivial, since it may be regarded as a) map from a compact holomorphic manifold into a complex Euclidean space.

"Small" open sets in $\mathbf{C}^n$, $n \geq 2$, exhibit subtle analytic behaviour; slightly deforming the boundary of a convex domain in $\mathbf{C}^n$ gives an uncountably infinite-dimensional family of mutually non-biholomorphic structures on the ball, for example. □

Example 1.7 One of the most important compact n-manifolds is the *complex projective space* $\mathbf{P}^n$. Intuitively, a point of $\mathbf{P}^n$ is a line through the origin in $\mathbf{C}^{n+1}$. More precisely, the (non-discrete) group $\mathbf{C}^\times$ acts on $\mathbf{C}^{n+1}\backslash\mathbf{0}$ by scalar multiplication. If the orbit space is given the quotient topology, then the holomorphic structure of $\mathbf{C}^{n+1}$ descends. The equivalence class of a point $Z = (Z^0, \dots, Z^n) \in \mathbf{C}^{n+1} \setminus \mathbf{0}$ is denoted $[Z] = [Z^0 : \cdots : Z^n]$, and the Euclidean coordinates of Z constitute so-called *homogeneous coordinates* of $[Z]$. While Z^α is not a well-defined holomorphic function on $\mathbf{P}^n$, the equation $Z^\alpha = 0$ is unambiguous. Furthermore, every quotient Z^α/Z^β *is* well-defined, and holomorphic except where $Z^\beta = 0$. There is an atlas consisting of $n+1$ charts: For each $\alpha = 0, \dots, n$, let $U_\alpha = \{[Z] \in \mathbf{P}^n : Z^\alpha \neq 0\}$, and use local coordinates

$$z_\alpha^0 = \frac{Z^0}{Z^\alpha}, \dots, \widehat{z_\alpha^\alpha}, \dots, z_\alpha^n = \frac{Z^n}{Z^\alpha}.$$

On $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the overlap map—essentially multiplication by Z^β/Z^α—is a biholomorphism, so $\mathbf{P}^n$ admits a holomorphic structure. To see that $\mathbf{P}^n$ is compact, observe that the unit sphere in $\mathbf{C}^{n+1}$ is mapped onto $\mathbf{P}^n$ by the quotient map.

If V is a finite-dimensional complex vector space, then the *projectivization* of V, denoted $\mathbf{P}(V)$, is formed as above by removing the origin and dividing by the action of $\mathbf{C}^\times$. This construction, while less concrete than the construction of $\mathbf{P}^n = \mathbf{P}(\mathbf{C}^{n+1})$, captures functorial properties of V, and can be applied fibrewise in vector bundles.

Many concepts from linear algebra (linear subspaces and spans, intersections, and the language of points, lines, and planes) carry over in the obvious way to projective space; for example, the line $\overline{xy}$ determined by a pair of points in $\mathbf{P}^n$ is the image of the plane in $\mathbf{C}^{n+1}$ spanned by the lines representing x and y. Disjoint linear subspaces of $\mathbf{P}^n$ are said to be *skew*. For example, there exist pairs of skew lines in $\mathbf{P}^3$, while every line in $\mathbf{P}^3$ intersects every plane in $\mathbf{P}^3$ in at least one point. A pair of skew linear subspaces of $\mathbf{P}^n$ is *maximal* if the respective inverse images in $\mathbf{C}^{n+1}$ are of complementary dimension in the usual sense. The "prototypical" maximal skew pairs $\mathbf{P}^k \sqcup \mathbf{P}^{n-k} \subset \mathbf{P}^{n+1}$ are indexed by an integer $k = 0, \dots, n$, and are of the form

$$\{[X^0 : \cdots : X^k], [Y^0 : \cdots : Y^{n-k}]\} \longmapsto [X^0 : \cdots : X^k : Y^0 : \cdots : Y^{n-k}].$$

(Linear Maps of Projective Spaces) Every linear automorphism of $\mathbf{C}^{n+1}$ induces a biholomorphism from $\mathbf{P}^n$ to $\mathbf{P}^n$; Proposition 6.13 below asserts, conversely, that every automorphism of $\mathbf{P}^n$ is induced by a linear automorphism of $\mathbf{C}^{n+1}$. Other linear transformations on $\mathbf{C}^{n+1}$ descend to interesting holomorphic maps defined on subsets of $\mathbf{P}^n$. Simple but geometrically important examples are furnished by *projection maps.* Let $\{P_1, P_2\}$ be a maximal skew pair of linear subspaces. There is a holomorphic map $\pi : \mathbf{P}^n \setminus P_1 \to P_2$, called *projection away from* P_1 *onto* P_2, with the following geometric description. For each point $x \notin P_1$, the linear span $\overline{xP_1}$ intersects P_2 in a unique point $\pi(x)$, which is by definition the image of x under projection. There is an algebraic description, namely that such a projection is exactly induced by a linear projection (in the usual sense) on $\mathbf{C}^{n+1}$; see Exercise 1.1. □

Example 1.8 (Submanifolds of Projective Space) A closed holomorphic submanifold of $\mathbf{P}^n$ is called a *projective* manifold. There is an intrinsic necessary and sufficient criterion—given by the *Kodaira Embedding Theorem*—for a compact holomorphic manifold to be projective; see Theorem 10.10 below. Hopf manifolds do not satisfy this criterion, while compact tori are projective if and only if the lattice Λ satisfies (explicit) integrality conditions. A compact, projective torus is an *Abelian variety.*

A *projective algebraic variety* is the image in $\mathbf{P}^n$ of the common zero set of a finite set of *homogeneous* polynomials on $\mathbf{C}^{n+1}$. Every *smooth* projective algebraic variety is a compact holomorphic manifold. Remarkably (*Chow's Theorem*, 6.14 below), the converse is true: Every compact holomorphic submanifold of $\mathbf{P}^n$ is the zero locus of a finite set of homogeneous polynomials in the homogeneous coordinates.

Complex *hypersurfaces*—those cut out by a single equation—are among the best-understood projective varieties. As will be seen, the set of *smooth* projective hypersurfaces in $\mathbf{P}^n$ having fixed degree is a "family" in the sense of deformation theory; the idea is to parametrize the set of hypersurfaces by the coefficients of defining polynomials. The set of singular hypersurfaces has complex codimension at least 1, so its complement is connected. Thus, for example, two (smooth) quintic hypersurfaces in $\mathbf{P}^4$ are diffeomorphic.

The simplest non-linear projective manifold is a *complex hyperquadric*, cut out by a single irreducible quadratic in $z^0, \dots, z^n$. Every non-degenerate quadratic form is equivalent, after a linear change of coordinates, to the standard diagonal form

$$(z^0)^2 + (z^1)^2 + \cdots + (z^n)^2 = 0,$$

so up to equivalence there is only one smooth, non-degenerate hyperquadric, denoted Q_n. The orthogonal group $O(n, \mathbf{C})$ preserves the diagonal quadratic form, and the induced action on Q_n is transitive. These manifolds are of considerable interest, both classically and recently. The *conic curve* in $\mathbf{P}^2$ is abstractly isomorphic to the projective line $\mathbf{P}^1$; to see this, pick a point on the conic, a line not containing this point, and project from the point to the line. Every line in $\mathbf{P}^2$ intersects the conic exactly twice (counting multiplicity), so the projection is 1-1 (and holomorphic), hence a biholomorphism. This is nothing but classical stereographic projection, a.k.a. the rational parametrization of the conic curve. The conic surface in $\mathbf{P}^3$ is abstractly isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, as is readily seen from the representation $\{Z^0Z^3 - Z^1Z^2 = 0\}$, namely the image of the embedding

$$([X^0 : X^1], [Y^0 : Y^1]) \mapsto [X^0Y^0 : X^0Y^1 : X^1Y^0 : X^1Y^1] \in \mathbf{P}^3.$$

The hyperbolic paraboloid in $\mathbf{R}^3$ is the affine real part of the quadric surface, and its two families of rulings correspond to the factors in the product $\mathbf{P}^1 \times \mathbf{P}^1$. explained, smooth hypersurfaces of □

Example 1.9 A *Riemann surface* or *algebraic curve* is a one-dimensional holomorphic manifold. Apart from the *rational curve* $\mathbf{P}^1$, the simplest curves are *elliptic curves*, namely quotients of $\mathbf{C}$ by a lattice Λ; of rank two. The smooth manifold underlying an elliptic curve is the real 2-torus, but the holomorphic structure depends on Λ; as will be shown now, elliptic curves correspond exactly to lattices modulo complex scaling.

Suppose E_1 and E_2 are elliptic curves, and write $E_i = \mathbf{C}/\Lambda_i$. Pick bases $\{\omega_i^1,\ \omega_i^2\}$ for Λ_i. If there is a non-constant holomorphic map $f : E_1 \to E_2$, then f lifts to an entire function $\tilde{f}$ that satisfies

$$\begin{aligned} \tilde{f}(z + \omega_1^1) &= \tilde{f}(z) + m_1\omega_2^1 + n_1\omega_2^2, \\ \tilde{f}(z + \omega_1^2) &= \tilde{f}(z) + m_2\omega_2^1 + n_2\omega_2^2 \end{aligned}$$

for some integers m_i and n_i. The derivative $\tilde{f}' : \mathbf{C} \to \mathbf{C}$ is therefore doubly-periodic, hence constant by Liouville's Theorem. Consequently, $\tilde{f}$ is affine: There exist complex numbers $\alpha \neq 0$ and β such that $\tilde{f}(z) = \alpha z + \beta$. By translating if necessary, $\beta = 0$ without loss of generality. In short, every holomorphic map between elliptic curves is—up to translation—covered by a homothety $\tilde{f}$ of $\mathbf{C}$ that carries Λ_1 to Λ_2 (not surjectively in general). This homothety is an isomorphism of lattices exactly when f is a biholomorphism.

The set of lattices modulo scaling has a well-known description as a quotient space, namely the upper half plane $\mathfrak{H}$ divided by $SL(2, \mathbf{Z})$. To see this, fix a lattice Λ_0, and let ω_1 and ω_2 be generators; thus $\omega_2/\omega_1 = \tau$ is non-real, and without loss of generality has positive imaginary part. The lattice Λ obtained by dividing by ω_1 is generated by 1 and τ, and every basis of Λ is of the form

$$\{a + b\tau,\ c + d\tau\}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}).$$

The orbit space of $\mathfrak{H}$ under this action of $SL(2, \mathbf{Z})$ has the structure of a one-dimensional holomorphic manifold; this is clear except at the two points stabilized by non-trivial elements of $SL(2, \mathbf{Z})$—so-called "orbifold" points—where the local structure is that of a disk divided by the action of a finite cyclic group of rotations. However, a branch of $z \mapsto \sqrt{z-i}$ gives a holomorphic chart for the orbit space near the equivalence class of i, and a similar cube root uniformizes the quotient at the non-real cube root of 1. The orbits are in one-to-one correspondance with biholomorphism classes of elliptic curves, and the orbit space is the "moduli space" of elliptic curves.

To each lattice Λ of rank two is associated a *Weierstrass $\wp$-function*, defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{1.4}$$

The following facts are not difficult to establish. (See, for example, L. Ahlfors, *Complex Analysis*, pp. 272 ff.) Setting $G_k = \sum_{\omega \in \Lambda^\times} \omega^{-2k}$, the $\wp$-function satisfies the first-order differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - 60 G_2\, \wp(z) - 140 G_3. \tag{1.5}$$

Consequently, the elliptic curve $\mathbf{C}/\Lambda$ embeds as a cubic curve in $\mathbf{P}^2$ via the mapping

$$z \notin \Lambda \longmapsto [1 : \wp(z) : \wp'(z)], \qquad z \in \Lambda \longmapsto [0 : 0 : 1].$$

The discrete group $\Gamma = \mathbf{Z} \times \mathbf{Z}$ acts on $\mathfrak{H} \times \mathbf{C}$ by

$$\gamma_{m,n}(\tau, z) = (\tau, z + m + n\tau).$$

The quotient is a (non-compact) complex surface S equipped with a holomorphic projection map $\pi : S \to \mathfrak{H}$ whose fibres are elliptic curves. Indeed, the fibre over $\tau \in \mathfrak{H}$ is the elliptic curve associated to the lattice generated by 1 and τ. While distinct fibres are deformation equivalent, they are not necessarily biholomorphic.

An algebro-geometric version of this picture is easily constructed from the modular invariant $\lambda : \mathfrak{H} \to \mathbf{C} \setminus \{0, 1\}$; the surface S is thereby realized as the zero locus in $\mathbf{P}^2 \times \mathfrak{H}$ of a cubic polynomial whose coefficients are analytic functions of λ. □

Remark 1.10 *In the study of real manifolds, a basic tool is existence of smooth submanifolds passing through an arbitrary point, and having arbitrary tangent space. The "rigidity" of the holomorphic category makes this tool available only to a limited extent for holomorphic manifolds. If $\Delta \subset \mathbf{C}$ is the unit disk, M is a complex manifold, and $(p, v) \in TM$ is an arbitrary one-jet, then there need not exist a holomorphic map $f : \Delta \to M$ with $f(0) = p$ and $f'(0) = v$, though it is always possible to arrange that $f'(0) = \varepsilon v$ for $\varepsilon \ll 1$. It is usually difficult to determine whether or not a holomorphic manifold M' embeds holomorphically in another manifold M. Even if M' is one-dimensional, existence of an embedding depends in a* *global way on the holomorphic structure of M; the prototypical result is Liouville's theorem, which asserts that every bounded, entire function (a.k.a. holomorphic map $f : \mathbf{C} \to \Delta$) is constant. Existence of compact holomorphic curves in M has been an area of active interest since the mid-1980's, following the work of Mori in complex geometry and the work of Gromov and McDuff in symplectic geometry. It is also of interest to determine whether or not there exist embeddings of $\mathbf{C}$ into M; this is related to the study of "hyperbolic" complex manifolds and value distribution theory.*

Exercises

Exercise 1.1 A *projection* on $\mathbf{C}^{n+1}$ is a linear transformation Π with $\Pi^2 = \Pi$. Prove that every such linear transformation induces a holomorphic map—projection away from $P_1 = \mathbf{P}(\ker \Pi)$ onto $P_2 = \mathbf{P}(\operatorname{im} \Pi)$—as described in Example 1.7. In particular, if Π has rank $\ell + 1$ as a linear transformation, then after a linear change of coordinates projection away from $\mathbf{P}(\ker \Pi)$ has the form

$$[Z] = [Z^0 : \cdots : Z^n] \longmapsto [Z^0 : \cdots : Z^\ell : 0 : \cdots : 0],$$

and the image is $\mathbf{P}^\ell \subset \mathbf{P}^n$. ◇

Exercise 1.2 Let $p : \mathbf{C}^{n+1} \setminus \mathbf{0} \to \mathbf{P}^n$ be the natural projection. Prove that there is no holomorphic map $s : \mathbf{P}^n \to \mathbf{C}^{n+1} \setminus \mathbf{0}$ with $p \circ s = \text{identity}$. (In fact, there is no *continuous* map with this property, but the latter requires some algebraic topology.) ◇

Exercise 1.3 Give an example of a non-compact complex manifold M such that every holomorphic function on M is constant. ◇

Exercise 1.4 Let $\widetilde{f} : \Delta \to \mathbf{C}^{n+1}$ be a non-constant holomorphic map of the unit disk into $\mathbf{C}^{n+1}$ with $\widetilde{f}(0) = \mathbf{0}$. Prove that the induced map $f : \Delta^\times \to \mathbf{P}^n$ on the punctured unit disk extends to the origin. ◇

Exercise 1.5 Fix $\tau \in \mathfrak{H}$, and let $E = E_\tau$ be the associated elliptic curve. The curve E has *complex multiplication* if there is a holomorphic map $f : E \to E$ that is covered by a homothety $z \mapsto \alpha z$ with α non-real. Show that E has complex multiplication if and only if τ satisfies a quadratic equation with *integral* coefficients. Find all curves that admit an *automorphism* by complex multiplication. ◇

2

Complex Structures and Integrability

In order to apply the machinery of differential geometry and bundle theory to the study of holomorphic manifolds, it is useful to express holomorphic atlases in bundle-theoretic terms. The first task is to study "pointwise" objects, that is, to construct complex linear algebra from real linear algebra. These constructions are then applied fibrewise to tensor bundles over smooth manifolds equipped with some additional structure.

2.1 Complex Linear Algebra

Let V be an m-dimensional real vector space. A *complex structure* on V is an operator $J : V \to V$ with $J^2 = -I_V$. Complex scalar multiplication is defined in terms of J by $(a + b\sqrt{-1})v = av + b\,Jv$. The operator $-J$ is also a complex structure on V, called the *conjugate* structure, and the space $(V, -J)$ is often denoted $\overline{V}$ for brevity. The standard complex vector space is $V = \mathbf{C}^n$ with J induced by multiplication by $\sqrt{-1}$.

Lemma 2.1 *If V admits a complex structure, then V is even-dimensional and has an induced orientation.*

Proof Since J is a real isomorphism, taking the determinant of $J^2 = -I$ gives $0 < (\det J)^2 = (-1)^m$, from which it follows $m = 2n$ is even. If $\{\mathbf{e}_i\}_{i=1}^n$ is a complex basis of V (so that $\{\mathbf{e}_i, J\mathbf{e}_i\}_{i=1}^n$ is a real basis), then the sign of the volume element $\mathbf{e}_1 \wedge J\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \wedge J\mathbf{e}_n$ is independent of $\{\mathbf{e}_i\}_{i=1}^n$. (See also Remark 2.2 and the remarks following Proposition 2.5 below.) $\square$

The *complexification* of a real vector space V is $V_{\mathbf{C}} := V \otimes \mathbf{C}$, with $\mathbf{C}$ regarded as a real two-dimensional vector space. It is customary to write $v \otimes 1 = v$ and $v \otimes i = iv$. If V has an complex structure J, then J extends to $V_{\mathbf{C}}$ by $J(v \otimes \alpha) = Jv \otimes \alpha$, and $V_{\mathbf{C}}$ decomposes into $\pm i$-eigenspaces:

$$\begin{aligned} V^{1,0} &= \{Z \in V_{\mathbf{C}} : JZ = iZ\} = \{X - iJX : X \in V\}, \\ V^{0,1} &= \{Z \in V_{\mathbf{C}} : JZ = -iZ\} = \{X + iJX : X \in V\}. \end{aligned}$$

The complex vector space (V, J) is $\mathbf{C}$-linearly isomorphic to $(V^{1,0}, i)$ via

$$X = 2\mathrm{Re}\, Z \longmapsto \frac{1}{2}(X - iJX) = Z =: X^{1,0}. \tag{2.1}$$

Similarly, $\overline{V} = (V, -J)$ is $\mathbf{C}$-linearly isomorphic to $(V^{0,1}, -i)$. Complex conjugation induces a real-linear isomorphism of $V_{\mathbf{C}}$ that exchanges $V^{1,0}$ and $V^{0,1}$. The fixed point set is exactly the maximal totally real subspace $V = V \otimes 1$. The complexification of (V, J) may be *efined* to be $V \oplus \overline{V}$ together with the real structure that exchanges the two factors.

If V is equipped with a complex structure J, then the dual pairing induces a complex structure on V^*—also denoted by J—via

$$\langle Jv, \lambda\rangle = \langle v, J\lambda\rangle \qquad (\text{or } \lambda(Jv) = J\lambda(v)). \tag{2.2}$$

The associated eigenspace decomposition of $V_{\mathbf{C}}^* = V^* \otimes \mathbf{C}$ is

$$\begin{aligned} V_{1,0}^* &= \{\lambda \in V_{\mathbf{C}}^* : J\lambda = i\lambda\} = \{\xi + iJ\xi : \xi \in V^*\}, \\ V_{0,1}^* &= \{\lambda \in V_{\mathbf{C}}^* : J\lambda = -i\lambda\} = \{\xi - iJ\xi : \xi \in V^*\}. \end{aligned}$$

By equation (2.2), the space $V_{1,0}^*$ is the annihilator of $V^{0,1}$; similarly $V_{0,1}^*$ annihilates $V^{1,0}$.

Let $\{\mathbf{e}^i\}_{i=1}^n$ be a complex basis of V^*. The exterior algebra $\bigwedge V_{\mathbf{C}}^*$ (note the complexification) has a decomposition into tensors of *type* (p, q), namely those that contain "p of the $\mathbf{e}^i$'s and q of the $\bar{\mathbf{e}}^j$'s. A little more precisely, start with $\bigwedge^p V_{1,0}^* \otimes \bigwedge^q V_{0,1}^*$ and skew-symmetrize. The space of skew-symmetric (p, q)-tensors is denoted $\bigwedge^{p,q} V^*$ (note the lack of complexification). Thus

$$\textstyle\bigwedge^r V_{\mathbf{C}}^* = \bigoplus_{p+q=r} \bigwedge^{p,q} V^*, \qquad \bigwedge^{q,p} V^* = \overline{\bigwedge^{p,q} V^*}.$$

For dimensional reasons, $0 \le r \le 2n = m$, while $0 \le p,\ q \le n$.

Remark 2.2 *The "standard" complex vector space* $\mathbf{C}^n$ *has coordinates* $z = (z^1, \dots, z^n)$. *There are two useful (real-linear) isomorphisms with* $\mathbf{R}^{2n}$*; if* $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ *with* x^α *and* y^α *real, then* $z \in \mathbf{C}^n$ *is associated with either*

$$(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n) \qquad \text{or} \qquad (x^1, y^1, \dots, x^n, y^n) \tag{2.3}$$

in $\mathbf{R}^{2n}$*. With respect to the first set of real coordinates, a complex-linear transformation is represented by a* 2×2 *block matrix with* $n \times n$ *blocks (see the remarks following Proposition 2.5 below). With respect to the second set of real coordinates, a complex-linear transformation is represented by an* $n \times n$ *block matrix of* 2×2 *blocks. This representation is preferable when working with metrics and volume forms. In either case, the* 2×2 *blocks have the form (1.1).*

2.2 Complex Manifolds

A *complex manifold*[1] is a smooth manifold M equipped with a smooth endomorphism field $J : TM \to TM$ satisfying $J_x^2 = -I_x$ for all $x \in M$. The linear algebra introduced above may be applied pointwise to the tangent bundle of M. The complexified tangent bundle is $T_{\mathbf{C}}M = TM \otimes \mathbf{C}$, where $\mathbf{C}$ is regarded as a trivial vector bundle. The tensor field J splits the complexified tangent bundle into bundles of eigenspaces

$$T_{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

and the complex vector bundles (TM, J) and $(T^{1,0}M, i)$ are complex-linearly isomorphic. Complex conjugation induces an involution of $T_{\mathbf{C}}M$ that exchanges the bundles of eigenspaces. A (local) section Z of $T^{1,0}M$ is called a *vector field of type* $(1, 0)$, though Z is not a vector field on M in the sense of being tangent to a curve in M. If ordinary tangent vectors are regarded as real differential operators, then $(1, 0)$ vectors are complex-valued differential operators.

The splitting of the set of complex-valued skew-symmetric r-tensors into skew-symmetric (p, q)-tensors gives rise to spaces of (p, q)*-forms*. If A^r and $A^{p,q}$ denote the space of smooth r-forms and the space of smooth (p, q)-forms respectively, then

$$A^r = \bigoplus_{p+q=r} A^{p,q}.$$

In local coordinates, $A^{p,q}$ is generated by the forms $dz^{I_1} \wedge d\bar{z}^{I_2}$ with $|I_1| = p$ and $|I_2| = q$.

Example 2.3 Euclidean space $\mathbf{C}^n$ is a complex manifold. Explicitly, let $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ be the usual coordinates on $\mathbf{C}^n$, identified with coordinates (x, y) on $\mathbf{R}^{2n}$. The real tangent bundle and its complexification have the standard frames

$$\left\{ \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha} \right\},$$

[1] Recall that this terminology is not standard; see Table 1.

$$\left\{\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right), \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right)\right\},$$

while the real cotangent bundle and its complexification have coframes

$$\{dx^\alpha,\ dy^\alpha\}, \qquad \{dz^\alpha = dx^\alpha + \sqrt{-1}dy^\alpha, \quad d\bar{z}^\alpha = dx^\alpha - \sqrt{-1}dy^\alpha\}.$$

The complex structure J acts *only on tangent spaces*, not on the coordinates. Thus "Jz" is meaningless, while

$$J\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha}, \qquad J\frac{\partial}{\partial y^\alpha} = -\frac{\partial}{\partial x^\alpha}. \tag{2.4}$$

The tensor field J has constant components with respect to the usual coordinate system.

The exterior derivative $d : A^r \to A^{r+1}$ maps $A^{p,q}$ to $A^{p+1,q} \oplus A^{p,q+1}$, and the corresponding "pieces" are denoted ∂ and $\bar{\partial}$. On functions,

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} =: \partial f + \bar{\partial} f.$$

Observe that f is holomorphic if and only if $\bar{\partial} f = 0$. More generally, a *holomorphic p-form* is a $(p, 0)$-form η with $\bar{\partial}\eta = 0$. The fact that "holomorphic functions are $\bar{\partial}$-constant" has deep ramifications, as will become apparent in Chapter 4. □

Example 2.4 Even-dimensional spheres are perhaps the simplest candidates for examples of compact complex manifolds. It is not difficult to show that if S^{2n} admits a complex structure, then TS^{2n+1} is trivial. Deep results from algebraic topology imply that $n = 0$, 1, or 3. (If $\mathbf{R}^{2n+2}$ admits the structure of a normed division algebra, then TS^{2n+1} is trivial since the unit sphere S^{2n+1} is a Lie group under the algebra product. The classical theorem of Hurwitz to the effect that $n = 0$, 1, or 3 lends some circumstantial credence to the asserted restrictions on n. This relationship is not accidental, as explained below.)

Conversely, the spheres S^2 and S^6 admit complex structures. The 2-sphere is the complex projective line, so it already has a *holomorphic* structure, but there is an alternative description that illuminates the complex structure on S^6. It is currently unknown whether or not S^6 admits a holomorphic structure.

The "standard" complex structures on S^2 and S^6 are closely related to the quaternions and Cayley numbers respectively. Geometrically, the complex structure on S^2 rotates each tangent plane is through an angle $\pi/2$, in a fashion consistent with a choice of orientation. The following algebraic description considerably illuminates the corresponding construction for S^6. Consider the usual embedding $S^2 \subset \mathbf{R}^3$; a point $x \in S^2$ may be regarded as a unit vector in $\mathbf{R}^3$, and via the cross product induces a linear transformation $J_x(y) = x \times y$ on $\mathbf{R}^3$. If $y \perp x$ (i.e. if $y \in T_xS^2$), then $(x \times y) \perp x$ as well, so this linear transformation is an endomorphism J_x of the tangent space T_xS^2. It is easy to verify that $J_x^2 = -I$, so that this field of endomorphisms is a complex structure on S^2. Regarding $\mathbf{R}^3$ as the space of pure imaginary quaternions, the cross product arises as the imaginary part of quaternion multiplication.

Similarly, multiplication by a pure imaginary Cayley number defines a "cross product" on $\mathbf{R}^7$, and the analogous construction defines a complex structure on S^6. The set of complex structures on S^6 is very large, since small perturbations of a given structure give rise to another (usually non-equivalent) structure. By contrast, S^2 has only one complex structure up to equivalence. This is shown later, as it requires a bit of machinery. □

A map $f : (M, J) \to (M', J')$ between complex manifolds is *complex* or *pseudoholomorphic*/ if $(f_*)J = J'(f_*)$. The following is a straightforward consequence of the chain rule and the Cauchy-Riemann equations.

Proposition 2.5 *Let $\Delta \subset \mathbf{C}^n$ be a polydisk. A map $f : \Delta \to \mathbf{C}^m$ is complex if and only if f is holomorphic.*

There are several useful applications, of which two deserve immediate mention:

- A complex manifold has a natural orientation.

- A holomorphic manifold has a natural complex structure.

To prove the first assertion, suppose $m = n$ and f is complex. Using the isomorphism $z \in \mathbf{C}^n \longleftrightarrow (x, y) \in \mathbf{R}^{2n}$, and writing $f = u + iv$ with u and v real-valued, the Jacobian f_* has matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \sim_{\mathbf{C}} \begin{bmatrix} A + \sqrt{-1}B & 0 \\ 0 & A - \sqrt{-1}B \end{bmatrix},$$

with $A, B \in \mathbf{R}^{n \times n}$. This matrix has positive determinant, proving that a complex manifold is orientable. The *natural orientation* is by definition the orientation compatible with an ordered basis $\{\mathbf{e}_1, J\mathbf{e}_1, \dots, \mathbf{e}_n, J\mathbf{e}_n\}$, cf. the proof of Lemma 2.1.

To prove the second assertion, let M be a holomorphic manifold, and let (φ, U) be a chart near $x \in M$. The standard complex structure of $\mathbf{C}^n$ induces a complex structure on U, and by Proposition 2.5 this complex structure is well-defined.

Combining these observations, a mapping $f : M \to M'$ between holomorphic manifolds is holomorphic if and only if f is pseudoholomorphic with respect to the induced complex structures.

Complex Structures, Orientations, and Conjugate Atlases

When $(M, \mathfrak{J})$ is a holomorphic manifold, J is tacitly taken to be the induced complex structure and the orientation is taken to be that induced by J. In the absence of such a convention, there are potentially confusing relationships between complex structures, choices of orientation, and pseudoholomorphic maps, which the following remarks are intended to clarify. The distinction is not merely pedantic; increasingly in symplectic geometry and geometric field theory it is useful to fix an orientation on M, then seek compatible complex structures.

Let M be a real $2n$-dimensional smooth manifold admitting a qcomplex structure $J : TM \to TM$. There is an induced orientation as just observed; this may be realized concretely as a non-vanishing smooth $2n$-form on M. Let M_+ denote the corresponding *oriented* $2n$-manifold, and M_- denote the oppositely oriented manifold. There is also a *conjugate* complex structure $-J : TM \to TM$, and the pair $(M, -J)$ equipped with the induced orientation is denoted $\overline{M}$.

There are four choices of "orientation and complex structure" on the smooth manifold M, which will be denoted (see also Remark 2.6 below)

$$\begin{array}{rclcrcl} M & := & (M_+, J), & & -M & := & (M_-, J), \\ \overline{M}_+ & := & (M_+, -J), & & \overline{M}_- & := & (M_-, -J). \end{array}$$

The pair (M_+, J) is "compatible" in the sense that the orientation is induced by the complex structure. Since the orientation induced by $-J$ is $(-1)^n$ times the orientation induced by J, either the third or fourth pair is compatible—i.e. is equal to $\overline{M}$—depending on whether the complex dimension n is even or odd.

If $(M, \mathfrak{J})$ is a holomorphic manifold, then there is a holomorphic atlas $\overline{\mathfrak{J}}$ on the smooth manifold M whose charts are complex conjugates of the charts in $\mathfrak{J}$. More precisely, if (U, φ) is a chart in $\mathfrak{J}$, then $(U, \overline{\varphi})$ is, by fiat, a chart in $\overline{\mathfrak{J}}$. Observe that $\overline{\mathfrak{J}}$ is a holomorphic atlas because the *overlap maps* are holomorphic. Further, it is clear that if J is the complex structure induced by $\mathfrak{J}$, then $-J$ is induced by $\overline{\mathfrak{J}}$. Suppressing complex structures, if M is a holomorphic manifold, then $\overline{M}$ is a holomorphic manifold having the same underlying smooth manifold. The identity map is an antiholomorphic diffeomorphism from M to $\overline{M}$. The manifolds M and $\overline{M}$ may or may not be biholomorphic, see Exercises 2.4 and 2.5. An overkill application of Theorem 2.10 below gives an alternate, less constructive proof that $\overline{M}$ is a holomorphic manifold.

Remark 2.6 *Let P^n be the smooth, real $2n$-manifold underlying the complex projective space $\mathbf{P}^n$, and let J be the usual complex structure. By Exercise 2.4, the manifolds (P^n, J) and $(P^n, -J)$ are biholomorphic. Thus when n is odd both orientations on P^n are compatible with a holomorphic* *atlas. When n is even this need not be the case; there is no complex structure on* P^2 *inducing the "anti-standard" orientation. Unfortunately, it is nearly universal in the literature to write* $\overline{\mathbf{P}^2}$ *for the oriented manifold here denoted* $-\mathbf{P}^2$*. Forming the oriented connected sum of a four-manifold* N *with* $M = -\mathbf{P}^2$ *is the topological analogue of "blowing up" a point of* N*. This analytic operation is defined precisely in Exercise 3.5 when* $N = \mathbf{C}^2$*; other exercises in that section justify the topological claim just made.*

In general, nothing can be said about the oriented manifold M_-. There may or may not exist a compatible complex structure, to say nothing of a compatible holomorphic atlas. Further, there is no general relationship between the holomorphic manifolds M and $\overline{M}$; they may or may not be biholomorphic, regardless of the respective induced orientations. Exercise 2.5 investigates the case when M is an elliptic curve.

2.3 Integrability Conditions

Every holomorphic manifold comes equipped with an induced complex structure. Conversely, it is of interest to characterize complex structures that arise from a holomorphic atlas. It is to be expected that some differential condition on J is necessary, since J is an algebraic object while a holomorphic atlas contains analytic information.

On an arbitrary complex manifold, the exterior derivative has four type components, namely

$$d : A^{p,q} \longrightarrow A^{p-1,q+2} \oplus A^{p,q+1} \oplus A^{p+1,q} \oplus A^{p+2,q-1} \subset A^{p+q+1}.$$

This is easily seen from $dA^{1,0} \subset A^{2,0} \oplus A^{1,1} \oplus A^{0,2}$ and induction on the total degree. Under a suitable first-order differential condition, the "unexpected" components are equal to zero. To introduce this condition, first define the *Nijenhuis*[2] (or *torsion*) *tensor* N_J of J by

$$N_J(X, Y) = 2\big([JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]\big) \tag{2.5}$$

for local vector fields X and Y. The torsion tensor measures involutivity of the i-eigenspace bundle $T^{1,0}M$ in the following sense.

Lemma 2.7 *Let* X *and* Y *be local vector fields, and set*

$$Z = [X - iJX, Y - iJY].$$

Then $2(Z + iJZ) = -N_J(X, Y) - iJN_J(X, Y)$*.*

[2]Pronounced **Nī** yen haus.

In words, the torsion tensor is gotten by sending a pair of real vector fields to the corresponding $(1,0)$ fields and taking minus the real part of the $(0,1)$-component of their Lie bracket.

Theorem 2.8 *The following are equivalent:*

(a) $T^{1,0}M$ *is involutive, i.e., for all* $(1,0)$ *vector fields* Z *and* W*, the bracket* $[Z,W]$ *is of type* $(1,0)$*.*

(b) $T^{0,1}M$ *is involutive.*

(c) $dA^{1,0} \subset A^{2,0} \oplus A^{1,1}$ *and* $dA^{0,1} \subset A^{1,1} \oplus A^{0,2}$*.*

(d) $dA^{p,q} \subset A^{p+1,q} \oplus A^{p,q+1}$ *for all* p*,* $q \geq 0$*.*

(e) $N_J(X,Y) = 0$ *for all local vector fields* X *and* Y*.*

Proof (a) is equivalent to (b) by taking complex conjugates. To show (a) and (b) are equivalent to (c), recall that for an arbitrary one-form η,

$$2d\eta(Z,W) = Z\big(\eta(W)\big) - W\big(\eta(Z)\big) - \eta\big([Z,W]\big). \tag{2.6}$$

If (a) and (b) hold, and if $\eta \in A^{1,0}$, then the right side of (2.6) vanishes for all Z, W of type $(0,1)$, proving $d\eta$ has no component of type $(0,2)$, i.e. (c) holds.

Conversely, suppose (c) holds and let $\eta \in A^{0,1}$. For all Z, W of type $(1,0)$, (2.6) implies $\eta([Z,W]) = 0$. Thus $[Z,W]$ is of type $(1,0)$ and (a) holds.

(c) implies (d) by induction, while (d) implies (c) trivially. Finally, (a) and (e) are equivalent by Lemma 2.7. ∎

Consequently, if N_J vanishes identically then there is a decomposition $d = \partial + \bar\partial$ as in Example 2.3 above. Considering types and using $d^2 = 0$, it follows that

$$\partial^2 = 0, \qquad \partial\bar\partial = -\bar\partial\partial, \qquad \bar\partial^2 = 0. \tag{2.7}$$

Theorem 2.9 *Let* $f : (M,J) \to (M',J')$ *be a mapping of complex manifolds. The following are equivalent.*

(a) f_*Z *is of type* $(1,0)$ *for every local vector field* Z *of type* $(1,0)$*.*

(b) f_*Z *is of type* $(0,1)$ *for every local vector field* Z *of type* $(0,1)$*.*

(c) *Pullback by* f *preserves types, i.e.* $f^* : A^{p,q}(M') \to A^{p,q}(M)$*.*

(d) f *is pseudoholomorphic.*

Proof Since f_* commutes with conjugation, (a) and (b) are equivalent. Elements of $A^{p,q}$ are locally generated by elements of $A^{1,0}$ and $A^{0,1}$, so (a)/(b) are equivalent to (c). Finally, $f_*(X - iJX) = f_*X - if_*JX$ is of type $(1,0)$ if and only if $f_*JX = J'f_*X$, i.e. if and only if f is complex. ∎

Vanishing of the torsion tensor is a necessary condition for a complex structure J to be induced by a holomorphic atlas; since the components of J are constant in a holomorphic coordinate system by equation (2.4), the torsion of the induced complex structure vanishes identically. More interestingly, vanishing of the torsion (together with a mild regularity condition) is *sufficient* for a complex structure to be induced by a holomorphic atlas. When (M, J) is real-analytic, this amounts to the Frobenius theorem. When (M, J) satisfies less stringent regularity conditions, the theorem is a difficult result in partial differential equations, and is loosely known as the *Newlander-Nirenberg Theorem.* The weakest hypothesis is that (M, J) be of Hölder class $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$.

Theorem 2.10 *Let (M, J) be a real-analytic complex manifold, and assume N_J vanishes identically. Then there exists a holomorphic atlas on M whose induced complex structure coincides with J.*

Proof (Sketch) The idea is to "complexify" M. Then the Frobenius theorem may be applied.

Suppose $\psi : \mathbf{R}^{2n} \to \mathbf{R}$ is real-analytic. Then—locally—there is a holomorphic extension $\psi_{\mathbf{C}} : \Delta \subset \mathbf{C}^{2n} \to \mathbf{C}$ obtained by expressing ψ as a convergent power series and regarding the variables as complex numbers.

Complexifying the charts of M gives the following: For each $x \in M$, there exists a coordinate neighborhood U of x and a holomorphic manifold $U_{\mathbf{C}}$ isomorphic to $U \times B$, B a ball in $\mathbf{R}^{2n}$, whose charts extend the charts of M. The tangent bundle $TU_{\mathbf{C}}$, when restricted to U, is the complexification of TU, and in particular contains the involutive (by hypothesis) subbundle $T^{1,0}U$. By the Frobenius theorem, there is an integral manifold through x, and local holomorphic coordinates on this leaf induce holomorphic coordinates on U. ∎

Example 2.11 The complex structure on S^6 induced by Cayley multiplication has non-zero torsion tensor. It is not known at present whether or not S^6 admits a holomorphic atlas, though there is circumstantial and heuristic evidence against, and the answer is generally believed to be negative. □

The Newlander-Nirenberg theorem characterizes holomorphic atlases in terms of real data, namely a complex structure with vanishing torsion. It is desirable to have a similar description of "holomorphic vector fields" on a holomorphic manifold. A *holomorphic vector field* on a holomorphic manifold is a $(1, 0)$-vector field Z such that Zf is holomorphic for every local holomorphic function f. In local coordinates,

$$Z = \sum_{j=1}^{n} \xi^{\alpha} \frac{\partial}{\partial z^{\alpha}}$$

with ξ^{α} local holomorphic functions.

A holomorphic vector field is not, in the sense of real manifolds, a vector field on M. In spite of this it is often convenient to identify holomorphic vector fields and certain real vector fields; the next two propositions make this identification precise. The real counterpart of a holomorphic vector field is an *infinitesimal automorphism* of a complex structure J, namely, a vector field X for which the Lie derivative $L_X J$ vanishes.

Proposition 2.12 *A vector field X is an infinitesimal automorphism of J if and only if $[X, JY] = J[X, Y]$ for all Y. If X is an infinitesimal automorphism of J, then JX is an infinitesimal automorphism if and only if $N_J(X, Y) = 0$ for all Y.*

Proof By the Leibnitz rule, $[X, JY] = L_X(JY) = (L_X J)(Y) + J[X, Y]$ for all Y, proving the first assertion. To prove the second assertion, note that if X is an infinitesimal automorphism of J, then $N(X, Y) = 2\big([JX, JY] - J[JX, Y]\big)$, □

Proposition 2.13 *Let M be a holomorphic manifold with induced complex structure J. Then X is an infinitesimal automorphism of J if and only if $Z = X^{1,0} := (1/2)(X - iJX)$ is a holomorphic vector field. Furthermore, the map $X \longmapsto X^{1,0}$ is an isomorphism of complex Lie algebras.*

Exercises

Exercise 2.1 Consider the following descriptions of the complex projective line $\mathbf{P}^1$:

- The unit sphere $\{(u, v, w) \in \mathbf{R}^3 \mid u^2+v^2+w^2 = 1\}$, which is identified with the "Riemann sphere" $\mathbf{C} \cup \infty$ by stereographic projection.

- Two copies of the complex line $\mathbf{C}$ suitably glued together. More precisely, let z^0 and z^1 be complex coordinates in the two copies of $\mathbf{C}$, and identify z^0 with $1/z^1$. In this picture, the origin in each copy of $\mathbf{C}$ is the point at infinity in the other copy.

- The set of non-zero pairs of complex numbers $(Z^0, Z^1) \in \mathbf{C}^2 \setminus (0, 0)$, with the equivalence relation $(Z^0, Z^1) \sim (W^0, W^1)$ if and only if $Z^0 W^1 = Z^1 W^0$, i.e. the points (Z^0, Z^1) and (W^0, W^1) lie on the same complex line through $(0, 0)$. In other words, a *point* of $\mathbf{P}^1$ is a *line* through the origin in $\mathbf{C}^2$.

Show that these three descriptions are equivalent by using the identification $z^0 = Z^0/Z^1$. (A sketch of the real points may be helpful.) Describe the space of holomorphic vector fields on $\mathbf{P}^1$ in terms of each presentation; determine, in particular, which vector fields correspond to rotations of the sphere. Describe the space of holomorphic 1-forms on $\mathbf{P}^1$ and the space of meromorphic 1-forms dual to holomorphic vector fields. ◇

Exercise 2.2 For a holomorphic manifold N, denote by $H^0(N, TN)$ the space of holomorphic vector fields on N. Let M_i, $i = 1, 2$ be *compact* holomorphic manifolds, $M = M_1 \times M_2$ their product. Prove that

$$H^0(M, TM) \simeq H^0(M_1, TM_1) \oplus H^0(M_2, TM_2).$$

Roughly, every vector field on M is uniquely the sum of a vector field on M_1 and a vector field on M_2. Give examples to show that if compactness is dropped then the result may or may not hold. ◇

Exercise 2.3 Fix a complex number τ and a complex number α with $|\alpha| > 1$. The linear transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} \alpha z_1 + \tau z_2 \\ \alpha z_2 \end{bmatrix} = \begin{bmatrix} \alpha & \tau \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

generates a properly discontinuous action of $\mathbf{Z}$ on $\mathbf{C}^2 \setminus (0, 0)$ whose quotient $X_\alpha(\tau)$ is a complex surface, called a *Hopf surface*. Show that the Hopf surface $X_\alpha(\tau)$ is diffeomorphic to $S^1 \times S^3$ for every (α, τ). Find necessary and sufficient conditions under which $X_\alpha(\tau)$ and $X_{\alpha'}(\tau')$ are biholomorphic. Calculate the dimension $d(\alpha, \tau)$ of the space of holomorphic vector fields on $X_\alpha(\tau)$. Show that d is upper semicontinuous, i.e. $\{(\alpha, \tau) \mid d(\alpha, \tau) \geq c\}$ is closed for every $c \in \mathbf{R}$. Describe the space of holomorphic 1-forms on $X_\alpha(\tau)$. Essentially by construction, two Hopf surfaces are deformation equivalent, but the dimension of the space of holomorphic vector fields "jumps" as the parameters α and τ vary. ◇

Exercise 2.4 Let $\mathbf{P}^n$ denote the space of complex lines through the origin in $\mathbf{C}^{n+1}$, with the holomorphic atlas $\mathfrak{J}$ described in Example 1.7, and let M be the underlying real $2n$-dimensional smooth manifold. Prove that complex conjugation on $\mathbf{C}^{n+1}$ induces a diffeomorphism $f : M \to M$ such that $J \circ f_* = -J$; describe the fixed points of f. In other words, if $\overline{\mathbf{P}}^n$ denotes the holomorphic manifold whose charts are complex conjugates of the standard charts, then $\mathbf{P}^n$ and $\overline{\mathbf{P}}^n$ are biholomorphic.

The diffeomorphism f preserves orientation if $n = 2$; in fact, it is known that there is no holomorphic atlas on the smooth manifold underlying $\mathbf{P}^2$ that induces the orientation opposite to the standard orientation. Briefly, "$-\mathbf{P}^2$ is not a holomorphic manifold." ◇

Exercise 2.5 Let E_τ be the elliptic curve associated to $\tau \in \mathfrak{H}$, as in Exercise 1.5. Prove that E_τ is biholomorphic to $\overline{E}_\tau$ if and only if $2\mathrm{Re}\,\tau$ is an integer. This involves several easy arguments very much in the spirit of Example 1.9. Products of suitable elliptic curves give examples of compact tori for which M and $\overline{M}$ have the same orientation but are not biholomorphic. ◇

Exercise 2.6 Let C_1 and C_2 be compact, connected Riemann surfaces, and let $f : C_1 \to C_2$ be a non-constant holomorphic map. Prove that f is onto, and that if f is one-to-one, then f is a biholomorphism. Prove that there is a finite set $R \subset C_2$ such that f restricted to $f^{-1}(C_2 \setminus R)$ is a covering map. Conclude that there is a positive integer d—the *degree* of f—such that f is d-to-one except for a finite (possibly empty) subset of C_1.
Suggestion: Let R be the image of the set of critical points of f. $\diamond$

A *branch point* of a (non-constant holomorphic) map between Riemann surfaces is a point $p \in C_1$ at which f is not locally one-to-one (i.e. there does not exist a neighborhood V of p with $f|_V$ one-to-one). The prototypical branch point is the origin for the mapping $z \mapsto z^n$; if p is a branch point of f and if $f(p) = q$, then there exist charts ϕ near p and ψ near q such that $\psi \circ f \circ \phi^{-1}(z) = z^n$ for some integer $n > 1$ independent of ϕ and ψ. The number $n - 1$ is called the *ramification number* or *branching order* of f at p, and is zero except possibly at finitely many points. Consequently, the *total branching order* b (the sum of all branching orders) is well-defined when C_1 is compact.

Exercise 2.7 Let $f : C_1 \to C_2$ be a holomorphic map of Riemann surfaces. Prove that branch points of f are exactly critical points, i.e. points where $df = 0$. Assume further that C_1 and C_2 are compact Riemann surfaces of respective topological genera g_1 and g_2, and that f has degree d. Prove the *Riemann-Hurwitz formula*:

$$g_1 = d(g_2 - 1) + 1 + \frac{b}{2}. \tag{2.8}$$

Conclude that $g_2 \leq g_1$. Calculate the total branching order of a rational function of degree d (by direct counting), regarded as a map $\mathbf{P}^1 \to \mathbf{P}^1$, and verify the Riemann-Hurwitz formula in this case.
Suggestion: Triangulate C_2 so that every image of a critical point is a vertex, then lift to C_1 and compute the Euler characteristic. $\diamond$

Exercise 2.8 Let d be a positive integer, and let $f : \mathbf{C}^3 \to \mathbf{C}$ be a the *Fermat polynomial* $f(Z^0, Z^1, Z^2) = (Z^0)^d + (Z^1)^d + (Z^2)^d$ of degree d. The zero locus $C \subset \mathbf{P}^2$ is a smooth curve. Find the Euler characteristic and genus of C in terms of d.
Suggestion: Project away from $[0 : 0 : 1] \notin C$ to get a branched cover of $\mathbf{P}^1$, then use the Riemann-Hurwitz Formula. $\diamond$

Exercise 2.9 (A first attempt at meromorphic functions) Let M be a holomorphic manifold, and $x \in M$. A "meromorphic function" is given in a neighborhood U of x as the quotient of a pair of holomorphic functions $\phi_0, \phi_\infty : U \to \mathbf{C}$ with ϕ_∞ not identically zero. Show that a meromorphic function on a Riemann surface may be regarded as a holomorphic map to $\mathbf{P}^1$; Exercise 1.4 may be helpful. There is no general analogue of this assertion if M has dimension at least two; on $\mathbf{C}^2$, it is impossible to extend z^1/z^2 to the origin even if ∞ is allowed as a value. Meromorphic functions are defined precisely in Chapter 6. ◇

Exercise 2.10 The *great dodecahedron* is the regular (Keplerian) polyhedron, not embedded in $\mathbf{R}^3$, whose 1-skeleton is that of a regular icosahedron and whose twelve regular pentagonal faces intersect five at a vertex. The edges of a single face are the link of a vertex of the 1-skeleton. Realize the great dodecahedron as a three-sheeted cover of the sphere ($\mathbf{P}^1$) branched at twelve points, and use the Riemann-Hurwitz formula to compute the genus. ◇

3

Sheaves and Vector Bundles

A holomorphic function on a domain in **C** determines a collection of power series, each of which contains, in "pointwise" form, *local* information about the function. For this reason, a power series is called a "germ" of an analytic function. The concept of a sheaf is motivated by this picture, and provides a powerful bookkeeping tool for patching together global data from local data. This is due principally to existence of cohomology theories with coefficients in a sheaf and the attendant homological machinery. Many geometric objects (line bundles and infinitesimal deformations of pseudogroup structures, for example) can be expressed in terms of higher sheaf cohomology. Additionally, there are theorems that relate sheaf cohomology to de Rham or singular cohomology, and theorems that guarantee vanishing of sheaf cohomology under various geometric hypotheses. These are used to extract isomorphisms from long exact sequences, thereby expressing solutions to geometric questions in algebraic or topological terms that can be computed.

3.1 Presheaves and Morphisms

Let X be a paracompact Hausdorff space. A *presheaf* $\mathcal{F}$ *of Abelian groups* over X is an association of an Abelian group $\mathcal{F}(U)$ to each open set $U \subset X$ and of a "restriction" homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair $V \subset U$ of nested open sets, subject to the compatibility conditions

$$\rho_{UU} = \text{Identity}, \qquad \rho_{VW}\rho_{UV} = \rho_{UW} \quad \text{if } W \subset V \subset U.$$

The notation $\mathcal{F}$ follows the French *faisceau*. An element $s \in \mathcal{F}(U)$ is a *section* of $\mathcal{F}$ over U; elements of $\mathcal{F}(X)$ are *global sections* of $\mathcal{F}$. It is useful to allow the group $\mathcal{F}(U)$ to be *empty* (as opposed to trivial), as in Example 3.4 below.

Remark 3.1 *Associated to a topological space $(X, \mathfrak{G})$ is a category $\mathfrak{G}$ whose objects are open sets and whose morphisms are inclusions of open sets. A presheaf of Abelian groups on X is then a contravariant functor from $\mathfrak{G}$ to the category of Abelian groups and group homomorphisms. Some find this point of view illuminating.*

It is often useful to consider presheaves of commutative rings or algebras; these are defined in the obvious way. Further, a *presheaf $\mathfrak{M}$ of modules over a presheaf $\mathcal{R}$ of rings* is an association, to each open set U, of an $\mathcal{R}(U)$-module $\mathfrak{M}(U)$ in a manner compatible with restriction maps; for obvious reasons, the coefficient ring should depend on the open set U.

There is nothing in the definition to force the "restriction maps" to be actual restriction maps. However, with the exception of Example 3.6 below, "restriction" will be taken literally in the sequel, and restriction maps will be denoted with more standard notation.

Example 3.2 The simplest presheaves over X are *constant* presheaves $\underline{G}$, whose sections are locally constant G-valued functions. In other words, if G is endowed with the *discrete* topology, then a section of $\underline{G}$ is a continuous, G-valued function.

The presheaf $\mathcal{C}^0_X$ of continuous (complex-valued) functions is defined by taking $\mathcal{C}^0_X(U)$ to be the space of complex-valued, continuous (with respect to the usual topology on $\mathbf{C}$) functions on U. If M is a smooth manifold, then $\mathfrak{A}^r_M$ denotes the presheaf of smooth r-forms; it is customary to write $\mathfrak{A}_M$ instead of $\mathfrak{A}^0_M$. These are all presheaves of rings. A surprisingly interesting example is the presheaf $\mathfrak{A}^\times_M$ of non-vanishing smooth functions, which is a presheaf of Abelian groups under pointwise multiplication of functions. □

Example 3.3 Let (M, J) be a complex manifold. The presheaf of smooth (p, q)-forms on M is denoted $\mathfrak{A}^{p,q}_M$. If M is holomorphic, then the $\bar{\partial}$ operator is a morphism of presheaves, $\bar{\partial} : \mathfrak{A}^{p,q}_M \to \mathfrak{A}^{p,q+1}_M$, and the kernel—consisting of $\bar{\partial}$-closed (p, q)-forms—is denoted $\mathcal{Z}^{p,q}$. The presheaf of local holomorphic functions is denoted $\mathcal{O}_M$, and the presheaf of local holomorphic p-forms is denoted Ω^p_M. □

Example 3.4 Let $\pi : Y \to X$ be a surjective mapping of topological spaces. The *presheaf of continuous sections* of π associates to each open set $U \subset X$ the (possibly empty) set $\Gamma(U)$ of continuous maps $s : U \to Y$ with $\pi \circ s =$ Identity. The presheaf restrictions are the ordinary restriction maps. Generally there is no algebraic structure on $\Gamma(U)$. □

A *morphism* ϕ of presheaves is a collection of group homomorphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{E}(U)$ that are compatible with the restriction maps. A morphism is *injective* if each map $\phi(U)$ is injective. The *kernel* of a presheaf morphism is defined in the obvious way. Surjectivity and cokernels are more conveniently phrased with additional terminology, and are discussed later, see also Exercise 3.1.

Example 3.5 Suppose $f : X \to Y$ is a continuous map of topological spaces and $\mathcal{E}$ is a presheaf on X. The *pushforward* of $\mathcal{E}$ by f is the presheaf on Y defined by $f_*\mathcal{E}(V) = \mathcal{E}(f^{-1}(V))$. If $\mathcal{F}$ is a presheaf on Y, the *pullback* is the presheaf on X defined by $f^*\mathcal{F}(U) = \mathcal{F}\big(f(U)\big)$. □

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a family of open sets. Intersections will be denoted by multiple subscripts; thus $U_{\alpha\beta} = U_\alpha \cap U_\beta$, for example. A presheaf over X is *complete* if two additional properties are satisfied:

i. For every open set $U \subset X$, if $s, t \in \mathcal{F}(U)$ and $s|_V = t|_V \in \mathcal{F}(V)$ for all proper open sets $V \subset U$, then $s = t \in \mathcal{F}(U)$. In words, sections are determined by their values locally.

ii. For every open set $U \subset X$, if $\mathcal{U} = \{U_\alpha\}$ is a cover of U by open sets, and if there exist *compatible* sections $s_\alpha \in \mathcal{F}(U_\alpha)$, i.e. such that

$$s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}} \quad \text{whenever } U_\alpha \cap U_\beta \text{ is non-empty,}$$

then there is a section $s \in \mathcal{F}(U)$ with $s|_{U_\alpha} = s_\alpha$ for all α. In words, compatible local data may be patched together.

The presheaves described in Examples 3.2–3.4 are complete. Example 3.6 below shows how each of these axioms may fail.

There is a natural way of "completing" an arbitrary presheaf. Associated to a presheaf $\mathcal{F}$ over X is a *sheaf of germs of sections*, which is a topological space equipped with a surjective local homeomorphism to X. The inverse image of a point $x \in X$ is called the *stalk* at x, and is defined to be the direct limit

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U). \tag{3.1}$$

To elaborate on this definition, note that the set of neighborhoods of x forms a directed system under inclusion of sets, and because restriction maps satisfy a compatibility (or functoriality) condition, it makes sense to declare $s \in \mathcal{F}(U)$ to be equivalent to $t \in \mathcal{F}(V)$ if there is a neighborhood $W \subset U \cap V$ of x with $s|_W = t|_W$. The stalk $\mathcal{F}_x$ is the set of equivalence classes—or *germs*. Any algebraic structure possessed by the spaces $\mathcal{F}(U)$ will be inherited by the stalks.

Let $\mathcal{F} = \bigcup_{x\in X} \mathcal{F}_x$ be the union of the stalks. A basis for a topology on $\mathcal{F}$ is defined as follows: For each open set $U \subset X$, there is a natural map $\mathcal{F}(U) \to \bigcup_{x\in U} \mathcal{F}_x$ that maps a section $s \in \mathcal{F}(U)$ to the set of germs $\{s_x \in \mathcal{F}_x : x \in U\}$. The image of such a map is declared to be a basic open set. The *projection map* $\pi : \mathcal{F} \to X$ sending each stalk $\mathcal{F}_x$ to $x \in X$ is a local homeomorphism.

A *sheaf* is a complete presheaf. The "completion" process just described[1] associates a sheaf to an arbitrary presheaf. The presheaf of continuous sections of the completion is isomorphic to the original presheaf, so it usually unnecessary to be scrupulous in distinguishing a presheaf and its completion.

Example 3.6 There are presheaves that fail to satisfy each of the completeness axioms. Let $X = \mathbf{R}$ with the usual topology, and let $\mathcal{C}(U)$ denote the set of complex-valued functions on U. Define ρ_{UV} to be the zero map if V is a proper subset of U. This defines a presheaf $\mathcal{C}$ that fails to satisfy axiom i. Intuitively, the restriction maps of $\mathcal{C}$ lose local information.

If $\mathcal{B}$ is the presheaf of bounded holomorphic functions over $\mathbf{C}$, then $\mathcal{B}$ fails to satisfy axiom ii: If U_n denotes the disk of radius n centered at $0 \in \mathbf{C}$, then the function z lies in $\mathcal{B}(U_n)$ for every n, but these local functions do not give a globally defined bounded function. Intuitively, this presheaf is defined by non-local information. The completion of $\mathcal{B}$ is $\mathcal{O}_{\mathbf{C}}$, whose stalk at z_0 is the ring $\mathcal{O}_{\mathbf{C},z_0}$ of locally convergent power series centered at z_0. □

Remark 3.7 *Some authors define a sheaf to be a topological space $\mathcal{F}$ together with a surjective local homeomorphism to X satisfying additional conditions. This is an especially useful point of view in algebraic geometry, but for the relatively naive purposes below, the concrete definition is simpler.*

A *morphism* ϕ of sheaves of Abelian groups over X is a continuous map $\phi : \mathcal{E} \to \mathcal{F}$ of topological spaces that maps stalks homomorphically to stalks. A morphism of sheaves of Abelian groups is injective/surjective (by definition) if and only if ϕ is injective/surjective on stalks. The *quotient* of a sheaf by a subsheaf is similarly defined on stalks. Sheaves of appropriate algebraic structures may be direct summed, tensored, dualized, and so on. A sequence $\mathcal{S} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{Q}$ is *exact at $\mathcal{F}$* if the image of ϕ is equal to the kernel of ψ.

Example 3.8 For a complex-valued function f, set $\operatorname{Exp}(f) = e^{2\pi\sqrt{-1}f}$. If M is a smooth manifold, then the sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \xrightarrow{i} \mathfrak{A} \xrightarrow{\operatorname{Exp}} \mathfrak{A}^{\times} \longrightarrow 0 \tag{3.2}$$

is exact. If M is a holomorphic manifold, there is a short exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \xrightarrow{i} \mathcal{O} \xrightarrow{\operatorname{Exp}} \mathcal{O}^{\times} \longrightarrow 0 \tag{3.3}$$

called the *exponential sheaf sequence*. These sequences are geometrically important in view of the cohomological interpretation of line bundles discussed later. □

[1] Sometimes called "sheafification," from the verb "to sheafify."

Let M be a holomorphic manifold. An *analytic* sheaf on M is a sheaf of $\mathcal{O}_M$-modules. An analytic sheaf $\mathcal{F}$ is *finitely generated* if there is an exact sequence $\bigoplus^k \mathcal{O}_M \to \mathcal{F} \to 0$, and is *coherent* if, in addition, the kernel of the first map is finitely generated, that is, there is an exact sequence

$$\bigoplus^\ell \mathcal{O}_M \to \bigoplus^k \mathcal{O}_M \to \mathcal{F} \to 0.$$

It is perhaps worth emphasizing that these sequences are essentially local, in the sense that exactness holds only at the level of stalks. Thus a finitely generated analytic sheaf has the property that for every $x \in M$, there is an open neighbourhood U such that $\bigoplus^k \mathcal{O}_M(U) \to \mathcal{F}(U) \to 0$ is an exact sequence of Abelian groups.

Example 3.9 Let $Y \subset X$ be a closed subspace. If $\mathcal{F}$ is a sheaf of Abelian groups on Y, then there is a sheaf on X obtained by *extending by zero*; to each open set $U \subset X$, associate the Abelian group $\mathcal{F}(U \cap Y)$ when this intersection is non-empty, and associate the trivial Abelian group otherwise.

An important special case is when Y is a closed complex submanifold of a holomorphic manifold M. The *ideal sheaf* $\mathfrak{I}_Y$ is the subsheaf of $\mathcal{O}_M$ consisting of germs of holomorphic functions that vanish on Y. The quotient $\mathcal{O}_M/\mathfrak{I}_Y$ is the extension by zero of $\mathcal{O}_Y$. Both of these sheaves are coherent.
□

A coherent analytic sheaf $\mathcal{E}$ is *locally free* if each $x \in M$ has a neighborhood U such that $\mathcal{E}(U)$ is a free $\mathcal{O}_M(U)$-module. In other words, each stalk $\mathcal{E}_x$ is isomorphic to a direct sum of finitely many copies of $\mathcal{O}_{M,x}$. Locally free sheaves are closely related to holomorphic vector bundles, see Example 3.10.

Coherent analytic sheaves are basic tools in algebraic geometry, several complex variables, and—to an increasing extent—differential geometry. As a category, coherent sheaves behave better than locally free sheaves. Kernels, images, quotients, and pushforwards of coherent sheaves are coherent, while analogous assertions about locally free sheaves are generally false because ranks of morphisms can vary from point to point.

3.2 Vector Bundles

Intuitively, a vector bundle over a smooth manifold M may be regarded as a family of vector spaces (*fibres*) parametrized by points of the manifold. The family is required to satisfy a "local triviality" condition which, among other things, implies that over each component of M the fibres are all isomorphic. A vector bundle is said to be real or complex, of rank k, according to the nature of the fibres.

Precisely, a *complex vector bundle of rank k* over a smooth manifold M is a smooth submersion $p : E \to M$ of smooth manifolds with the following properties:

i. For each $x \in M$, $E_x := p^{-1}(x) \simeq \mathbf{C}^k$.

ii. For each $x \in M$, there exists a neighborhood U of x and a diffeomorphism $\varphi : p^{-1}(U) \to U \times \mathbf{C}^k$, called a *vector bundle chart*, such that $p \circ \varphi^{-1}$ is projection on the first factor.

iii. If $(\varphi_\alpha, U_\alpha)$ and (φ_β, U_β) are vector bundle charts at x, then the *overlap map* $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_{\alpha\beta}} : U_{\alpha\beta} \times \mathbf{C}^k \to U_{\alpha\beta} \times \mathbf{C}^k$ is given by

$$(x, v) \mapsto \big(x, g_{\alpha\beta}(x)v\big), \qquad g_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbf{C});$$

the smooth, matrix-valued function $g_{\alpha\beta}$ is called the *transition function* from U_α to U_β. Observe that by definition,

$$g_{\beta\alpha} = g_{\alpha\beta}^{-1}, \quad \text{and} \quad g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = I_k \text{ on } U_{\alpha\beta\gamma}. \tag{3.4}$$

The manifold M is the *base space* of the vector bundle, and E is the *total space*. Though it is not unusual to speak of "the vector bundle E," this abuse of language can be substantially imprecise unless the projection map is dictated by context; logically preferable is "the vector bundle p," though this is almost never followed. If $p : E \to M$ is a complex vector bundle over a holomorphic manifold and the transition functions are holomorphic maps into the complex Lie group $GL(k, \mathbf{C})$, then $p : E \to M$ is a *holomorphic* vector bundle. In this case, the total space E is a holomorphic manifold, and the projection p is a holomorphic submersion.

The simplest vector bundles are presented as collections of charts and transition functions; examples are given in the exercises. Other useful examples, such as tangent bundles or tautological bundles, can be described geometrically but transition functions are tedious to write explicitly. For a generic holomorphic vector bundle, a description may consist of little more than the topological type of the underlying smooth vector bundle.

Example 3.10 Let $p : E \to M$ be a holomorphic vector bundle. The presheaf of local holomorphic sections gives rise to a locally free analytic sheaf. Conversely, a locally free analytic sheaf $\mathcal{E}$ determines a holomorphic vector bundle whose presheaf of sections induces the original sheaf. To accomplish the latter, choose sections $\{\mathbf{e}_j\}_{j=1}^k$ in $\mathcal{E}(U)$ that generate $\mathcal{E}(U)$ as an $\mathcal{O}_M(U)$-module, and use the correspondance

$$\sum_{j=1}^k a^j \mathbf{e}_j\Big|_z \in \mathcal{E}_z \longmapsto \big(z, a^1(z), \dots, a^k(z)\big) \in U \times \mathbf{C}^k. \tag{3.5}$$

As a first approximation, coherent sheaves may be regarded as families of vector spaces whose fibre dimension "jumps" from point to point, as in Example 3.9. □

Example 3.11 Let M be a holomorphic manifold, and let TM denote the (real) tangent bundle of the underlying smooth manifold. The bundle

$T^{1,0}M \subset TM \otimes \mathbf{C}$ is a holomorphic vector bundle, called the *holomorphic tangent bundle of* M, whose underlying complex vector bundle is complex-linearly isomorphic to TM endowed with the natural complex structure.

The tensor bundles $\bigotimes^p T^{1,0}M$ and $\bigwedge^p T^{1,0}M$ are holomorphic, while $\bigwedge^q T^{0,1}M$ is merely smooth. Since the composite of anti-holomorphic maps is not anti-holomorphic, there is no notion of an "anti-holomorphic" vector bundle. □

A *morphism* of vector bundles over M (or a *vector bundle map*) is a map $\phi : E \to F$ of smooth manifolds that is linear on fibres and is compatible with the projection maps. The kernel, cokernel, and image of a vector bundle map are defined in the obvious way. However, these families of vector spaces are not generally vector bundles; the fibre dimension may vary from point to point. Two vector bundles E and E' over M are *isomorphic* if there is a vector bundle map from $E \to E'$ that is an isomorphism on fibres.

It is sometimes useful to consider maps between vector bundles over different base spaces. In this case, it is necessary to have compatible mappings of the total spaces and the base spaces, of course. The prototypical example is the "pullback" of a vector bundle under a smooth map. Let $p : E \to M$ be a vector bundle, and let $f : M' \to M$ be a smooth map. There is a *pullback* vector bundle $f^*E \to M'$, whose total space is defined to be

$$f^*E = \{(x, v) \in M' \times E \mid f(x) = p(v)\} \subset M' \times E, \tag{3.6}$$

with projection induced by projection on the first factor. The fibre of f^*E over $x \in M'$ is the fibre $E_{f(x)}$ of E over $f(x)$, and the isomorphism class of f^*E is determined by the homotopy class of the map f.

Remark 3.12 *An important result of algebraic topology—of which only a very special case is given here—is that for each positive integer k, there exists a "classifying space" G_k and a "universal bundle" $U_k \to G_k$ with the following property: For every smooth, rank k vector bundle $p : E \to M$ over a finite-dimensional smooth manifold, there is a "classifying map" $\psi : M \to G_k$ such that $E = \psi^* U_k$, and the classifying map is unique up to smooth homotopy. Roughly, rank-k vector bundles over M are in natural one-one correspondance with homotopy classes of smooth maps $M \to G_k$. One way of constructing the classifying space is via "infinite-dimensional Grassmannians" (see Exercise 3.12 and Chapter 9).*

For $i = 1, 2$, let $p_i : E_i \to M$ be a complex vector bundle of rank k_i over a smooth manifold. Very loosely, a "differential operator" is a linear map $D : A^0(E_1) \to A^0(E_2)$ that, when expressed with respect to a chart, is a differential operator in the usual sense between k_1-tuples of functions and k_2-tuples of functions. The order of the local representative is independent of the choice of vector bundle charts.

For the relatively naive purposes here, a few elementary examples will suffice as illustration. Let M be a smooth manifold. For each integer $r \geq 0$, the exterior derivative $d : A^r \to A^{r+1}$ is a first-order differential operator, which in local coordinates satisfies $d(f\,dx^I) = df \wedge dx^I$. In this example, $E_1 = \bigwedge^r T^*M$ and $E_2 = \bigwedge^{r+1} T^*M$. If M is a holomorphic manifold, then the operators ∂ and $\bar\partial$ are similarly first-order differential operators. As with the exterior derivative, differential operators are often specified by their action on local sections. It is necessary in such an event to check that local expressions of sections "transform" correctly, i.e. are independent of chart. This is illustrated by the second example, which is elementary, but of sufficient importance to deserve special mention.

Proposition 3.13 *Let $p : E \to M$ be a holomorphic vector bundle. Then there is a differential operator $\bar\partial : A^{p,q}(E) \to A^{p,q+1}(E)$ acting on (p,q)-forms with values in E.*

Proof Let $U \subset M$ be a trivializing neighborhood for E, and let $\{\mathbf{e}_j\}_{j=1}^k$ be a local holomorphic frame over U. Every holomorphic frame $\{\mathbf{e}_j{}'\}_{j=1}^k$ over U is of the form $\widetilde{\mathbf{e}}_i = g_i^j \mathbf{e}_j$ for a non-singular matrix (g_i^j) of holomorphic functions on U. Let

$$s = \sum_{j=1}^k s^j \mathbf{e}_j = \sum_{i,j=1}^k \widetilde{s}^i g_i^j \mathbf{e}_j = \sum_{i=1}^k \widetilde{s}^i \widetilde{\mathbf{e}}_i$$

be a local smooth section of $\bigwedge^{p,q} \otimes E$. The expression $\bar\partial s = \sum_{j=1}^k \bar\partial s^j \otimes \mathbf{e}_j$, which is a local section of $\bigwedge^{p,q+1} \otimes E$, is independent of the choice of frame. Indeed, g_i^j is holomorphic in U, i.e. $\bar\partial g_i^j = 0$, so

$$\begin{aligned}\sum_{i=1}^k \bar\partial\widetilde{s}^i \otimes \widetilde{\mathbf{e}}_i &= \sum_{i,j=1}^k \bar\partial\widetilde{s}^i \otimes g_i^j \mathbf{e}_j = \sum_{i,j=1}^k (\bar\partial\widetilde{s}^i) g_i^j \otimes \mathbf{e}_j \\ &= \sum_{i,j=1}^k \bar\partial(\widetilde{s}^i g_i^j) \otimes \mathbf{e}_j = \sum_{j=1}^k \bar\partial s^j \otimes \mathbf{e}_j.\end{aligned}$$

It follows that $\bar\partial s$ is independent of vector bundle charts. ∎

The philosophical content of Proposition 3.13 is that holomorphic functions are constant with respect to $\bar\partial$, so if $E \to M$ is a holomorphic vector bundle, then there is an induced map on forms with values in E. There is no natural analogue of the exterior derivative d for forms with values in E because the only "d-constant" functions are genuine constants. It is clear that $\bar\partial^2 = 0$ as a differential operator acting on E-valued forms. This gives rise to a cohomology theory, studied in Chapters 4 and 8. There is no analogue for the ∂ operator because "∂-constant" functions are anti-holomorphic, and the set of anti-holomorphic functions does not form a pseudogroup.

The theory of differential operators on manifolds is vast and deep. A cornerstone is the "Atiyah-Singer Index Theorem" for elliptic operators, which expresses the index (dimension of kernel minus dimension of cokernel) of certain differential operators in terms of integrals over M of cohomology classes depending on the operator. In many cases of interest, these classes may be effectively computed. Hodge Theory is the prototypical application of the theory of differential operators, see Chapter 8.

Exercises

Exercise 3.1 Consider the condition: "For every open set $U \subset X$, the map $\phi(U) : \mathcal{S}(U) \to \mathcal{F}(U)$ is a surjective homomorphism of Abelian groups." Explain carefully how this condition differs from surjectivity for a presheaf morphism. It may be helpful to consider the map Exp in the exponential sheaf sequence, which is surjective as a presheaf morphism but does not generally satisfy the condition above. ◇

Holomorphic Line Bundles Over $\mathbf{P}^1$

Exercises 3.2–3.7 introduce several holomorphic line bundles over $\mathbf{P}^1$. It will be shown later that holomorphic line bundles over $\mathbf{P}^1$ are completely determined by an integer called their *degree*.

The holomorphic line bundle of degree $k \in \mathbf{Z}$ over $\mathbf{P}^1$ is constructed explicitly as follows. Using the standard atlas, take two "trivial" families over U_0 and U_1 with coordinates (z^0, ζ^0) and (z^1, ζ^1)—z^i is the *base coordinate* and ζ^i is the *fibre coordinate* over U_i—and "glue them together" over the set $\mathbf{C}^\times = \mathbf{P}^1 \setminus \{0, \infty\}$ by the identification

$$z^0 = \frac{1}{z^1}, \qquad \zeta^0 = \frac{\zeta^1}{(z^1)^k}. \tag{3.7}$$

This line bundle is usually denoted $\mathcal{O}_{\mathbf{P}^1}(k)$, or simply $\mathcal{O}(k)$ for brevity. The total space H^k of $\mathcal{O}(k)$ is the identification space; thus H^k consists of the disjoint union $U_0 \sqcup U_1$ of two copies of $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$, with respective coordinates (z^0, ζ^0) and (z^1, ζ^1), which have been glued together along $\mathbf{C}^\times \times \mathbf{C}$ according to (3.7). The transition function from z^0 to z^1 is

$$g_{01}(z^0) = \frac{1}{(z^0)^k};$$

the other is $g_{10}(z^1) = 1/(z^1)^k$. A holomorphic section of $\mathcal{O}(k)$ is represented by holomorphic functions $\zeta^0 = f_0(z^0)$ and $\zeta^1 = f_1(z^1)$ with

$$g_{01}(z^0) f_0(z^0) = f_1(z^1). \tag{3.8}$$

Every bundle has a *trivial* section, given by $\zeta^i = 0$; the graph of this section is often called the *zero section*. If there are no other sections, the bundle is said to have no sections.

Exercise 3.2 Find all holomorphic sections of $\mathcal{O}(k)$. The projection map $p : H^k \to \mathbf{P}^1$ is defined by sending (z^i, ζ^i) to z^i. Check that this map is well-defined, that H^k is a holomorphic manifold, and that a holomorphic section of $\mathcal{O}(k)$ is a holomorphic map $\sigma : \mathbf{P}^1 \to H^k$ that satisfies $p \circ \sigma =$ Identity on $\mathbf{P}^1$. In other words, a section is a (holomorphic) choice of point in each fibre. ◇

Exercise 3.3 Find the value of k for which $\mathcal{O}(k)$ is the bundle of holomorphic 1-forms on $\mathbf{P}^1$. (Suggestion: Use the identifications $\zeta^0 dz^0 \leftrightarrow (z^0, \zeta^0)$ and $-\zeta^1 dz^1 \leftrightarrow (z^1, \zeta^1)$.) Answer the same question for the tangent bundle of $\mathbf{P}^1$. ◇

Exercise 3.4 For each point $[Z]$ in $\mathbf{P}^1$, let $\ell_{[Z]}$ be the line in $\mathbf{C}^2$ represented by this point. Show that the family $\{\ell_{[Z]} : [Z] \in \mathbf{P}^1\}$ may be regarded as a holomorphic line bundle in the following manner: Set

$$L = \{([Z], \zeta) \in \mathbf{P}^1 \times \mathbf{C}^2 \mid \zeta \in \ell_{[Z]}\},$$

with projection to the first factor. This holomorphic line bundle is called the *tautological* bundle. Find k such that $L = \mathcal{O}(k)$. Use the fact that a holomorphic map $s : \mathbf{P}^1 \to \mathbf{C}^2$ must be constant to show that L has no sections (cf. Exercises 3.2 and 1.2). ◇

Exercise 3.5 (Blowing up) In the notation of Exercise 3.4, let $\pi : L \to \mathbf{C}^2$ be the projection on the second factor. Show that π maps the zero section of L to the origin $(0, 0)$, and that π maps the complement of the zero section biholomorphiocally to $\mathbf{C}^2 \setminus (0, 0)$. Thus, the total space of L is obtained from $\mathbf{C}^2$ by removing the origin and gluing in a $\mathbf{P}^1$; the geometric effect is to give each line through $(0, 0)$ a distinct origin. The total space of L is called the *blow-up* of $\mathbf{C}^2$ at the origin. ◇

Exercise 3.6 Let $[Z^0 : Z^1 : Z^2]$ be homogeneous coordinates on $\mathbf{P}^2$. The projection map $p : [Z^0 : Z^1 : Z^2] \mapsto [Z^0 : Z^1 : 0]$ is defined everywhere except at the point $[0 : 0 : 1]$, and the image is the $\mathbf{P}^1$ in $\mathbf{P}^2$ with equation $Z^2 = 0$ (see also Exercise 1.1). Let $H = \mathbf{P}^2 \setminus [0 : 0 : 1]$. Show that $p : H \to \mathbf{P}^1$ is a holomorphic line bundle, and find the corresponding value of k. Show that the space of holomorphic sections is the set of linear functions $\{a_0 Z^0 + a_1 Z^1 \mid a_i \in \mathbf{C}\}$. ◇

Exercise 3.7 Let L be the tautological bundle (Exercise 3.4), and H the *hyperplane* bundle (Exercise 3.6). Show that $H^\times$—the complement of the zero section of H—is biholomorphic to $\mathbf{C}^2 \setminus (0, 0)$, which is biholomorphic to $L^\times$ by Exercise 3.5. Find an analogous statement for the bundles $\mathcal{O}(k)$ and $\mathcal{O}(-k)$. Prove that $\mathcal{O}^\times(k)$ is the quotient of $H^\times$ by the action of a cyclic group of order k. ◇

Let $\mathcal{U} = \{U_\alpha\}_{\alpha=0}^n$ be the standard atlas on $\mathbf{P}^n$. The line bundle $\mathcal{O}_{\mathbf{P}^n}(k)$ is defined by the transition functions $g_{\alpha\beta}([z]) = (z^\beta / z^\alpha)^k$.

Exercise 3.8 Show that the line bundle $L = \mathcal{O}_{\mathbf{P}^n}(-1)$ is the tautological bundle, that is, the subbundle of $L \subset \mathbf{P}^n \times \mathbf{C}^{n+1}$ whose fibre at $[Z] \in \mathbf{P}^n$ is the line $\ell_Z \subset \mathbf{C}^{n+1}$ through 0 and Z. Let Sym_n^d denote the space of homogeneous polynomials of degree d on $\mathbf{C}^n$. Verify that the space of holomorphic sections of $\mathcal{O}_{\mathbf{P}^n}(k)$ is equal to Sym_{n+1}^k if $k \geq 0$ and is trivial otherwise. In particular, the homogeneous coordinate functions are sections of the line bundle $H = L^*$. ◇

Projectivization, Tautological Bundles, and Blowing Up

Let $p : E \to M$ be a holomorphic vector bundle of rank $r > 1$ over a compact complex manifold. The group $\mathbf{C}^\times$ acts by scalar multiplication in the fibres on the total space of E, and the action is free on the complement $E^\times$ of the zero section.

Exercise 3.9 Prove that the quotient space $\mathbf{P}(E) = E^\times/\mathbf{C}^\times$—called the *projectivization* of E—is a compact holomorphic manifold, and that the projection $p : E \to M$ factors through the quotient map. The induced map $\pi : \mathbf{P}(E) \to M$ is a fibre bundle projection whose fibres are projective spaces. Show that if L is a line bundle over M, then $\mathbf{P}(E \otimes L)$ is biholomorphic to $\mathbf{P}(E)$. ◇

With notation as in Exercise 3.9, consider the rank r vector bundle $\pi^*E \to \mathbf{P}(E)$. A point of $\mathbf{P}(E)$ is represented by a point $x \in M$ and a line ℓ_x in the fibre E_x. Let $\tau_E \subset \pi^*E$ be the line subbundle whose fibre is the line ℓ_x. The bundle τ_E is called the *tautological bundle* of E (cf. Exercise 3.8, where M is a point).

Exercise 3.10 Prove that the restriction of τ_E to a fibre of $\mathbf{P}(E)$ is a tautological bundle over a projective space. Let $\varpi : L \to M$ be a line bundle. Prove that the tautological bundle of $E \otimes L$ is $\tau_E \otimes \varpi^*L$. (This equation involves a minor abuse of notation.) Prove that $E^\times$ and $\tau_E^\times$ are biholomorphic; in other words, the complement of the zero section in E is the same as the complement of the zero section of τ_E. The total space of τ_E is said to be obtained from the total space of E by *blowing up the zero section.* ◇

Exercise 3.11 Let $\mathcal{O}$ denote the trivial line bundle over M. Prove that the total space of E embeds in $\mathbf{P}(E \oplus \mathcal{O})$—the *completion* of E—as an open, dense submanifold. Describe the complement of E in $\mathbf{P}(E \oplus \mathcal{O})$. ◇

Grassmannian Manifolds

Let $0 < k < n$ be integers. The set of k-dimensional linear subspaces of $\mathbf{C}^n$ can be made into a compact complex manifold, called the (complex) *Grassmannian* $\mathbf{G}_k(\mathbf{C}^n) = \mathbf{G}_{k,n}$. Consider first the *Stiefel manifold* $S_{k,n}$ of k-frames in $\mathbf{C}^n$, which may be realized as the open set in $\mathbf{C}^{n \times k}$ consisting of matrices of rank k. The group $GL(k,\mathbf{C})$ acts by right multipication, and an orbit of this action is exactly the set of frames that span a fixed subspace. The Grassmannian is defined to be the quotient $S_{k,n}/GL(k,\mathbf{C})$ with the quotient topology. Observe that $\mathbf{G}_{1,n+1} = \mathbf{P}^n$.

Exercise 3.12 Prove that the Grassmannian $\mathbf{G}_{k,n}$ has the structure of a compact holomorphic manifold, and that the tangent space at $W \subset \mathbf{C}^n$ is $\mathrm{Hom}(W, \mathbf{C}^n/W)$. (Intuitively, the subspace W can be perturbed in the transverse directions, and these are parametrized by $\mathbf{C}^n/W$; see also equation (3.9) below. This description is preferable to taking the orthogonal complement with respect to an inner product, as the latter is "non-holomorphic," see also Exercise 3.14.) ◇

Exercise 3.13 Let $U_{k,n} \to \mathbf{G}_{k,n}$ be the *universal* (k-plane) bundle, that is, the subbundle of $\mathbf{G}_{k,n} \times \mathbf{C}^n$ whose fibre at $W \in \mathbf{G}_{k,n}$ is W. If Q is the quotient bundle in the exact sequence $0 \to U_{k,n} \to \mathbf{G}_{k,n} \times \mathbf{C}^n \to Q \to 0$, prove that

$$T\mathbf{G}_{k,n} = \mathrm{Hom}(U_{k,n}, Q). \tag{3.9}$$

In particular, there is a surjection $\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathbf{C}^{n+1} \to T\mathbf{P}^n$ of holomorphic vector bundles. Describe the kernel, and use this description to show that the holomorphic manifolds $\mathbf{P}(T\mathbf{P}^n)$ and $\mathbf{P}^n \times \mathbf{P}^{n-1}$ are diffeomorphic. (They are not deformation equivalent, much less biholomorphic.) ◇

Exercise 3.14 Show that the Euclidean inner product on $\mathbf{C}^n$ induces a diffeomorphism $\mathbf{G}_{k,n} \simeq \mathbf{G}_{n-k,n}$ that associates a k-plane to its orthogonal complement, but that this map is not a biholomorphism.

Let V be a complex vector space of dimension n. Prove that $\mathbf{G}_k(V)$ and $\mathbf{G}_{n-k}(V^*)$ are biholomorphic. Show that if $f : V \to V_0$ is a vector space isomorphism with inverse ϕ, then the induced maps

$$f : \mathbf{G}_k(V) \to \mathbf{G}_k(V_0) \quad \text{and} \quad \phi : \mathbf{G}_{n-k}(V^*) \to \mathbf{G}_{n-k}(V_0^*)$$

are compatible with the respective biholomorphisms. ◇

The construction of the Grassmannian and universal bundle is functorial (see Exercise 3.12); it therefore makes sense to form bundles of Grassmannians from vector bundles, and to take universal bundles over bundles of Grassmannians. There are useful notions of "blowing up" that can be defined using these ideas. For example, if $f : N \to M$ is a holomorphic immersion, then the *Gauss map* of f is the map $\widehat{f} : N \to G_k(TM)$ that sends a point x to $f_*T_xN \subset T_{f(x)}M$, the tangent space of its image. The *Nash blow-up* of f is closure of the graph of $\widehat{f}$. Geometrically, the Nash blow-up separates points of N that are mapped to the same point of M but with different tangent spaces.

4
Cohomology

Sheaves are a natural tool for piecing together global data (e.g. holomorphic functions) from local data (e.g. germs). Potential obstructions to the possibility of patching are measured by Čech cohomology, which is approximated by the simplicial cohomology of the "nerve" of a locally finite open cover. Dolbeault cohomology, by contrast, is the complex analogue of de Rham cohomology on a complex manifold. There is a salient difference between Dolbeault and de Rham theories, however: Dolbeault cohomology of a manifold M depends on the holomorphic structure and not merely on the topology, *even (possibly) when M is compact*.

The Dolbeault Theorem (Theorem 4.11 below) asserts the isomorphism of certain Čech and Dolbeault spaces. Coupled with the Hodge Decomposition on compact "Kähler" manifolds, the Dolbeault isomorphism allows calculation of sheaf cohomology, and forges a powerful link between complex analysis (sheaves), differential geometry (Dolbeault theory), and topology (singular theory).

4.1 Čech Cohomology

Let X be a paracompact Hausdorff space. To a locally finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ is associated a simplicial complex, the *nerve* of $\mathcal{U}$, whose r-simplices are non-empty $(r+1)$-fold intersections of sets in $\mathcal{U}$. As usual, multiple subscripts denote intersections of sets. The typical r-simplex is denoted $U_{\alpha_0 \cdots \alpha_r}$.

Let $\mathcal{F}$ be a sheaf of Abelian groups (written additively) over X. A *Čech*

r-cochain with values in $\mathcal{F}$ is a collection of sections $f_{\alpha_0\cdots\alpha_r} \in \mathcal{F}(U_{\alpha_0\cdots\alpha_r})$, one for each *ordered* $(r+1)$-tuple of indices. The $(r+1)!$ sections associated to a fixed r-simplex are assumed to be completely skew-symmetric in their indices. The group of r-cochains is denoted $C^r(\mathcal{U}, \mathcal{F})$.

The *coboundary operator* $\delta_r : C^r(\mathcal{U}, \mathcal{F}) \to C^{r+1}(\mathcal{U}, \mathcal{F})$ is defined by

$$(\delta_r f)_{\alpha_0\cdots\alpha_{r+1}} = \sum_{j=0}^{r+1} (-1)^j \rho_j \, f_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_{r+1}};$$

here ρ_j denotes the presheaf restriction from the r-simplex $U_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_{r+1}}$ to the $(r+1)$-simplex $U_{\alpha_0\cdots\alpha_{r+1}}$. The group of *Čech cocycles* $\mathcal{Z}^r(\mathcal{U}, \mathcal{F})$ is defined to be the kernel of δ_r, and the group of *Čech coboundaries* $\mathcal{B}^r(\mathcal{U}, \mathcal{F})$ is the image of δ_{r-1}. The usual calculation shows that $\delta_r \, \delta_{r-1} = 0$, so every coboundary is a cocycle, and the r-dimensional cohomology of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is defined to be $H^r(\mathcal{U}, \mathcal{F}) = \mathcal{Z}^r(\mathcal{U}, \mathcal{F})/\mathcal{B}^r(\mathcal{U}, \mathcal{F})$.

The term "higher" cohomology refers to cohomology in dimension greater than zero. The zero-dimensional Čech cohomology of an open cover is independent of the cover, a simple fact of recurring importance.

Proposition 4.1 *Let $\mathcal{F} \to X$ be a sheaf of Abelian groups over a paracompact Hausdorff space. For every cover $\mathcal{U}$ of X by open sets, the zero-dimensional cohomology $H^0(\mathcal{U}, \mathcal{F})$ is naturally isomorphic to the group $H^0(X, \mathcal{F})$ of global sections of $\mathcal{F}$.*

Proof The coboundary operator δ_0 sends a 0-cochain $f = \{f_\alpha\}$ to the 1-cochain

$$\delta_0 f = \{(\delta_0 f)_{\alpha\beta}\}, \qquad (\delta_0 f)_{\alpha\beta} = f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}}.$$

Consequently, a 0-cocycle is exactly a compatible collection of local sections. □

A *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ together with a map $\mu : J \to I$ of index sets such that $V_\beta \subset U_{\mu(\beta)}$ for all $\beta \in J$. Every refinement induces a map μ_r on r-cochains by "including open sets into $\mathcal{U}$, evaluating cochains, and restricting." These maps commute with the respective coboundary maps, so there is an induced map μ_r^* in cohomology. The hope is to define the r-dimensional *Čech cohomology of X with coefficients in $\mathcal{F}$* as the direct limit over successive refinements, $H^r(X, \mathcal{F}) = \varinjlim H^r(\mathcal{U}, \mathcal{F})$. This is sensible because different refinement maps between $\mathcal{V}$ and $\mathcal{U}$ induce the same map in cohomology:

Proposition 4.2 *Let $\tau, \mu : J \to I$ be refinements. Then $\tau_r^* = \mu_r^*$ for all $r \geq 0$.*

Proof (Sketch) Let $\sigma = V_{\beta_0\cdots\beta_{r-1}}$ be an $(r-1)$-simplex of $\mathcal{V}$. For each index $j = 0, \ldots, r-1$, let $\widetilde{\sigma}_j$ be the r-simplex

$$\widetilde{\sigma}_j = V_{\mu(\beta_0)} \cap \cdots \cap V_{\mu(\beta_j)} \cap V_{\tau(\beta_j)} \cap \cdots \cap V_{\tau(\beta_{r-1})},$$

and define $h_r : C^r(\mathcal{U}, \mathcal{F}) \to C^{r-1}(\mathcal{V}, \mathcal{F})$ by "refining, evalutaing cochains on $\widetilde{\sigma}_j$, and taking the alternating sum." An involved but straightforward calculation shows that $\{h_r\}$ is a homotopy operator, that is,

$$h_{r+1} \circ \delta_r + \delta_{r-1} \circ h_r = \tau_r - \mu_r$$

as maps on cochains. As usual, this shows the induced maps on cohomology coincide. □

In order to have a computable theory, it is necessary to have criteria for an open cover under which the map to the direct limit is an isomorphism, or to have theorems which relate Čech cohomology to other cohomology theories (e.g. singular or de Rham). Such a criterion can be given immediately, and useful isomorphism theorems will be developed presently.

A locally finite open cover $\mathcal{U}$ is *acyclic* or *Leray* for the sheaf $\mathcal{F}$ if every simplex in the nerve of $\mathcal{U}$ has trivial higher cohomology with coefficients in $\mathcal{F}$. The following general result, due to Leray, is proven after Theorem 4.11 below in the special case $\mathcal{F} = \Omega^p$, the sheaf of germs of holomorphic p-forms.

Theorem 4.3 *If $\mathcal{U}$ is an acyclic cover of X for the sheaf $\mathcal{F}$, then the map to the direct limit induces an isomorphism from $H^r(\mathcal{U}, \mathcal{F})$ to the Čech cohomology $H^r(X, \mathcal{F})$.*

Let X be a simplicial complex. For every Abelian group G, the Čech cohomology $H^r(X, \underline{G})$ of X with coefficients in the constant sheaf $\underline{G}$ is isomorphic to the r-dimensional simplicial cohomology of X. The idea is to associate a locally finite open cover to the simplicial complex, in an inverse procedure to forming the nerve of a cover. This is accomplished by taking, for each simplex σ, the interior of the union of all simplices incident on σ. The combinatorics of the simplicial complex are the same as that of the resulting cover.

The topological space underlying an arbitrary smooth manifold admits the structure of a simplicial complex ("every smooth manifold is triangulable"). Further, the simplicial cohomology of a simplicial complex is isomorphic to the singular cohomology of the underlying topological space. The observation of the previous paragraph implies the following.

Proposition 4.4 *Let G be an Abelian group, and let $\underline{G} \to X$ be the associated constant sheaf over a compact smooth manifold. Then the Čech cohomology $H^r(X, \underline{G})$ is isomorphic to the r-dimensional singular cohomology of X with coefficients in G.*

Let $\mathcal{F}$ be a sheaf of Abelian groups on X. A *partition of unity* of $\mathcal{F}$ subordinate to a locally finite open cover $\mathcal{U}$ is a collection of sheaf morphisms $\{\eta_\alpha\}_{\alpha \in I}$ such that

i. η_α is the zero map on a neighborhood of $X \setminus U_\alpha$ (i.e. is supported in U_α), and

ii. $\sum_{\alpha \in I} \eta_\alpha$ is the identity map of $\mathcal{F}$.

A sheaf admitting a partition of unity subordinate to every locally finite open cover of X is *fine*.

Theorem 4.5 *If $\mathcal{F}$ is a fine sheaf over X, then $H^r(X, \mathcal{F}) = 0$ for $r \geq 1$.*

Proof (Sketch) It suffices to prove that $H^r(\mathcal{U}, \mathcal{F}) = 0$ for every locally finite open cover. Take a subordinate partition of unity $\{\eta_\beta\}$. If f is an r-cochain, then set

$$(h_r f)_{\alpha_0 \cdots \alpha_{r-1}} = \sum_{\beta \in I} \eta_\beta (f_{\beta \alpha_0 \cdots \alpha_{r-1}}),$$

where $\eta_\beta(f_{\beta\alpha_0\cdots\alpha_{r-1}})$ has been extended to $U_{\alpha_0\cdots\alpha_{r-1}}$ by zero off the support of η_β. (It is an instructive exercise to write $h_1 f$ explicitly.) A straightforward calculation shows that $h_{r+1} \circ \delta_r + \delta_{r-1} \circ h_r$ is the identity map on r-cochains. If f is an r-cocycle, $r \geq 1$, then $g = h_r f$ is an $(r-1)$-cochain with $\delta_{r-1} g = f$, so f is a coboundary. □

If X is a smooth manifold, the sheaves $\mathfrak{A}$ and $\mathfrak{A}^r$ (smooth complex-valued functions and r-forms, respectively) are fine. Similarly, on a holomorphic manifold, the sheaves $\mathfrak{A}^{p,q}$ are fine. In each case, choose an ordinary smooth partition of unity and define sheaf maps by pointwise multiplication. By contrast, the sheaves $\underline{G}$, $\mathcal{O}$, and Ω^p (constants, holomorphic functions, and holomorphic p-forms) are not fine.

Let M be a smooth manifold. The sheaf $\mathfrak{A}^\times$ of germs of smooth, non-vanishing, complex-valued functions on M is a sheaf of Abelian groups under pointwise multiplication. By Proposition 4.6 below, $\mathfrak{A}^\times$ does not generally have vanishing cohomology in dimension one, hence cannot be fine. Intuitively, an attempt to define a partition of unity by "exponentiating to the power of an ordinary partition of unity" will be met with failure of the complex logarithm to be single-valued.

Proposition 4.6 *Let M be a smooth manifold. There is a natural one-one correspondance between equivalence classes of smooth complex line bundles over a complex manifold M and elements of $H^1(M, \mathfrak{A}^\times)$.*

Proof The idea is extremely simple: Transition functions for a line bundle constitute a one-cocycle with coefficients in $\mathfrak{A}^\times$, see equation (3.4), and isomorphic bundles differ by a coboundary.

In more detail, let $p : L \to M$ be a smooth complex line bundle, and let $\mathcal{U}$ be a locally finite trivializing cover. The transition functions of L take values in the Abelian group $GL(1, \mathbf{C}) = \mathbf{C}^\times$, so $\{g_{\alpha\beta}\}$ may be interpreted as a Čech 1-cochain $\psi_L \in C^1(\mathcal{U}, \mathfrak{A}^\times)$. The compatibility condition $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$ in $U_{\alpha\beta\gamma}$ is exactly the cocycle condition for ψ_L.

Suppose L' is a line bundle isomorphic to L, and let $\mathcal{U}$ be a locally finite cover of M that simultaneously trivializes L and L'. Over each open set

$U_\alpha \in \mathcal{U}$, an isomorphism $\phi : L \to L'$ is given by multiplication by a smooth, non-vanishing function ϕ_α; the collection $\{\phi_\alpha\}$ constitutes a zero-cochain ϕ, whose coboundary is immediately verified to satisfy $\psi_{L'} = \psi_L + \delta_0\phi$.

In summary, these observations prove that for every locally finite cover $\mathcal{U}$ of M by open sets, there is a natural one-one correspondance between equivalence classes of complex line bundles trivialized over $\mathcal{U}$ and elements of the Čech cohomology group $H^1(\mathcal{U}, \mathfrak{A}^\times)$. Passing to the direct limit finishes the proof. ∎

The cohomology space $H^1(M, \mathfrak{A}^\times)$ comes equipped with a group structure. The induced group operation on line bundles is the tensor product, since the tensor product of two numbers (1×1 matrices) is their ordinary product. In particular, every complex line bundle L has an inverse L^{-1} whose transition functions are reciprocals of the transition functions of L. Completely analogous remarks are true for holomorphic line bundles, which correspond to elements of $H^1(M, \mathcal{O}_M^\times)$. Algebraic geometers often speak of *invertible sheaves* instead of line bundles.

The standard homological tool for a cohomology theory is the long exact sequence associated to a short exact sequence of cochain complexes. As in singular or simplicial theory, the connecting homomorphism is defined by a diagram chase, and exactness is verified by standard arguments.

Proposition 4.7 *Let $0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{Q} \to 0$ be a short exact sequence of sheaves over X. Then there is a long exact sequence*

$$\begin{aligned} 0 \to H^0(X, \mathcal{S}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{Q}) \\ \to H^1(X, \mathcal{S}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{Q}) \to \cdots \end{aligned} \tag{4.1}$$

in sheaf cohomology.

Example 4.8 (Chern classes of line bundles) Let M be a compact, smooth manifold, and let $0 \to \underline{\mathbf{Z}} \to \mathfrak{A} \to \mathfrak{A}^\times \to 0$ be the smooth exponential sheaf sequence. The associated long exact sequence contains the terms

$$H^1(M, \mathfrak{A}) \to H^1(M, \mathfrak{A}^\times) \xrightarrow{c_1} H^2(M, \underline{\mathbf{Z}}) \to H^2(M, \mathfrak{A}).$$

Because $\mathfrak{A}$ is a fine sheaf, the first and last terms vanish. Thus the coboundary operator c_1 is an isomorphism between the group of smooth complex line bundles on M and $H^2(M, \underline{\mathbf{Z}})$, which by Proposition 4.4 is the ordinary integral singular cohomology of M. The image of a line bundle L in $H^2(M, \mathbf{Z})$ is called the *first Chern class* $c_1(L)$.

Similar considerations for holomorphic line bundles imply there is an exact sequence

$$H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M^\times) \xrightarrow{c_1} H^2(M, \underline{\mathbf{Z}}) \to H^2(M, \mathcal{O}_M).$$

However, the groups $H^i(M, \mathcal{O}_M)$ are not generally trivial, reflecting the fact that *a fixed complex line bundle may admit many inequivalent holomorphic* *structures*. This sequence is investigated further in Example 4.13 when M is a complex curve. □

4.2 Dolbeault Cohomology

Let M be a holomorphic manifold. The complex structure splits the exterior derivative d into an operator ∂ of type $(1,0)$ and an operator $\bar{\partial}$ of type $(0,1)$. As noted previously, the latter is of greater importance because holomorphic functions are "constant with respect to $\bar{\partial}$". The cohomology theory of the chain complex $\bar{\partial} : A^{p,q} \to A^{p,q+1}$ is called the *Dolbeault cohomology* of M, and is a complex analogue of the de Rham cohomology of a smooth manifold. The space $H^{p,q}_{\bar{\partial}}(M, \mathbf{C})$ of $\bar{\partial}$-closed (p,q)-forms modulo exact forms is a complex vector space.

It is not uncommon to blur the distinction between a vector bundle, its sheaf of germs of sections, and the space of global sections, especially when the bundle is not holomorphic. When precision demands, $\bigwedge^{p,q}$ denotes the bundle of smooth skew-symmetric (p,q)-tensors, $\mathfrak{A}^{p,q}$ is the sheaf of germs of smooth sections, and $A^{p,q}$ is the space of smooth global sections. If $\pi : E \to M$ is a smooth complex vector bundle, then the bundle of *E-valued (p,q)-forms* is the tensor product $\bigwedge^{p,q} \otimes E$ of $\mathfrak{A}_M$-modules. The sheaf of germs of sections of $\bigwedge^{p,q} \otimes E$ is denoted $\mathfrak{A}^{p,q}_M(E)$ and the space of global sections is denoted $A^{p,q}(M,E)$ or simply $A^{p,q}(E)$ if no confusion is likely. Recall Proposition 3.13, which asserts that if $\pi : E \to M$ is a *holomorphic* vector bundle, then there is a $\bar{\partial}$-operator acting on (p,q)-forms with values in E. These $\bar{\partial}$-operators are coboundary operators, and the resulting cohomology theory is called *E-valued Dolbeault cohomology.*

The basic property of Dolbeault cohomology—the *Dolbeault-Grothendieck Lemma*—is analogous to the Poincaré Lemma for de Rham theory; polydisks have no higher cohomology for the sheaf of (p,q)-forms.

Theorem 4.9 *Let $\Delta \subseteq \mathbf{C}^n$ be a polydisk, and let α be a smooth $\bar{\partial}$-closed (p,q) form on Δ, $q \geq 1$. Then there is a smooth $(p,q-1)$-form β on Δ with $\bar{\partial}\beta = \alpha$.*

Proof (Outline) It suffices to assume Δ is centered at the origin. The result is first shown using an integral formula for a slightly smaller polydisk. This is accomplished by an explicit calculation in one variable, together with a slightly delicate induction on dimension. Next choose an exhaustion of Δ by smaller polydisks. The requisite detail is to ensure that the solutions found on the smaller polydisks patch together. This is not automatic since the equation $\bar{\partial}\beta = \alpha$ is nowhere close to having unique solutions, but conversely is facilitated by the extreme latitude in choosing solutions. The sketched details follow.

• Smaller polydisk, $q = 1$: Let $\alpha = f(z)\,d\bar{z}$ on $\Delta = \Delta_r$, and let $r' < r$. Motivated by Stokes' Theorem, set

$$\beta(z) = \frac{1}{2\pi i} \int_{\Delta'_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

A short calculation—essentially application of Stokes' Theorem to the differential form $d\big(f(\zeta)\log|\zeta - z|^2 d\bar{\zeta}\big)$—proves that $\bar{\partial}\beta = f(z)d\bar{z}$, and this solution is unique up to an added holomorphic function. If D is a domain in $\mathbf{C}^{n-1}$, ω is a holomorphic p-form on D, and if $\alpha = f(z)\,d\bar{z}\wedge\omega$, then with β chosen as before, $\bar{\partial}(\beta(z)\wedge\omega) = \alpha$.

• Smaller polydisk, $q > 1$: Let $A_j^{p,q}$ be the space of (p, q)-forms on $\Delta_{r'}$ that do not contain $d\bar{z}^{j+1}, \dots, d\bar{z}^n$. Consider the induction hypothesis:

(H)$_j$ If $\alpha \in A_j^{p,q}$ and $\bar{\partial}\alpha = 0$, then there is a $\beta \in A_j^{p,q-1}$ with $\bar{\partial}\beta = \alpha$.

(H)$_1$ has just been established. Assume (H)$_{j-1}$, and let $\alpha \in A_j^{p,q}$ be $\bar{\partial}$-closed.

Write $\alpha = \widetilde{\alpha} + d\bar{z}^j \wedge \gamma$ with $\gamma \in A_{j-1}^{p,q-1}$, $\widetilde{\alpha} \in A_{j-1}^{p,q}$. Since $\bar{\partial}\alpha = 0$, the coefficients of γ and $\widetilde{\alpha}$ are holomorphic in $z^{j+1}, \dots, z^n$. By the $q = 1$ step, for each pair of multi-indices I and J (with $|I| = p$, $|J| = q - 1$, and each index of J at most $j - 1$), there is a function γ'_{IJ} with $\partial\gamma'_{IJ}/\partial\bar{z}^j = \gamma_{IJ}$. Set $\eta = \bar{\partial}\gamma' - d\bar{z}^j \wedge \gamma \in A_{j-1}^{p,q}$, so $\bar{\partial}(\widetilde{\alpha} - \eta) = \bar{\partial}(\alpha + \bar{\partial}\gamma') = 0$. By the induction hypothesis, there is a $\upsilon \in A_{j-1}^{p,q-1}$ with $\bar{\partial}\upsilon = \widetilde{\alpha} - \eta$. Thus $\beta = \gamma' + \upsilon$ satisfies $\bar{\partial}\beta = \alpha$.

• Enlarging the polydisk for $q \geq 2$: The above argument shows that if α is a $\bar{\partial}$-closed (p, q)-form on Δ, then for each Δ_r compactly contained in Δ, there exists a $(p, q-1)$-form β on Δ with $\bar{\partial}\beta = \alpha$ on Δ_r.

Choose an exhaustion sequence $r_k \nearrow r$, and let Δ_k denote the corresponding polydisk. Choose β_k with $\operatorname{supp}\beta_k \subset \Delta_{k+1}$ and $\bar{\partial}\beta_k = \alpha$ on Δ_k. There is a γ_k with $\bar{\partial}\gamma_k = \alpha$ in Δ_{k+1}, so $\bar{\partial}(\beta_k - \gamma_k) = 0$ in Δ_k. Again by the first part, there is a θ_k of type $(p, q-2)$ with $\bar{\partial}\theta_k = \beta_k - \gamma_k$ on Δ_{k-1}. Set $\beta_{k+1} = \gamma_k + \bar{\partial}\theta_k$, so that $\bar{\partial}\beta_{k+1} = \alpha$ on Δ_{k+1} and $\beta_k = \beta_{k+1}$ on Δ_{k-1}. The sequence $\{\beta_k\}$ thereby constructed converges uniformly on compact sets, to β by declaration.

• Enlarging the polydisk for $q = 1$: Proceed as above to choose β_k and γ_k. Use holomorphicity of $\beta_k - \gamma_k$ in Δ_k to approximate uniformly by a polynomial θ_k in Δ_{k-1}, and set $\beta_{k+1} = \gamma_k + \theta_k$. The only difference between this and the previous case is that $\beta_{k+1} - \beta_k$ does not vanish on Δ_{k-1}, but is merely uniformly small; this however is sufficient to guarantee convergence of the sequence $\{\beta_k\}$. ∎

Theorem 4.9 holds when $r = \infty$, i.e. when $\Delta = \mathbf{C}^n$. The theorem also holds when the polydisk is replaced by a product of disks and punctured (one-dimensional) disks, as is seen with easy modification of the argument by using annuli in the $q = 1$ step, or by using Laurent series to compute the Čech cohomology.

Theorem 4.10 *Let $\Delta^\times$ denote a punctured disk, and let k and ℓ be non-negative integers. Then*

$$H_{\bar\partial}^{p,q}\big((\Delta)^k \times (\Delta^\times)^\ell\big) = 0$$

for $q \geq 1$.

The analogue of Theorem 4.9 does *not* hold for punctured polydisks; as a complex vector space, $H_{\bar\partial}^{0,1}(\mathbf{C}^2 \setminus \mathbf{0}) \simeq H^1(\mathbf{C}^2 \setminus \mathbf{0}, \mathcal{O})$ is infinite-dimensional by Exercise 4.1 below. This example highlights the analytic (as opposed to topological) nature of Čech and Dolbeault cohomologies, since $\mathbf{C}^2 \setminus \mathbf{0}$ has the homotopy type of S^3, but the one-dimensional cohomology is infinite-dimensional.

A basic consequence of the Dolbeault-Grothendieck Lemma is the isomorphism between Dolbeault and Čech cohomology, usually called the *Dolbeault Theorem.*

Theorem 4.11 *Let M be a holomorphic manifold. For each pair of non-negative integers p and q, the Dolbeault cohomology $H_{\bar\partial}^{p,q}(M)$ is isomorphic to the Čech cohomology $H^q(M, \Omega^p)$.*

Proof On a holomorphic manifold, the sheaf sequence

$$0 \to \mathcal{Z}_{\bar\partial}^{p,q} \hookrightarrow \mathfrak{A}^{p,q} \xrightarrow{\bar\partial} \mathcal{Z}_{\bar\partial}^{p,q+1} \to 0 \tag{4.2}$$

is exact; surjectivity of $\bar\partial$ is the only non-obvious assertion, and is exactly the content of Theorem 4.9. Taking $q = 0$, the long exact sequence contains, for $r \geq 1$,

$$H^{r-1}(M, \mathfrak{A}_M^{p,0}) \to H^{r-1}(M, \mathcal{Z}_{\bar\partial}^{p,1}) \to H^r(M, \mathcal{Z}_{\bar\partial}^{p,0}) \to H^r(M, \mathfrak{A}_M^{p,0}). \tag{4.3}$$

When $r > 1$, the first and last terms vanish, and the middle spaces are isomorphic. Continuing inductively,

$$H^r(M, \mathcal{Z}_{\bar\partial}^{p,0}) \simeq H^{r-1}(M, \mathcal{Z}_{\bar\partial}^{p,1}) \simeq \cdots \simeq H^1(M, \mathcal{Z}_{\bar\partial}^{p,r-1}). \tag{4.4}$$

When $r = 1$, (4.3) yields the isomorphism

$$H^1(M, \mathcal{Z}_{\bar\partial}^{p,r-1}) \simeq H^0(M, \mathcal{Z}_{\bar\partial}^{p,r})/\bar\partial H^0(M, \mathfrak{A}_M^{p,r-1}),$$

which is exactly the Dolbeault cohomology space $H_{\bar\partial}^{p,r}(M)$. Replacing r by q and using the fact that $\bar\partial$-closed $(p, 0)$ forms are exactly holomorphic p-forms completes the proof. □

The argument in the proof of the Dolbeault Theorem may be used to establish a version of the Leray Theorem (Theorem 4.3 above) for the sheaf of germs of holomorphic p-forms. The idea is to prove a higher Čech cohomology is isomorphic to a quotient of two other zero-dimensional Čech cohomology spaces, which are identied with global sections independently of the cover.

Proposition 4.12 *Let $\mathcal{U}$ be a Leray cover for the sheaf Ω^p. Then*

$$H^q(\mathcal{U}, \Omega^p) \simeq H^q(M, \Omega^p).$$

Proof By hypothesis, each r-simplex $U_{\alpha_0 \cdots \alpha_r}$ has trivial higher cohomology, i.e. $\mathcal{Z}_{\bar\partial}^{p,q+1}(U_{\alpha_0\cdots\alpha_r}) = \bar\partial \mathfrak{A}^{p,q}(U_{\alpha_0\cdots\alpha_r})$. Thus there is a short exact sequence of chain complexes analogous to (4.2) but with cochains on the nerve of $\mathcal{U}$. By the same long exact sequence argument that yields (4.4), and which relies only on fineness of the sheaf $\mathfrak{A}_M^{p,q}$,

$$H^q(\mathcal{U}, \Omega^p) \simeq H^0(\mathcal{U}, \mathcal{Z}_{\bar\partial}^{p,q})/\bar\partial H^0(\mathcal{U}, \mathfrak{A}_M^{p,q-1}) \simeq H_{\bar\partial}^{p,q}(M) \simeq H^q(M, \Omega^p)$$

as claimed. □

The dimension of $H^q(M, \mathcal{F})$ is denoted $h^q(M, \mathcal{F})$; the *Hodge numbers* of a compact holomorphic manifold M are the dimensions $h^{p,q}$ of the Dolbeault spaces $H_{\bar\partial}^{p,q}(M)$.

Example 4.13 (Holomorphic line bundles on Riemann surfaces) If M is a compact Riemann surface, then $h^{1,0} = h^0(M, \Omega^1)$ is called the *genus* of M and is usually denoted by g. There is a real-linear isomorphism $H_{\bar\partial}^{1,0}(M) \simeq_{\mathbf{R}} H_{\bar\partial}^{0,1}(M)$ induced by complex conjugation on forms. By the Dolbeault Theorem, $H^0(M, \Omega^1) \simeq_{\mathbf{R}} H^1(M, \mathcal{O})$.

On a compact manifold, holomorphic functions are constant, so the sheaf morphism $\mathrm{Exp} : \mathcal{O} \to \mathcal{O}^\times$ induces a surjection $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}^\times)$. If M is a curve, then there are no non-zero $(0, 2)$-forms on M, so by the Dolbeualt Theorem $H^2(M, \mathcal{O}) = 0$. The interesting portion of the long exact sequence associated to the exponential sheaf sequence on a Riemann surface M is therefore

$$0 \to H^1(M, \mathbf{Z}) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^\times) \xrightarrow{c_1} H^2(M, \mathbf{Z}) \to 0. \tag{4.5}$$

As seen in Example 4.8, the group of smooth complex line bundles on M is isomorphic to $\mathbf{Z}$ via the Chern class map c_1. More precisely, if $L \to M$ is a complex line bundle, then the value $c_1(L)$ of the Chern class map is a 2-dimensional cohomology class, whose pairing with the fundamental class of M is an integer, called the *degree* of L:

$$\deg L = \langle c_1(L), [M] \rangle \in \mathbf{Z}.$$

Consider the Chern class map on the space $H^1(M, \mathcal{O}^\times)$ of holomorphic line bundles. The kernel of c_1 is a subgroup, consisting of degree zero line bundles. These may be regarded as non-trivial holomorphic structures on the topologically trivial complex line bundle; while the total space of a degree zero line bundle is diffeomorphic to $M \times \mathbf{C}$, a non-trivial degree zero bundle has no holomorphic sections other than the zero section. Exactness of the sequence (4.5) implies that the kernel of c_1 is isomorphic to

$$H^1(M, \mathcal{O})/H^1(M, \mathbf{Z}) =: J_0(M), \tag{4.6}$$

called the *degree zero Jacobian* of M. It is not difficult to see that the Jacobian is a compact complex g-torus: Fix a basis for the one-dimensional singular homology $H_1(M, \mathbf{Z})$; it is customary to choose curves a_i and b_j, $1 \leq i, j \leq g$, whose oriented intersection numbers are given by

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

Topologically the "a-curves" are "latitudes" around the handles of M and the "b-curves" are "longitudes" with suitable orientations. The space $H^1(M, \mathcal{O})$ is real-linearly isomorphic to the space of holomorphic one-forms, which has complex dimension g, and the space $H^1(M, \mathbf{Z})$ may be regarded as the rank-$2g$ lattice of holomorphic one-forms whose integrals over the curves a_i and b_j are integers. Such forms are said to have *integral periods.*

When $M = \mathbf{P}^1$ $(g = 0)$, the Jacobian is trivial; holomorphic line bundles are classified by their degree. When M is an elliptic curve $(g = 1)$, it turns out that M and $J_0(M)$ are isomorphic, not only as complex curves but also as Abelian groups, see Example 6.9 below.

There is a holomorphic embedding of a compact Riemann surface of genus $g \geq 1$ into its Jacobian, given by integrating holomorphic 1-forms over (real) curves. There is also an algebro-geometric interpretation in terms of "divisors" on M. This mapping generalizes the isomorphism between an elliptic curve and its degree zero Jacobian. The definition of this embedding by "period integrals" originated in the late 19th Century, while the algebro-geometric interpretation arose only in the mid-20th Century. □

4.3 Elementary Deformation Theory

The first step in studying the set of holomorphic structures on a fixed smooth manifold is to understand "infinitesimal" deformations of a holomorphic atlas. The idea and basic results are due to Kodaira and Kodaira-Spencer. Kodaira's idea was to view a holomorphic manifold as a collection of polydisks glued together with biholomorphisms. Perturbing the gluings should deform the holomorphic structure; however, some perturbations arise from global biholomorphisms. Čech theory is an appropriate tool to investigate these matters.

Let (M, J) be a fixed holomorphic manifold, and let $\mathcal{U}$ be a locally finite cover of M that is acyclic for the structure and tangent sheaves $\mathcal{O}_M$ and Ξ_M. If U_α and U_β are open sets with non-empty intersection, and if φ_α and φ_β, are the respective charts to polydisks in $\mathbf{C}^n$, then the transition function

$$g_{\alpha\beta} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta}) \tag{4.7}$$

is a biholomorphism that specifies the manner in which U_α and U_β are to be glued. A deformation of the holomorphic structure should be regarded as a one-parameter family $\{g_{\alpha\beta}(t)\}$ of transition functions, and the time derivatives $\dot{g}_{\alpha\beta}$ represent an "infinitesimal deformation" of the holomorphic structure and constitute a Čech one-cocycle with values in the tangent sheaf Ξ_M. The space of infinitesimal deformations of the complex structure of M is therefore identified with the cohomology space $H^1(M, \Xi_M)$. In the most optimistic circumstance, the set of complex structures on M which are deformation equivalent to (M, J) will be a complex manifold—the *moduli space* of complex structures on M, and the tangent space at a complex structure will be exactly the cohomology $H^1(M, \Xi)$.

There are several technical problems which may arise. First the moduli space may not be smooth. There is a purely cohomological criterion which guarantees smoothness. It may also occur that some infinitesimal deformation may not arise from an actual one-parameter family of deformations. In this event, the infinitesimal deformation is said to be *non-integrable*.

Exercises

Exercise 4.1 Let $M = \mathbf{C}^2 \backslash \mathbf{0}$. By considering the open cover $U_0 = \mathbf{C} \times \mathbf{C}^\times$, $U_1 = \mathbf{C}^\times \times \mathbf{C}$, which is acyclic for the sheaf $\mathcal{O}$, show that $H^1(M, \mathcal{O})$ is infinite-dimensional. Observe that $H^q(M, \mathcal{O})$ is trivial for $q > 1$. ◇

Exercise 4.2 Compute the Čech cohomology $H^q(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$, $q \geq 1$, using the standard atlas (which is a Leray cover for $\mathcal{O}$). Compute $H^q(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}^\times)$. (Use the fact that $H^q(\mathbf{P}^1, \mathbf{Z})$ is isomorphic to the singular cohomology of $\mathbf{P}^1$; the standard cover is *not* acyclic for the constant sheaf $\underline{\mathbf{Z}}$.) Finally, show that the first Chern class of $\mathcal{O}_{\mathbf{P}^1}(1)$ is the positive generator of $H^2(\mathbf{P}^1, \mathbf{Z})$, and conclude that *Every holomorphic line bundle over* $\mathbf{P}^1$ *is of the form* $\mathcal{O}(k)$ *for some integer* k. ◇

A similar result is true for holomorphic line bundles over $\mathbf{P}^n$, but the bookkeeping makes direct calculation of $H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) = H^{0,q}_{\bar\partial}(\mathbf{P}^n, \mathbf{C})$ tedious. However, the following result (see also Corollary 8.19 below), which gives the Dolbeault cohomology of $\mathbf{P}^n$, is plausible and is so useful that it is convenient to state it now for future use.

Proposition 4.13 *The Dolbeault cohomology of complex projective space is*

$$H^{p,q}_{\bar\partial}(\mathbf{P}^n, \mathbf{C}) = \begin{cases} \mathbf{C} & \textit{if } 0 \leq p = q \leq n \\ 0 & \textit{otherwise} \end{cases}$$

Proof (Idea) The singular cohomology of $\mathbf{P}^n$ is one-dimensional in even dimensions up to $2n$ and is zero otherwise. But for general reasons, the singular cohomology $H^r(\mathbf{P}^n, \mathbf{C})$ is isomorphic to the direct sum of the Dolbeault cohomology spaces $H^{p,q}_{\bar\partial}(\mathbf{P}^n, \mathbf{C})$ with $p+q = r$, and further $H^{p,q}_{\bar\partial}$ and $H^{q,p}_{\bar\partial}$ are real-linearly isomorphic. Proposition 4.13 follows at once. □

Exercise 4.2 Let $\mathcal{U} = \{U_\alpha\}_{\alpha=0}^n$ be the standard atlas on $\mathbf{P}^n$. Recall that the line bundle $\mathcal{O}_{\mathbf{P}^n}(k)$ has transition functions from U_α to U_β given by $g_{\alpha\beta}([Z]) = (z^\beta/z^\alpha)^k$. Use Proposition 4.13 to prove that every holomorphic line bundle on $\mathbf{P}^n$ is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(k)$ for some integer k. $\diamond$

5

Analytic and Algebraic Varieties

Many examples of smooth real manifolds arise as level sets of mappings; indeed, this is the historical source of interest in manifolds. Similarly, examples of complex manifolds arise as level sets of holomorphic mappings; the Inverse and Implicit Function Theorems have direct holomorphic analogues.

Theorem 5.1 *Let $\Delta \subset \mathbf{C}^n$ be a polydisk containing the origin, and let $f : \Delta \to \mathbf{C}^n$ be a holomorphic map with $Df(0)$ non-singular. Then there exists a polydisk $\Delta' \subset \Delta$ such that $f|_{\Delta'}$ is a biholomorphism onto its image.*

Proof By the ordinary Inverse Function Theorem in $\mathbf{R}^{2n}$, there is a polydisk $\Delta' \subset \Delta$ such that $f|_{\Delta'}$ has a smooth local inverse g and $Df(z)$ is non-singular for $z \in \Delta'$. Holomorphicity is a simple consequence of the chain rule. Indeed, let $w = f(z)$ with an obvious use of multi-index notation. Then $z = g(w) = g\big(f(z)\big)$ on Δ', and

$$0 = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

The first term vanishes since f is holomorphic, while the matrix $(\partial\bar{f}/\partial\bar{z})$ in the second term is non-singular on Δ'. Thus $\partial g/\partial\bar{w} = 0$, i.e. g is holomorphic. □

Theorem 5.2 *Let $\Delta \subset \mathbf{C}^n$ be a polydisk containing the origin, and let $f : \Delta \to \mathbf{C}^k$ be a holomorphic map with $\mathrm{rk}_{\mathbf{C}} Df(0) = k$. Then, after shrinking Δ and permuting coordinates if necessary, there exists a polydisk*

$\Delta' \subset \mathbf{C}^{n-k}$ and a holomorphic map $\varphi : \Delta' \to \mathbf{C}^k$ such that $f(z, w) = 0$ for $(z, w) \in \Delta$ if and only if $w = \varphi(z)$.

One striking difference between the smooth and holomorphic categories is a strengthening of the Inverse Function Theorem which asserts that a one-one holomorphic map is a biholomorphism. In other words, there is no analogue of the smooth map $x \in \mathbf{R} \mapsto x^3$, which is one-one but not a diffeomorphism.

Theorem 5.3 *If $f : U \to V$ is a one-one holomorphic map between open subsets of $\mathbf{C}^n$, then $f^{-1} : V \to U$ is holomorphic, i.e.* $\det Df \neq 0$.

Proof It suffices to assume $f(0) = 0$ and to prove f^{-1} is a biholomorphism on a neighborhood of 0. The strategy is to prove, by induction on n, that:

If $f : \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, and if $Df(0)$ is singular, then $Df(0) = 0$.

This is clear when $n = 1$. Suppose $f : U \to \mathbf{C}^n$ is a holomorphic map on a neighborhood of $0 \in \mathbf{C}^n$ and $Df(0)$ has rank $k < n$. By permuting coordinates if necessary, f followed by projection to $\mathbf{C}^k$ is a biholomorphism of neighborhoods of the origin in $\mathbf{C}^k$. However, restricted to the orthogonal complement $\mathbf{C}^{n-k}$, f has singular Jacobian at the origin. By the inductive hypothesis $k = 0$, i.e. the Jacobian vanishes everywhere it is singular. In this case, f maps each component of the zero locus of Df to a point. But the function $\det Df$ cannot have an isolated zero, so if the Jacobian vanishes at one point, then it vanishes at infinitely many points, so f is not one-one.
$\square$

A simple consequence of Theorem 5.2 is a criterion for determining smoothness of the zero locus in $\mathbf{P}^n$ of a single homogeneous polynomial f of degree d. It is useful to introduce, for each coordinate chart U_α on $\mathbf{P}^n$, the polynomial f_α defined by

$$f(Z^0, \ldots, Z^n) = (Z_\alpha)^d f_\alpha(z_\alpha^0, \ldots, \widehat{z_\alpha^\alpha}, \ldots, z_\alpha^n);$$

in words, f_α is obtained by "setting $Z^\alpha = 1$," and defines the same zero set as f in U_α.

Proposition 5.4 *If $f : \mathbf{C}^{n+1} \to \mathbf{C}$ is a homogeneous polynomial, and if $\nabla f(Z) \neq 0$ when $Z \neq 0$, then the zero locus of f in $\mathbf{C}^{n+1} \setminus \mathbf{0}$ is a smooth submanifold and projects to a smooth, compact submanifold $V_f \subset \mathbf{P}^n$.*

Proof Smoothness of the zero locus in $\mathbf{C}^{n+1}$ is immediate from Theorem 5.2. To show the image in $\mathbf{P}^n$ is smooth, consider the coordinate neighborhood $U_0 = \{Z^0 \neq 0\}$, with coordinates $z^\alpha = Z^\alpha/Z^0$, and write $f(Z) = (Z^0)^d f_0(z)$ on U_0. By the chain rule,

$$\nabla f(Z) = \Big(d(Z^0)^{d-1} f_0(z), (Z^0)^d \nabla f_0(z)\Big) \neq 0.$$

But $f_0(z) = 0$ exactly when $z \in V_f \cap U_0$, so $\nabla f_0(z)$ cannot vanish on $V_{f_0} = V_f \cap U_0$. By Theorem 5.2, the set $V_f \cap U_0$ is a smooth manifold. A similar argument proves $V_f \cap U_\alpha$ is smooth for $\alpha = 1, \dots, n$. Finally, V_f is a closed subset of $\mathbf{P}^n$, hence is compact. ∎

The set V_f is called an *algebraic hypersurface* of degree d. More generally, if the component functions of $f : \mathbf{C}^{n+1} \to \mathbf{C}^k$ are homogeneous polynomials of degree $d = (d_1, \dots, d_k)$, and if $\operatorname{rk}_{\mathbf{C}} Df(Z) = k$ except at $Z = 0$, then the zero locus of f projects to a closed submanifold of $\mathbf{P}^n$, called a *complete intersection* of (multi-)degree d.

Example 5.5 The *Fermat hypersurface* of degree d in $\mathbf{P}^n$ is the zero locus of the polynomial $f(Z) = (Z^0)^d + \cdots + (Z^n)^d$. The Fermat quadric in $\mathbf{P}^3$ is equivalent, after a linear change of coordinates, to the quadric surface $Q \subset \mathbf{P}^3$ defined by $Z^0Z^3 - Z^1Z^2 = 0$. The *Segré* mapping

$$[X^0 : X^1] \times [Y^0 : Y^1] \in \mathbf{P}^1 \times \mathbf{P}^1 \longmapsto [X^0Y^0 : X^0Y^1 : X^1Y^0 : X^1Y^1] \in \mathbf{P}^3$$

is a biholomorphism from $\mathbf{P}^1 \times \mathbf{P}^1$ to Q.

The *twisted cubic curve* $C \subset \mathbf{P}^3$, defined as the image of the map

$$[s : t] \in \mathbf{P}^1 \longmapsto [s^3 : s^2t : st^2 : t^3] \in \mathbf{P}^3$$

is locally a complete intersection, but is not a complete intersection. A proof of this fact requires examination of commutative-algebraic local data near the point $[0 : 0 : 0 : 1]$ and is not carried out here. □

It is often the case in algebraic geometry that families of objects are parametrized by points of a projective variety. The simplest example is the set of linear subspaces of dimension $k < n$ in $\mathbf{P}^n$, which is the Grassmannian of $(k+1)$-planes in $\mathbf{C}^{n+1}$. A property possessed by some members of such a family is *generic* if the set of members failing to satisfy the property lies in a proper, closed subvariety of the family. For example, "a generic $n \times n$ matrix is invertible" since the set of non-invertible matrices lies in the zero locus of the determinant function.

The set of algebraic hypersurfaces of degree d in $\mathbf{P}^n$ is a projective space: The set of monomials of degree d in $n+1$ variables is a vector space of dimension $\binom{n+d}{d} =: N + 1$, and two monomials define the same projective variety if and only if their quotient is a non-zero constant. The set of degree d hypersurfaces in $\mathbf{P}^n$ is therefore $\mathbf{P}^N$.

Proposition 5.6 *A generic hypersurface of degree d in $\mathbf{P}^n$ is smooth.*

5.1 The Local Structure of Analytic Hypersurfaces

By definition, an *analytic variety* is a subset V of a holomorphic manifold M that is locally the common zero locus of a finite set of analytic functions. A *hypersurface* is locally the zero locus of a single function.

Locally, an analytic hypersurface is either a manifold, or else is a finite-sheeted branched cover of a polydisk, branched along an analytic subvariety. The prototypical behavior is exhibited by the zero locus of a *Weierstrass polynomial*, namely a holomorphic function of the form

$$f(z,w) = w^d + a_{d-1}(z)w^{d-1} + \cdots + a_1(z)w + a_0(z), \tag{5.1}$$

with $z \in \mathbf{C}^{n-1}$, $w \in \mathbf{C}$, and $a_i : \Delta \subset \mathbf{C}^{n-1} \to \mathbf{C}$ holomorphic.

Theorem 5.7 *Let $f : U \subset \mathbf{C}^n \to \mathbf{C}$ be a holomorphic function on a neighborhood of the origin, not identically zero, but with $f(0) = 0$. Then there exists a polydisk Δ containing 0, a Weierstrass polynomial $g : \Delta \to \mathbf{C}$, and a holomorphic function $h : \Delta \to \mathbf{C}$ with $h(0) \neq 0$ and $f = gh$ on Δ.*

Proof After a linear change of coordinates, it may be assumed that $f(0,w)$ is not the zero function, i.e. that the w-axis is not contained in the zero locus of f. There is a non-zero constant a with $f(0,w) = aw^d + o(w^d)$ near $w = 0$. Choose $\varepsilon > 0$ so that $f(0,w) \neq 0$ for $|w| = \varepsilon$, and choose a multi-radius $r > 0$ such that $f(z,w) \neq 0$ for $|w| = \varepsilon$ and $|z| < r$. For each z with $|z| < r$, there are exactly d zeros of f in the disc $\{z\} \times \{|w| < \varepsilon\}$. (This is clear for $z = 0$, and follows for $|z| < r$ by continuity, since $f(z,w) \neq 0$ on $|w| = \varepsilon$.) Let these zeros be denoted by $b_1, \ldots, b_d$. For each fixed z, and for an arbitrary positive integer k, the Residue Theorem gives

$$\sum_{j=1}^{d} b_j^k = \frac{1}{2\pi i} \int_{|w|=\varepsilon} w^k \, \frac{1}{f(z,w)} \frac{\partial f}{\partial w}(z,w)\, dw.$$

Thus while the roots b_j are not necessarily analytic functions of z, their power sums are. Let $\sigma_1, \ldots, \sigma_d$ be the elementary symmetric polynomials and set

$$g(z,w) = w^d - \sigma_1(z)w^{d-1} + \cdots + (-1)^d\, \sigma_d(z).$$

The function g is holomorphic on the polydisc Δ where $|z| < r$ and $|w| < \varepsilon$, and vanishes exactly where f vanishes.

It remains to check that the function $h = f/g$ extends to a holomorphic function on Δ. But for each z in the disk $|z| < r$, the function $h(w) = f(z,w)/g(z,w)$ has only removable singularities, hence is defined everywhere. The integral representation

$$h(z,w) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{h(z,\xi)}{z - w}\, d\xi$$

proves that h is holomorphic in z. $\square$

Once the coordinates (z,w) have been chosen in the proof of Theorem 5.7, it is clear that the decomposition $f = gh$ is unique. In addition, the proof establishes a Riemann extension theorem for functions of several variables:

Proposition 5.8 *Let $f : \Delta \to \mathbf{C}$ be a holomorphic function on a polydisc, and assume $g : \Delta \setminus \{f = 0\} \to \mathbf{C}$ is a bounded holomorphic function. Then there is an extension $\widetilde{g} : \Delta \to \mathbf{C}$ of g.*

Let $\mathcal{O}_{n,z_0}$ denote the ring of germs of local holomorphic functions at $z_0 \in \mathbf{C}^n$. Understanding the local structure of an analytic variety $V \subset \mathbf{C}^n$ is tantamount to understanding the zero locus of a germ $f \in \mathcal{O}_n$, the ring of germs at the origin. Units in $\mathcal{O}_n$ are exactly germs that do not vanish at 0, i.e. locally convergent power series with non-zero constant term.

A (non-zero) non-unit in $\mathcal{O}_n$ is sometimes said to be the *germ of an analytic variety* at $0 \in \mathbf{C}^n$. An analytic variety V is *reduced* if the local defining functions for V vanish to first order, is *irreducible* if V is not the union of (non-empty) proper, closed subvarieties, and is *locally irreducible at* $p \in V$ if there is a neighborhood of p on which V is irreducible. Thus, an analytic variety $V \subset \mathbf{C}^n$ is reduced and irreducible at $0 \in \mathbf{C}^n$ if and only if the germ of V at 0 is an irreducible element of $\mathcal{O}_n$. Corollary 5.12 below implies that locally, an analytic hypersurface is a finite union of irreducible hypersurfaces.

Proposition 5.9 *The ring $\mathcal{O}_n$ is a unique factorization domain.*

Proof (Sketch) The assertion is true when $n = 1$, and the Weierstrass Preparation Theorem says every $f \in \mathcal{O}_n$ is the product of a unit h and a Weierstrass polynomial $g \in \mathcal{O}_{n-1}[w]$. By Gauss' lemma, the proposition follows by induction on n. □

Proposition 5.10 *If f and g are relatively prime in $\mathcal{O}_n$, then there is a polydisk Δ on which f and g define holomorphic functions whose germs are relatively prime in $\mathcal{O}_{n,z}$ for all $z \in \Delta$.*

Proof (Sketch) It suffices to take f and g to be Weierstrass polynomials in $\mathcal{O}_{n-1}[w]$; let $\gamma \in \mathcal{O}_{n-1}$ be their resultant, and let Δ be a polydisk on which f, g, and γ define holomorphic functions. If the germs of f and g have a common factor h in $\mathcal{O}_{n,z}$ for some $z \in \Delta$, then $h \,|\, \gamma$, implying $h \in \mathcal{O}_{n-1}$. Since f does not vanish identically along the w-axis, h must be the zero element in $\mathcal{O}_{n-1}$. □

The next result, the Weierstrass Division Theorem, gives as a corollary the divisibility criterion for a germ to vanish on the zero locus of another germ. This proves the assertion made above: locally an analytic hypersurface is a union of irreducible hypersurfaces.

Theorem 5.11 *Let $g \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree d in w. If $f \in \mathcal{O}_n$, then there is an $h \in \mathcal{O}_n$ and a polynomial $r \in \mathcal{O}_{n-1}[w]$ of degree $< d$ in w with $f = gh + r$.*

Proof (Sketch) Choose $\varepsilon > 0$ so that $g(0, w) \neq 0$ for $|w| = \varepsilon$, and choose a multi-radius $r > 0$ such that $g(z, w) \neq 0$ for $|w| = \varepsilon$ and $|z| < r$. For each

$z \in \mathbf{C}^{n-1}$ with $|z| < r$, define

$$h(z,w) = \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{f(z,\xi)}{g(z,\xi)}\frac{d\xi}{\xi - w}. \tag{5.2}$$

Then h is holomorphic (cf. proof of Theorem 5.7), and so is $r = f - gh$. But

$$\begin{aligned} r(z,w) &= \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{f(z,\xi)}{g(z,\xi)}\,\frac{g(z,\xi) - g(z,w)}{\xi - w}\,d\xi \\ &=: \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{f(z,\xi)}{g(z,\xi)}\,p(z,\xi,w)\,d\xi, \end{aligned}$$

with $p(z,\xi,w)$ a polynomial of degree $< d$ in w. □

Corollary 5.12 *If $f \in \mathcal{O}_n$ is irreducible and if $h \in \mathcal{O}_n$ vanishes on V_f, then $f \,|\, h$ in $\mathcal{O}_n$.*

Proof Assume without loss of generality that $f \in \mathcal{O}_{n-1}[w]$ is a Weierstrass polynomial of degree d in w. Since f is irreducible, f and $\partial f/\partial w$ are relatively prime in $\mathcal{O}_{n-1}[w]$; let $\gamma \in \mathcal{O}_{n-1}$ be their resultant. Since γ is non-vanishing on some polydisk Δ about the origin by Proposition 5.10, $f(z,\cdot)$ has distinct roots for $z \in \mathbf{C}^{n-1} \cap \Delta$.

Using Theorem 5.11, write $f = gh + r$ with $r \in \mathcal{O}_{n-1}[w]$ of degree $< d$ in w. On slices of Δ where z is fixed, the function h has the same roots as f by equation (5.2), and in particular has d distinct roots. Since r has degree $< d$ in w, $r \equiv 0$. □

5.2 Singularities of Algebraic Varieties

Let M be a holomorphic manifold, with atlas $\{U_\alpha, \varphi_\alpha\}$. Recall that a subset $V \subset M$ is an analytic subvariety if, for every $p \in M$, there is a neighborhood U of p and a finite collection of holomorphic functions f^i on U—called *local defining functions* for V in U—such that the common zero locus $\{f^i = 0\} \subset U$ is exactly $V \cap U$. An analytic subvariety is necessarily a closed subset of M, and if $p \notin V$, then it is conventional to take $U = M \setminus V$, with local defining function $f = 1$.

These definitions may be phrased globally in terms of the structure sheaf $\mathcal{O}_M$; define the *ideal sheaf* of V to be the subsheaf $\mathcal{I}_V \subset \mathcal{O}_M$ whose stalk at $p \in M$ consists of germs of holomorphic functions that vanish on V. The *structure sheaf* of V is defined by the exact sequence

$$0 \to \mathcal{I}_V \to \mathcal{O}_M|_V \to \mathcal{O}_V \to 0.$$

The ideal and structure sheaves of an analytic subvariety are coherent.

In order to study locally the singularities of a variety V, it suffices to work near the origin in $\mathbf{C}^n$. Let $\{f^i\}_{i=1}^k$ be local defining functions for V at $\mathbf{0}$, chosen so that there is a neighborhood U of $\mathbf{0}$ on which the differentials df^i are not everywhere linearly dependent. The origin is a *smooth point* of V if the map $Df(\mathbf{0}) : \mathbf{C}^n \to \mathbf{C}^k$ has rank k. By the Implicit Function Theorem, this is equivalent to V being a manifold at $\mathbf{0}$. Otherwise $\mathbf{0}$ is a *singular point* of V; the set of all singular points is denoted $\operatorname{Sing} V$, and the *smooth locus* $V^* \subset V$ is the complement of the singular set. The smooth locus is a holomorphic manifold and the dimension of V is defined to be the dimension of V^*.

Proposition 5.13 *Let V be an analytic subvariety of a holomorphic manifold M. Then the set of singular points* $\operatorname{Sing} V$ *is an analytic subvariety of M.*

Proof It suffices to work locally. Suppose $\mathbf{0} \in V$ is a singular point, and let $U \subset \mathbf{C}^n$ be a neighborhood of $\mathbf{0}$ on which $\{f^i\}_{i=1}^k$ is a set of local defining functions for V. The intersection $U \cap \operatorname{Sing} V$ is cut out by $\{f^i\}$ together with the set of determinants of $k \times k$ minors of $Df : U \to T^*\mathbf{C}^n$. The latter are local holomorphic functions in U. $\square$

Proposition 5.14 *An analytic variety V is irreducible if and only if the smooth locus V^* is connected.*

Proof (Sketch) If $V = W_1 \cup W_2$ is a union of non-empty, proper closed subvarieties, then $W_1 \cap W_2 \subset \operatorname{Sing}(W_1 \cup W_2)$, so the sets $W_i \setminus (\operatorname{Sing} V)$ separate V^*. Conversely, the closure of a component of V^* is a proper, non-empty, closed subvariety of V. $\square$

Locally, a hypersurface $V = V_f$ is cut out by a single holomorphic function f. The *multiplicity* of V at p is defined to be the order of vanishing of f at p:

$$\operatorname{mult}_p(V) = \max \left\{ m \in \mathbf{N} \;\middle|\; \frac{\partial^I f}{\partial z^I}(p) = 0 \text{ for all } I \text{ with } |I| < m \right\}. \tag{5.3}$$

If f is expanded as a power series centered at p with homogeneous terms f_i of weight i in z, then $m = \operatorname{mult}_p(V)$ is the degree of the first non-vanishing term: $f = f_m + f_{m+1} + \cdots$. The *tangent cone* of V at p is the zero locus of f_m in $T_p\mathbf{C}^n$, which is alternately characterized as the set of tangent vectors

$$T_pV = \left\{ \sum_{i=1}^n X^i \frac{\partial}{\partial z^i} \;\middle|\; f_m(X^1, \dots, X^n) = 0 \right\}.$$

For example, if V is smooth at p, then T_pV is the usual tangent space, while the tangent cone at the origin of the zero locus of $f(x, y) = y^2 - x^2 - x^3$ is the union of the lines $y = \pm x$.

Affine Varieties and Coordinate Rings

Let R be a commutative ring with unit element. A *radical ideal* $\mathfrak{r} \subset R$ is an ideal with the property that if $x^n \in \mathfrak{r}$ for some $n > 1$, then $x \in \mathfrak{r}$. In words, $\mathfrak{r}$ contains the nth roots in R of all its elements.

Let $V \subset \mathbf{C}^n$ be an *algebraic* subvariety, that is, an analytic variety whose local defining functions are polynomials. By Proposition 6.4 below, V is the common zero set of finitely many polynomials on $\mathbf{C}^n$. The ideal $\mathfrak{r}(V) = \{f \in \mathbf{C}[z] : f|_V = 0\}$ of polynomials that vanish along V is the radical of $\mathcal{I}$, the ideal of global sections of the ideal sheaf of V, and the *affine coordinate ring* of V is the quotient $\mathcal{O}(V) = \mathbf{C}[z]/\mathfrak{r}(V)$. Conversely, given a radical ideal $\mathfrak{r} \subset \mathbf{C}[z]$, the associated *affine algebraic variety* is $V(\mathfrak{r}) = \{z \in \mathbf{C}^n \mid f(z) = 0 \text{ for all } f \in \mathfrak{r}\}$. There is a one-to-one correspondence between radical ideals in $\mathbf{C}[z]$ and affine algebraic varieties in $\mathbf{C}^n$. Under this correspondance, prime ideals correspond to irreducible varieties, and maximal ideals correspond to points.

A *projective algebraic variety* is a (necessarily closed) subset $V \subset \mathbf{P}^n$ whose intersection V_α with the affine coordinate chart U_α is an affine algebraic variety for each $j = 0, \dots, n$. Equivalently, a projective algebraic variety is the common zero locus of finitely many homogeneous polynomials in the homogenerous coordinates on $\mathbf{P}^n$.

In algebraic geometry, one often works with the *Zariski topology* on an affine algebraic variety. A Zariski-open set in an algebraic variety $V \subset \mathbf{C}^n$ is the complement of an algebraic subvariety of V. Regarding $\mathbf{C}$ as an affine variety, Zariski-open sets are exactly sets with finite complement. The Zariski topology on a complex variety of positive dimension is never Hausdorff. One advantage of the Zariski topology is that algebraic objects (e.g. meromorphic functions) are determined by their restriction to an arbitrary non-empty open set. With respect to their Zariski topologies, $\mathbf{P}^1$ and an elliptic curve E are not even locally isomorphic.

Example 5.15 The basic objects of algebraic geometry are "schemes." Under the correspondance just outlined, affine varieties correspond to radical ideals $\mathfrak{r} \subset \mathbf{C}[z]$, via their coordinate ring $\mathcal{O}(V)$. The points of V correspond to maximal ideals in $\mathcal{O}(V)$, and all questions about V may be phrased in terms of $\mathcal{O}(V)$.

It is natural to ask what sort of geometric object can be associated to an arbitrary commutative ring R with unity. To this end, let $\operatorname{spec} R$ denote the set of prime ideals of R. For each ideal $I \subset R$, the associated *basic closed set* is defined to be the set of prime ideals containing I; the complements of basic closed sets form a basis for the Zariski topology on $\operatorname{spec} R$. Finally, the topological space $\operatorname{spec} R$ is endowed with a sheaf $\mathcal{O}$ of rings; the stalk of $\mathcal{O}$ at a point (i.e. prime ideal) $\mathfrak{p} \in \operatorname{spec} R$ is the localization $R_\mathfrak{p}$. The topological space $\operatorname{spec} R$ together with the sheaf of rings is called an *affine scheme*.

Generally, a topological space X together with a sheaf of rings $\mathcal{O}_X$ is a scheme if every point $p \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X(U))$ is isomorphic to some affine scheme $\operatorname{spec} R$. Various geometric constructions (intersection theory, for example) work better in the category of schemes than in the category of varieties.

Maximal ideals in R correspond to *closed points* of $\operatorname{spec} R$. The "additional" points corresponding to non-maximal prime ideals $\mathfrak{p}$ are *generic points*, whose closures contain, among other things, the closed points of the subscheme cut out by $\mathfrak{p}$. An illustrative example is furnished by $R = \mathbf{C}[x, y]$, whose spectrum is the affine plane. Points $(x_0, y_0) \in \mathbf{C}^2$ are in one-to-one correspondance with maximal ideals $(x - x_0, y - y_0) \subset R$. Every irreducible polynomial, e.g. $x^2 - y$, generates a prime ideal; the closure of the point $\mathfrak{p} = (x^2 - y)$ contains all the closed points lying on the parabola $y = x^2$. □

An Introduction to Desingularization

One general goal in algebraic geometry is to find, for a given variety V, a variety $\widetilde{V}$ which is isomorphic to V away from a Zariski-closed set, and which has "nicer" singularities. A prototypical example is the following desingularization result.

Theorem 5.16 *Let $V \subset \mathbf{C}^n$ be an algebraic variety. Then there exists a smooth algebraic variety $\widetilde{V} \subset \mathbf{C}^N$ and a proper, surjective, holomorphic map $\pi : \widetilde{V} \to V$ which—on the complement of a divisor—is a biholomorphism onto its image.*

This "desingularization" can in fact be accomplished for arbitrary varieties defined over fields of characteristic zero, as shown by H. Hironaka in the late 1950's. Bierstone and Milman have recently given an explicit procedure for desingularization in characteristic zero. A more elementary—but already non-trivial—result can be obtained from commutative algebra via the correspondance between varieties and ideals, see Proposition 5.21 below.

Let $V \subset \mathbf{C}^n$ be an affine (algebraic) variety. To each $p \in V$, let $\mathfrak{m}_p$ denote the corresponding maximal ideal in the coordinate ring $\mathcal{O}(V)$. The *local ring* of V at p is the localization $\mathcal{O}_p(V)$ of $\mathcal{O}(V)$ at $\mathfrak{m}_p$. An element of the local ring should be regarded as a germ of a regular function at p; the image of $f \in \mathcal{O}(V)$ in $\mathcal{O}(V)/\mathfrak{m}_p \simeq \mathbf{C}$ is the value of f at p.

Generally, a *local ring* is a Noetherian, commutative ring with unity having a unique maximal ideal. The local ring of a variety is indeed a local ring in this sense. The *Krull dimension* of a commutative ring R is one less than the maximum length of a chain of nested prime ideals in R. For example, the polynomial ring $\mathbf{C}[z]$ in n variables has Krull dimension n. A local ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ is *regular* if $\dim \mathcal{O} = \dim(\mathfrak{m}/\mathfrak{m}^2)$.

Proposition 5.17 *A variety V is non-singular at p if and only if $\mathcal{O}_p(V)$ is a regular local ring.*

Example 5.18 Let V be the *cusp curve* $x^3 = y^2$ in $\mathbf{C}^2$. The affine coordinate ring of $\mathbf{C}^2$ is $\mathbf{C}[x,y]$, and the local ring at the origin p is $\mathfrak{a}_p = (x,y)$, the maximal ideal of $\mathbf{C}[x,y]$. The ideal $\mathfrak{a}_p^2$ is (x^2, xy, y^2), and the quotient is $\mathfrak{a}_p/\mathfrak{a}_p^2 = \mathbf{C}[x,y]$, so $\mathbf{C}^2$ is smooth at the origin, as expected.

By contrast, let $\mathfrak{m}_p = (x,y) \subset \mathcal{I}_V$ be the maximal ideal of the affine coordinate ring $\mathbf{C}[x,y]/(x^2 - y^3)$. Then

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = (x,y)/(x^2, xy, y^2, x^2 - y^3) = (x,y)/(x^2, xy, y^2),$$

which has Krull dimension two, while V has dimension one. Thus the local ring $\mathcal{O}_p(V)$ is not regular, so the cusp curve V is not smooth at the origin. □

Proposition 5.19 *Let $\mathfrak{a}_p$ be the maximal ideal of a regular local ring $\mathcal{O}_p$, let V be a variety containing p, and let $\mathfrak{m}_p$ be the maximal ideal in the coordinate ring of V at p. Then $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{a}/(\mathfrak{a}^2 + \mathcal{I}_V)$.*

Proof Let $\{f^i\}_{i=1}^k$ be local defining functions for V at p. Then the rank of the Jacobian $Df(p)$ is equal to $\dim(\mathfrak{a}_p/\mathfrak{a}_p^2) - \dim(\mathfrak{m}_p/\mathfrak{m}_p^2)$. But

$$\dim(\mathfrak{a}_p/\mathfrak{a}_p^2) - \dim(\mathfrak{m}_p/\mathfrak{m}_p^2) = \dim(\mathfrak{a}_p^2 + \mathcal{I}_V)/\mathfrak{a}_p^2,$$

so V is non-singular in the analytic sense if and only if the dimension is as expected. ▯

Normal Varieties

Let X be an affine algebraic variety. The quotient field $K(X)$ of the integral domain $\mathcal{O}(X)$ is the *function field* of X; elements of $K(X)$ are "meromorphic functions" on X. For each $p \in X$, there are inclusions $\mathcal{O}(X) \subset \mathcal{O}_p(X) \subset K(X)$.

If A and B are integral domains, then a ring homomorphism $\phi : A \to B$ induces a morphism spec $B \to$ spec A. Conversely, if X and Y are affine varieties, then a regular morphism $f : X \to Y$ induces a ring homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$. If f is surjective, then the induced map on coordinate rings is injective.

An element $x \in B$ is *integral over A* if there exists a monic polynomial $q \in A[\xi]$ with $q(x) = 0$. If every $x \in B$ which is integral over A actually lies in A, then A is *integrally closed* in B.

Lemma 5.20 *An integral domain R is integrally closed in its quotient field Q if and only if the localization $R_\mathfrak{m}$ is integrally closed in Q for every maximal ideal $\mathfrak{m}$.*

If X is irreducible, then $K(X)$—the quotient field of $\mathcal{O}(X)$—is the quotient field of $\mathcal{O}_p(X)$ for every $p \in X$. A variety X is *normal at p* if $\mathcal{O}_p(X)$ is integrally closed in $K(X)$, and is *normal* if normal at p for every $p \in X$.

By Lemma 5.20, X is normal if and only if $\mathcal{O}(X)$ is integrally closed in $K(X)$.

A *birational map* $f : X \to Y$ is a map which induces an isomorphism $K(Y) \to K(X)$ of function fields. In other words, a birational map is an isomorphism off a Zariski-closed set. A *finite* morphism is a morphism with finite fibres; more concretely, the preimage of each point is a finite set. Branched covers are finite, while "quadratic transformations" (blow-ups) are not. A *desingularization* of Y is a surjective, birational mapping $f : X \to Y$ with X a smooth variety. A weaker but more elementary construction gives a birational map $f : X \to Y$ with X normal: Let X be an irreducible affine variety, and let $\overline{\mathcal{O}(X)}$ be the integral closure of $\mathcal{O}(X)$ in $K(X)$. The associated variety, together with the induced map to X, is the *normalization* of X.

Proposition 5.21 *The variety $\widetilde{X}$ of closed points of* $\operatorname{spec}\overline{\mathcal{O}(X)}$ *is a normal variety endowed with a finite, surjective, birational morphism $\pi : \widetilde{X} \to X$.*

Theorem 5.22 *A regular local ring is integrally closed in its quotient field.*

By definition, X is smooth at p if and only if $\mathcal{O}_p(X)$ is a regular local ring, while X is normal at p if and only if $\mathcal{O}_p(X)$ is an integrally closed local ring. By Theorem 5.22, if X is smooth at p, then X is normal at p.

Proposition 5.23 *If $p \in X$ is a normal point, and if $f \in K(X)$ is regular on a deleted neighborhood of p, then f extends to p, i.e. $f \in \mathcal{O}_p(X)$.*

Example 5.24 Let $X = \{x^3 = y^2\} \subset \mathbf{C}^2$ be the cusp curve. The function $t = f(x, y) = y/x$ restricts to a meromorphic function on $X \setminus 0$, but $t^2 - x = 0 \in \mathcal{O}(X)[t]$, so t cannot be extended to a regular function on X. Thus the origin is not a normal point of X.

Intuitively, the normalization described in Proposition 5.21 "adds in" meromorphic functions which are defined away from a point. More precisely, the coordinate ring of $\widetilde{X}$ is

$$\mathcal{O}(X)[t]/(t^2 - x) = \Big(\mathbf{C}[x, y]/(y^2 - x^3)\Big)[t]/(t^2 - x),$$

and the normalization is accomplished by mapping the (x, t)-plane to the (x, y)-plane by $(x, t) \longmapsto (x, xt)$. Under this mapping, the equation $x^3 - y^2 = 0$ pulls back to $x - t^2 = 0$, whose zero locus is a smooth curve. This mapping is exactly the blow up at the origin expressed in an affine chart. $\square$

Proposition 5.25 *Let X be a normal variety. Then* $\operatorname{Sing} X$ *has codimension ≥ 2 in X.*

Consequently, if a curve is normal at p, then p is a smooth point. A normal variety of arbitrary (positive) dimension is locally irreducible.

Let X be a normal variety, $K = K(X)$ the function field, and let $L \to K$ be a finite field extension. The coordinate ring $\mathcal{O}(X)$ has an integral closure $\overline{\mathcal{O}(X)}$ in L, whose spectrum $\widetilde{X}$ is equipped with a finite, surjective morphism to X. Conversely, if $f : \widetilde{X} \to X$ is a finite, surjective morphism to a normal variety X, then the field extension $[K(\widetilde{X}), K(X)]$ is finite, and the integral closure of $\mathcal{O}(X)$ in $K(\widetilde{X})$ is the coordinate ring of $\widetilde{X}$.

The *degree* of f is by definition the degree $[L : K]$ of the field extension. For every $x \in X$, $\#\{f^{-1}(x)\} \leq \deg f$, and the maximum is achieved on a Zariski-open set. The algebraic subvariety

$$B = \{x \in X \mid \#\{f^{-1}(x)\} < \deg f\}$$

is called the *ramification locus* of f. The restriction $f : \widetilde{X} \setminus f^{-1}(B) \to X \setminus B$ is an unbranched covering, and the covering transformations permute points in the fibres of f.

Proposition 5.26 *The group of deck transformations acts transitively on fibres if and only if* $f_* \pi_1(\widetilde{X}) \lhd \pi_1(X)$, *if and only if the extension* $L \to K$ *is Galois.*

If $B \subset Y$ is normal and of pure codimension one, then to each subgroup of $\pi_1(Y \setminus B)$ is associated a covering space X which inherits the structure of an algebraic subvariety. The following is due to Enriques, Grauert-Remmert, and Grothendieck.

Theorem 5.27 *With the above notation, there is a unique normal algebraic variety* $\widetilde{Y}$ *which "completes"* X *and which admits a finite map to* Y *such that the diagram*

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ Y \setminus B & \hookrightarrow & Y \end{array}$$

commutes.

The set $\widetilde{B} \subset f^{-1}B \subset \widetilde{X}$ on which *Df* is not of maximal rank is the *branch locus* of f. The following remarkable result on the structure of the branch locus is due to Zariski.

Theorem 5.28 *The branch locus of* f *is of pure codimension one, that is, every irreducible component of* $\widetilde{B}$ *has codimension one.*

A variety X is *projectively normal* if there is a surjection $K(\mathbf{P}^n) \to K(X)$. The following is known as the *Stein Factorization Theorem.*

Theorem 5.29 *Let* $f : X \to Y$ *be a surjective, finite map of normal varieties of the same dimension, and let* $\widetilde{Y}$ *be the normalization of* Y *in* $K(X)$.

Then the natural map $X \to \widetilde{Y}$ *is a surjective, birational map with connected fibres, and the natural map* $\widetilde{Y} \to Y$ *is a finite, surjective morphism.*

6

Divisors, Meromorphic Functions, and Line Bundles

Let $M \subset \mathbf{P}^N$ be a holomorphic manifold embedded in a projective space. A *hyperplane section* of M is the intersection of M with a hyperplane in $\mathbf{P}^N$. The hyperplane sections of M carry information about the embedding. Conversely, given an arbitrary holomorphic manifold it is of interest to describe and classify "hyperplane sections" intrinsically. The relevant objects are called divisors; divisors, meromorphic functions, and line bundles are three ways of regarding essentially the same data from geometric, algebraic, and topological points of view.

6.1 Divisors and Line Bundles

Let M be a holomorphic manifold. A *divisor* D on M is a locally finite sum of closed, reduced, irreducible analytic hypersurfaces (the *components* of D) with non-zero integer coefficients. "Closed" means closed as subsets in the complex topology, "sum with integer coefficients" should be taken in the spirit of free Abelian groups, with the distinction that the sum here may be infinite (meaning there may be infinitely many components), and "locally finite" means that every $x \in M$ has a neighborhood U that intersects only finitely many components. A divisor is *effective* if every component has positive coefficient. An effective divisor is defined locally by a holomorphic function ϕ; the function ϕ vanishes along a union of irreducible analytic hypersurfaces, and the integer attached is the order of vanishing.

If M itself is compact, then a divisor is exactly an element of the free Abelian group on the set of closed, irreducible analytic hypersurfaces. The group of divisors is denoted $\operatorname{div}(M)$. The *support* of a divisor D is the union of the components of D.

Example 6.1 Let M be a compact Riemann surface. A divisor on M is a finite set of points weighted with integers. In other words, a divisor may be regarded as a sum $D = \sum_{x \in M} m_x \cdot x$, where the coefficient $m_x \in \mathbf{Z}$ is understood to be zero except for at most finitely many $x \in M$. The *degree* of D is defined to be $\deg D = \sum m_x$. □

Example 6.2 A divisor on $\mathbf{P}^n$ is a finite sum of algebraic hypersurfaces with integer coefficients. (Certainly, every such sum is a divisor; the converse is essentially Theorem 6.14 below.)

If $N \subset M$ is a closed, holomorphic submanifold and D is a divisor on M, then $N \cap D$ (taken in the obvious sense) is a divisor on N. Thus, projective manifolds "have many divisors." At the other end of the spectrum, there exist compact manifolds having *no* divisors. Generic compact tori provide examples of this counterintuitive behaviour. □

Assume D is an effective divisor on M, and let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover for which there exist holomorphic functions $\phi_\alpha \in \mathcal{O}(U_\alpha)$ that locally cut out D. If $U_{\alpha\beta}$ is non-empty, then there is a non-vanishing holomorphic function $\psi_{\alpha\beta}$ on $U_{\alpha\beta}$ such that

$$\psi_{\alpha\beta} = \frac{\phi_\beta}{\phi_\alpha}\Big|_{U_{\alpha\beta}}.$$

The collection $\{\psi_{\alpha\beta}\}$ is therefore a 1-cocycle of the nerve of $\mathcal{U}$ with coefficients in $\mathcal{O}_M^\times$, so an effective divisor D represents an element $[D] \in H^1(M, \mathcal{O}_M^\times)$. As noted in Cha[pter 4, this cohomology group may be interpreted as the group of holomorphic line bundles on M; addition of divisors corresponds to taking tensor products of line bundles. If $-D$ is effective, then $[-D]$ may be interpreted as the line bundle $[D]^* = [D]^{-1}$ dual to $[D]$. Finally, every divisor D is uniquely a difference of effective divisors, say $D = D_+ - D_-$, so an arbitrary divisor D represents the line bundle

$$[D] = [D_+] \otimes [D_-]^*.$$

This provides the following partial dictionary between divisors and holomorphic line bundles.

Proposition 6.3 *The map* $D \mapsto [D]$ *is a group homomorphism from* $\operatorname{div} M$ *to* $H^1(M, \mathcal{O}_M^\times)$.

This homomorphism is neither injective nor surjective in general; distinct divisors may give equivalent bundles, and not every holomorphic line bundle arises from a divisor. Happily, it will presently be possible to give a simple characterization of when these possibilities occur.

6.2 Meromorphic Functions

A meromorphic function on a Riemann surface is a holomorphic map to $\mathbf{P}^1$. This simple definition does not extend to holomorphic manifolds of dimension greater than one. A better approach is to regard meromorphic functions as "functions that are locally a quotient of holomorphic functions." Making this idea precise is not difficult, and the corresponding notion reduces to the expected one for Riemann surfaces. Explicitly, the difference between Riemann surfaces and higher-dimensional manifolds is that functions of one variable that vanish simultaneously have a common factor, while in two or more variables this is no longer automatic.

Let $\mathcal{M}_{\mathrm{pre}}$ be the presheaf on M whose field of sections over an open set $U \subset M$ is the quotient field of the integral domain $\mathcal{O}_M(U)$, and let $\mathcal{M}$ denote the completion. A *meromorphic function* on M is a global section of the sheaf $\mathcal{M}$. A non-zero meromorphic function f is therefore locally the quotient of holomorphic functions in the following sense: For every $x \in M$, there exists a neighborhood U of x and a pair of relatively prime holomorphic functions ϕ_0 and ϕ_∞ (in particular, neither is identically zero) such that f is represented on U by ϕ_0/ϕ_∞. The field of global sections of $\mathcal{M}$ is called the *function field* of M, and is often denoted $\mathcal{K}(M)$.

Up to multiplication by a non-vanishing local holomorphic function, ϕ_0 and ϕ_∞ depend only on $f \in \mathcal{M}$. Thus there is a *zero divisor* $(f)_0$, locally defined by ϕ_0, and a *polar divisor* $(f)_\infty$, locally cut out by ϕ_∞. The divisor $(f) = (f)_0 - (f)_\infty$ is the *principal divisor associated to* f. The *indeterminacy set* of f is the intersection of the zero and polar divisors of f, and is therefore an analytic variety of codimension two. Away from the indeterminacy set, f may be regarded as a holomorphic map to $\mathbf{P}^1$.

Proposition 6.4 *Every analytic hypersurface $D \subset \mathbf{C}^n$ is the zero locus of an entire function. Every meromorphic function f on $\mathbf{C}^n$ is a quotient of entire functions.*

Proof By Theorem 4.9, $H^q(\mathbf{C}^n, \mathcal{O}) = 0$ for $q \geq 1$. Moreover, $\mathbf{C}^n$ is contractible, so $H^{q+1}(\mathbf{C}^n, \underline{\mathbf{Z}}) = 0$ for $q \geq 0$. The long exact sequence induced by the exponential sheaf sequence contains

$$H^q(\mathbf{C}^n, \mathcal{O}) \to H^q(\mathbf{C}^n, \mathcal{O}^\times) \to H^{q+1}(\mathbf{C}^n, \underline{\mathbf{Z}}),$$

which for $q = 1$ implies $H^1(\mathbf{C}^n, \mathcal{O}^\times) = 0$. In particular the cocycle $[D]$, represented by a locally finite cover $\mathcal{U} = \{U_\alpha\}$ and local holomorphic functions ϕ_α on U_α, is a coboundary. After refining $\mathcal{U}$ if necessary, there exist local *non-vanishing* holomorphic functions h_α such that (omitting restrictions to $U_{\alpha\beta}$)

$$\frac{\phi_\alpha}{\phi_\beta} = \psi_{\alpha\beta} = \frac{h_\beta}{h_\alpha}.$$

The local functions $\{h_\alpha \phi_\alpha\}$ agree on overlaps, and therefore define an entire function.

The second assertion is now immediate: Choose entire functions cutting out the zero and polar divisors of f. ∎

Divisors D and D' on M are *linearly equivalent* if $D - D'$ is principle, that is, if there is a meromorphic function f on M with $D - D' = (f)$. On $\mathbf{P}^n$, a quotient of homogeneous polynomials of the same degree (in the homogeneous coordinates) defines a meromorphic function. Therefore, two algebraic hypersurfaces of the same degree define linearly equivalent divisors.

It is sometimes useful to regard a divisor as a global section of the quotient sheaf $\mathcal{M}^\times/\mathcal{O}^\times$; this highlights the group structure and functorial properties of $\operatorname{div} M$. The subgroup consisting of principal divisors is often denoted $\operatorname{div}_0(M)$, and the quotient

$$\operatorname{Pic}(M) := \operatorname{div}(M)/\operatorname{div}_0(M)$$

("divisors modulo linear equivalence) is called the *Picard group* of M. By Proposition 6.6 below, the Picard group of M can be viewed as the set of holomorphic line bundles on M that admit a "meromorphic section."

6.3 Sections of Line Bundles

Let $p : L \to M$ be a holomorphic line bundle. A *holomorphic section* of L is a holomorphic map $\sigma : M \to L$ with $p\,\sigma =$ identity. When a line bundle is presented as a 1-cocycle $\{\psi_{\alpha\beta}\}$, a section is realized concretely as a collection of holomorphic functions $\{\phi_\alpha\}$ satisfying $\phi_\alpha\psi_{\alpha\beta} = \phi_\beta$. For this reason, holomorphic sections are sometimes called *twisted functions*. If $L = [D]$ comes from a divisor, then $\mathcal{O}_M(D) = \mathcal{O}(L)$ is used to denote the sheaf of germs of sections of L. The following is immediate:

Lemma 6.5 *If $L = [D]$ is induced by an effective divisor, then each collection $\{\phi_\alpha\}$ of local defining functions for D gives a holomorphic section of L.*

A *meromorphic section* of a line bundle is defined to be a collection of local meromorphic functions $\{f_\alpha\}$ satisfying the compatibility condition $f_\alpha\psi_{\alpha\beta} = f_\beta$. In other words, a meromorphic section of L is a global section of the sheaf $\mathcal{O}(L)\otimes\mathcal{M}$. From Lemma 6.5, it is easy to derive a satisfactory supplement to Proposition 6.3.

Proposition 6.6 *A line bundle $L \in H^1(M, \mathcal{O}_M^\times)$ is of the form $[D]$ if and only if L admits a meromorphic section. Two line bundles $L = [D]$ and $L' = [D']$ are holomorphically equivalent if and only if $D - D'$ is principal.*

Proof Since every divisor is a difference of effective divisors, Lemma 6.5 implies that local defining functions for a divisor D may be viewed as a meromorphic section of $[D]$. Conversely, if L admits a meromorphic section, then the divisor D of this section induces L.

To see the second part, observe that $L \simeq L'$ if and only if $L^{-1} \otimes L'$ is holomorphically trivial, and a meromorphic section of the trivial line bundle is an ordinary meromorphic function. ∎

Example 6.7 Proposition 4.13 gives the cohomology of $\mathbf{P}^n$. Together with the Dolbeault Theorem, this implies that

$$H^1(\mathbf{P}^n, \mathcal{O}) = H^2(\mathbf{P}^n, \mathcal{O}) = 0.$$

From the long exact cohomology sequence of the exponential sheaf sequence on $\mathbf{P}^n$, the first Chern class map $c_1 : H^1(\mathbf{P}^n, \mathcal{O}^\times) \to H^2(\mathbf{P}_n, \mathbf{Z})$ is an isomorphism (cf. Example 4.13). Consequently, if H denotes the hyperplane class in $H_{2n-2}(\mathbf{P}^n, \mathbf{Z})$, then every line bundle on $\mathbf{P}^n$ comes from a divisor $D = dH$ for some integer d, called the *degree* of the bundle. □

Example 6.8 On a generic compact torus, only the trivial bundle comes from a divisor—the empty divisor—since there *are* no other divisors. The meromorphic sections are exactly constant functions. □

Example 6.9 Let E be an elliptic curve, and fix $0 \in E$. For each $x \in E$, the divisor $(0) - (x)$ has degree zero, but is principal if and only if $x = 0$. This may be seen with a bit of elementary complex analysis. If $f : E \to \mathbf{P}^1$ were a meromorphic function with principal divisor $(0)-(x)$, then f would have a simple pole at x and no other poles. The meromorphic 1-form $f(z)\,dz$ would therefore have non-zero total residue, which is impossible. Consequently, each point $x \in E$ induces a topologically trivial holomorphic line bundle, that is *holomorphically non-trivial* (i.e., has no meromorphic section) unless $x = 0$. By Example 4.13, these are exactly the degree zero holomorphic line bundles on E.

A choice of $0 \in E$ defines a group law with neutral element 0 (coming from a realization of E as a quotient $\mathbf{C}/\Lambda$). The lattice determines a $\wp$-function, and there is a holomorphic embedding $\phi : E \hookrightarrow \mathbf{P}^2$ defined in Example 1.9 whose image is a smooth cubic curve. The following illustrates the beautiful interplay between complex analysis, projective geometry, and algebraic geometry.

Proposition 6.10 *A divisor $\sum k_i(x_i)$ on E is principal if and only if $\sum k_i x_i = 0$ as elements of E. A divisor $(x_1) + (x_2) + (x_3)$ is the polar divisor of a meromorphic function on E if and only if the images of the points x_i under ϕ are collinear in $\mathbf{P}^2$.*

Proof (Sketch) Let $f : E = \mathbf{C}/\Lambda \to \mathbf{P}^1$ be a meromorphic function; the doubly-periodic meromorphic function on $\mathbf{C}$ that induces f will also be denoted f. Consider the meromorphic 1-form

$$\eta = \zeta \frac{f'(\zeta)}{f(\zeta)}\,d\zeta$$

on C. By the residue theorem, the sum of the residues of η on E is zero. To calculate the sum of the residues, choose a fundamental parallelogram $R \subset \mathbf{C}$ with edges ω_1 and ω_2 so that all the zeros and poles of f lie in the *interior* of R, and integrate over the boundary. The residue of η at a zero z of f of order k is kz, while the residue at a pole p of order k is $-kp$. Furthermore, cancellation over opposite edges of ∂R implies that

$$\int_{\partial R} \eta \in \Lambda$$

(i.e., is an integral linear combination of ω_1 and ω_2). In summary, if the principle divisor of f is $\sum k_i(x_i)$, then the total residue of η is $\sum k_i x_i = 0$. The converse follows from addition theorems for the $\wp$-function, see Ahlfors, *Complex Analysis*, p. 277. Intuitively, the condition that a divisor sum to zero (as an element of E) is an integrability condition for finding f.

The second part is an easy consequence of the principle that every meromorphic function on a smooth algebraic hypersurface extends to a meromorphic function on $\mathbf{P}^n$; this will be proven in Chapter 10. If $(x_1)+(x_2)+(x_3)$ is the polar divisor of a meromorphic function $f : E \to \mathbf{P}^1$, then f extends to a quotient of linear forms on $\mathbf{P}^2$, so the poles (and zeros) of f are collinear in $\mathbf{P}2$. ∎

A restatement of the first assertion in Proposition 6.10 is worth mentioning. As in equation (4.6) above, let $J_0(E)$ denote the group of degree-zero line bundles on E. The map $E \to J_0(E)$ that sends $x \in E$ to the line bundle associated to the divisor $(0)-(x)$ is an isomorphism of Abelian groups. □

6.4 Chow's Theorem

The theory of holomorphic manifolds is analogous in many ways to complex algebraic geometry. The rough principle, known as the GAGA Principle from Serre's *Géométrie algébrique et géométrie analytique*, is that compact analytic objects in projective space and morphisms between them are in fact algebraic. Chow's Theorem (Theorem 6.14 below) is the prototypical example of the GAGA Principle. As will become apparent, the underlying reason for GAGA is the simplicity of the analytic cohomology of $\mathbf{P}^n$.

Let M be a compact holomorphic manifold of (complex) dimension n. Because M is oriented, there is a *fundamental class* $[M] \in H_{2n}(M, \mathbf{Z})$ and a *Poincaré duality* isomorphism

$$H_k(M, \mathbf{Z}) \simeq H^{2n-k}(M, \mathbf{Z}).$$

The induced ring structure on homology is the *intersection pairing*, which is described in detail below for $\mathbf{P}^n$. If V is a k-cycle Poincaré dual to $\eta \in H^{2n-k}(M, \mathbf{Z})$, then

i. For all $\alpha \in H^k(M, \mathbf{Z})$, $\int_V \alpha = \int_M \eta \wedge \alpha = (\eta \cup \alpha)[M]$;

ii. For all $(2n-k)$-cycles W, $V \cdot W = \int_W \eta$.

Example 6.11 (Intersection theory in projective space.) Let $M = \mathbf{P}^n$. The cohomology ring is a truncated polynomial ring in one variable. More precisely, let $\Omega \in H^2(\mathbf{P}^n, \mathbf{Z})$ denote the cohomology class Poincaré dual to a hyperplane class $H \in H_{2n-2}(\mathbf{P}^n, \mathbf{Z})$. Then

$$H^*(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}[\Omega]/\langle \Omega^{n+1} \rangle \simeq \bigoplus_{k=0}^{n} \mathbf{Z}\Omega^k.$$

Since $\mathbf{P}^n$ is oriented, $H_{2n}(\mathbf{P}^n, \mathbf{Z})$ is naturally isomorphic to $\mathbf{Z}$, and this isomorphism is compatible with Poincaré duality and the description of the cohomology ring just given.

Let $V \subset \mathbf{P}^n$ be an algebraic hypersurface of degree d. A line in $\mathbf{P}^n$ intersects V in at most d points, and if intersections are counted with multiplicities, then *every* line intersects V exactly d times. From Property i. above it follows that V is Poincaré dual to $d\Omega$. By intersecting further with hyperplanes, it is easy to see that every *positive* cohomology class—namely, every $d\Omega^{n-k}$ with $d > 0$ and $0 \leq k \leq n$—is Poincaré dual to a homology class represented by an analytic cycle. More generally, if V is a complete intersection of multi-degree $(d_1, \dots, d_{n-k})$, then the homology class $[V]$ is Poincaré dual to $\eta_V = (\prod_i d_i)\Omega^{n-k}$. The coefficient is the number of points of intersection when V is intersected with a generic linear subspace of complementary dimension.

For arbitrary homology classes of complementary dimension, the intersection product $[V] \cdot [W]$ may be interpreted as an oriented intersection number, giving geometric significance to condition ii. Indeed, every nonzero homology class in $\mathbf{P}^n$ is Poincaré dual to $d\Omega^{n-k}$ for some $d \in \mathbf{Z}$, and may therefore be represented by $\pm V$ for a smooth complete intersection of complex dimension k. Two generic smooth, complete intersection varieties of complementary dimension intersect transversally (in finitely many points). □

In an arbitrary holomorphic manifold, not every homology class is representable by a smooth submanifold (even up to sign). However, it is always possible to represent a homology class by a cycle whose support is an embedded manifold except for a singular set of smaller dimension. Further, for each pair of homology classes of complementary dimension, representative cycles may be chosen that intersect transversally in finitely many manifold points. Let V and W be such representative cycles, and let p be a point of (transverse) intersection of their supports. If these cycles are oriented, then each tangent space T_pV, T_pW has an induced orientation, determined by ordered bases $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_{n-k}\}$. The *intersection number* $I_p(V, W)$ of V and W at p is defined to be ± 1, depending on whether the orientation on T_pM determined by the ordered basis $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\}$ agrees with the orientation of T_pM or not. The oriented intersection number of the homology classes $[V]$ and $[W]$ is defined to be

$$I(V, W) = [V] \cdot [W] = \sum_{p \in V \cap W} I_p(V, W).$$

If $[V]$ and $[W]$ are represented by transversely intersecting analytic cycles (i.e. embedded subvarieties of M), then $[V] \cdot [W] \geq 0$. This follows immediately from the observation that the usual orientation on $\mathbf{C}^n$ agrees with the orientation induced from a pair of complex subspaces of complementary dimension.

Example 6.12 Fix an integer k and let $H^k = \mathcal{O}_{\mathbf{P}^1}(k)$. The *Hirzebruch surface* $\mathbf{F}_k$ is the $\mathbf{P}^1$-bundle $\mathbf{P}(H^k \oplus \mathcal{O})$ obtained by adding an infinity section to the total space of H^k. The two-dimensional homology of $\mathbf{F}_k$ is generated by a fibre class $[F]$ and the class $[S_0]$ of the zero section. The intersection pairing is given by

$$[F] \cdot [F] = 0, \qquad [F] \cdot [S_0] = [S_0] \cdot [F] = 1, \qquad [S_0] \cdot [S_0] = k.$$

The homology class of the infinity section S_∞ is equal to $[S_0 - kF]$, as is readily checked by solving $[S_\infty] = [aF + bS_0]$ for a and b. In particular, $[S_\infty] \cdot [S_\infty] = -k$. □

Using intersection theory in projective space, it is easy to prove two basic but useful GAGA-type results.

Proposition 6.13 *Let $f : \mathbf{P}^n \to \mathbf{P}^n$ be a biholomorphism. Then there is a linear transformation $\widetilde{f} : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ that induces f. Briefly, every automorphism of $\mathbf{P}^n$ is linear.*

Proof Let $H \subset \mathbf{P}^n$ be a hyperplane, and let f be a biholomorphism of $\mathbf{P}^n$. It suffices to show that $f(H)$ is itself a hyperplane in $\mathbf{P}^n$. Certainly, $f(H)$ is an analytic subvariety homologous to a hyperplane. Let p_1 and p_2 be points in $f(H)$, and let $\ell \simeq \mathbf{P}^1$ be the line joining them. Then $\{p_1, p_2\} \subset \ell \cap H$, so if there were only finitely many points of intersection then $[\ell] \cdot [H] \geq 2$. But $[\ell] \cdot [H] = 1$ for homological reasons, so $\ell \cap H$ is infinite, hence must be all of ℓ since ℓ and H are analytic varieties. It follows that $\ell = \overline{p_1 p_2} \subset f(H)$ for every pair of points p_1, $p_2 \in f(H)$, so that $f(H)$ is actually a hyperplane. It is now a matter of linear algebra to verify that f must be covered by a linear isomorphism of $\mathbf{C}^{n+1}$. ▯

A surprisingly easy generalization of Proposition 6.13 is the prototypical GAGA theorem, usually known as *Chow's Theorem.*

Theorem 6.14 *Let $V \subset \mathbf{P}^n$ be an analytic subvariety. Then V is an algebraic subvariety.*

Proof The long exact sequence associated to the exponential sheaf sequence on $\mathbf{P}^n$ gives an isomorphism

$$H^1(\mathbf{P}^n, \mathcal{O}^\times) \simeq H^2(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z};$$

in words, holomorphic line bundles on $\mathbf{P}^n$ are classified by their first Chern class. Let $V \subset \mathbf{P}^n$ be an analytic hypersurface, and consider the holomorphic line bundle $L = \mathcal{O}(V)$. There is a positive integer d such that $L = \mathcal{O}_{\mathbf{P}^n}(d)$. The space of sections of $\mathcal{O}_{\mathbf{P}^n}(d)$ is exactly the space of homogeneous, degree d polynomials in the homogeneous coordinates. It follows that V is the zero locus of a single homogeneous polynomial, i.e. is an algebraic hypersurface. (This gives an alternate proof that the image of a hyperplane under an automorphism is a hyperplane.)

Now let V be an arbitrary analytic variety of dimension k in $\mathbf{P}^n$. The strategy is to show that for every $p \notin V$, there is a homogeneous polynomial F that vanishes along V, but with $F(p) \neq 0$. It then follows that the common zero locus of the set of polynomials vanishing along V—which is *a priori* larger than V—is equal to V, so that V is an algebraic variety.

Pick a hyperplane H not containing p, and project away from p. The image of V is an analytic subvariety in H. Proceed inductively until the image is a hypersurface in a linear space $\mathbf{P}^{k+1}$, which after linear change of coordinates may be taken to be the set of points of the form $[z^0 : \cdots : z^{k+1}]$. By the hypersurface case proven above, there is a homogeneous polynomial f of degree d in the variables $z^0, \ldots, z^{k+1}$ whose zero locus is the image of V in $\mathbf{P}^{k+1}$. The sequence of projections may actually be accomplished in a single step, by projecting away from a generic linear space of dimension $n - k - 2$ not containing p. The polynomial F defined by $F(z^0, \ldots, z^n) = f(z^0, \ldots, z^{k+1})$ is the desired polynomial. Geometrically, the zero locus of F is the cone on V with vertex $\mathbf{P}^{n-k-2}$. ∎

Exercises

Let $i : V \hookrightarrow M$ be a smooth complex submanifold of a complex manifold. There is an exact sequence

$$0 \to TV \to i^*TM \to \nu_{V/M} \to 0 \tag{6.1}$$

of vector bundles over V, whose quotient term is called the *normal bundle* of V in M. The dual of the normal bundle is the *conormal bundle*.

Exercise 6.1 Let $V \subset M$ be a smooth, irreducible hypersurface. Prove that the conormal bundle of V in M is isomorphic to the line bundle $i^*[-V]$; in words, this is the restriction to V of the line bundle on M induced by the divisor $-V$. This fact is sometimes called the *first adjunction formula*.

Suggestion: Let $\{f_\alpha\}$ be local defining functions for V; show that after restricting to V, the one-forms $\{df_\alpha\}$ constitute a global non-vanishing section of a certain line bundle. ◇

Exercise 6.2 Find the normal bundle of $\mathbf{P}^k \subset \mathbf{P}^n$, and prove that the exact sequence

$$0 \to T\mathbf{P}^k \to i^*T\mathbf{P}^n \to \nu_{\mathbf{P}^k/\mathbf{P}^n} \to 0$$

splits holomorphically. In fact, the total space of $\nu_{\mathbf{P}^k/\mathbf{P}^n}$ embeds as a Zariski-open subset of $\mathbf{P}^n$; identify this set and its complement geometrically. ◇

If $p : E \to M$ is a holomorphic vector bundle of rank k, then the *determinant bundle* $\det E$ is defined to be the top exterior power $\bigwedge^k E$. Thus $\det E$ is the holomorphic line bundle whose transition functions are determinants of the transition functions of E. Let M be a holomorphic manifold, and let $E = T^{1,0}M$ be the holomorphic tangent bundle of M. The *anti-canonical bundle* of M is $-K_M = \det E$, and the *canonical bundle* of M is $K_M = \det E^*$. Thus the canonical bundle of M is the bundle of holomorphic n-forms.

Exercise 6.3 Prove that $-K_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(n+1)$. ◇

Exercise 6.4 Let $0 \to S \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles over M. Prove that $\det E \simeq \det S \otimes \det Q$ as *holomorphic* line bundles. (Note that the splitting $E = S \oplus Q$ is generally not holomorphic.) ◇

Exercise 6.5 Let $V \subset M$ be a smooth hypersurface. Prove the *second adjunction formula*:

$$K_V = i^*(K_M \otimes [V]). \tag{6.2}$$

Suggestion: Use Exercises 6.1 and 6.4. ◇

When $V \subset M$ is a smooth hypersurface, the induced map $K_M \to K_V$ on global sections is called the *Poincaré residue map*, and is extremely useful for describing holomorphic forms on hypersurfaces in $\mathbf{P}^n$. Take $M = \mathbf{P}^n$, $n > 1$, and let $V = \{f = 0\}$ be a smooth, irreducible hypersurface of degree d. By the second adjunction formula, $K_V = i^*\mathcal{O}_{\mathbf{P}^n}(d - n - 1)$. A (local) section η of the bundle $\Omega^n_{\mathbf{P}^n} \otimes [V]$ is a (local) meromorphic n-form on $\mathbf{P}^n$ that has a single pole along V and is elsewhere holomorphic. (A global holomorphic section of $\Omega^n_{\mathbf{P}^n} \otimes [V]$ exists if and only if $d = \deg f \geq n + 1$.) The Poincaré residue of η is the unique holomorphic form η_0 on V such that

$$\eta = \frac{df}{f} \wedge \eta_0.$$

The kernel of the Poincaré residue map is the sheaf of germs of holomorphic n-forms; in other words, there is an exact sheaf sequence

$$0 \to \Omega^n_{\mathbf{P}^n} \to \Omega^n_{\mathbf{P}^n} \otimes [V] \to \Omega^{n-1}_V \to 0,$$

whose long exact cohomology sequence includes the terms

$$H^0(\mathbf{P}^n, \Omega^n_{\mathbf{P}^n} \otimes [V]) \to H^0(V, \Omega^{n-1}_V) \to H^1(\mathbf{P}^n, \Omega^n_{\mathbf{P}^n});$$

the last term is trivial by Proposition 4.13. Thus, every holomorphic $(n-1)$-form on V is the Poincaré residue of a global meromorphic form on $\mathbf{P}^n$.

Exercise 6.6 (Algebraic K3 Surfaces) Let V be a smooth quartic hypersurface in $\mathbf{P}^3$. Show that the canonical bundle of V is trivial, and find a non-vanishing holomorphic two-form on V. (By Theorem 10.5 below, the first homology of V is trivial. In fact, V is simply-connected. A simply-connected complex surface with trivial canonical bundle is called a *K3 surface*, after Klein, Kummer, and Kodaira.)

Let $M = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and let $V \subset M$ be a smooth hypersurface of degree $(2, 2, 2)$; this means V is the zero locus of an irreducible sextic polynomial that is quadratic in the homogeneous coordinates on each $\mathbf{P}^1$ factor. Prove that the canonical bundle of V is trivial, and find eight automorphisms of V. ◇

Exercise 6.7 Let $C \subset \mathbf{P}^2$ be a smooth curve of degree d. Calculate the dimension of the space of holomorphic one-forms on C using the Poincaré residue map, thereby giving an alternate proof of the degree formula (Exercise 2.8) for smooth plane curves. ◇

7

Metrics, Connections, and Curvature

In differential geometry, a (Riemannian) *metric* is a symmetric, positive-definite two-tensor g on a manifold M, namely, a smooth choice of inner product in the tangent spaces of M. The metric is used to define lengths of tangent vectors, and angles between pairs of vectors. The length of a path γ of class $\mathcal{C}^1$ is defined to be the integral of $\|\dot{\gamma}\|$, and the distance between two points is the infimum of the lengths of paths joining the points. A connection is an additional piece of data that specifies parallel transport along a piecewise $\mathcal{C}^1$ path, and thereby allows ("covariant") differentiation of tensor fields in a coordinate-independent manner. A Riemannian metric determines a Levi-Civita connection, for which parallel transport preserves orthonormality of frames and satisfies a symmetry condition. However, unlike partial derivatives, covariant derivatives in different directions do not generally commute. The failure of commutativity is measured by the "curvature tensor" of the connection.

A metric on a holomorphic manifold is usually required to satisfy additional restrictions, in order that the geometry of the metric reflect the holomorphic structure of the manifold. An "Hermitian" metric is algebraically compatible with the complex structure J; such a metric distinguishes *two* connections: The Levi-Civita connection, and the Chern connection, which is compatible with J. An Hermitian metric is "Kähler" if a further analytic condition holds. This condition has many interesting interpretations, one of which is that the Levi-Civita and Chern connections coincide.

7.1 Hermitian and Kähler Metrics

Let (V, J) be a complex vector space. An inner product g on V is *Hermitian* if the endomorphism J is g-orthogonal, that is, if $g(X, Y) = g(JX, JY)$ for all X and $Y \in V$. Every inner product may be averaged to yield an Hermitian inner product; if g is an arbitrary inner product, then

$$\tilde{g}(X, Y) = \frac{1}{2}\Big(g(X, Y) + g(JX, JY)\Big) \tag{7.1}$$

defines an Hermitian inner product.

An Hermitian inner product has a $\mathbf{C}$-bilinear extension $g_{\mathbf{C}}$ on $V_{\mathbf{C}}$, and it is immediate to verify that the eigenspaces $V^{1,0}$ and $V^{0,1}$ are isotropic with respect to $g_{\mathbf{C}}$. However, the complex-valued inner product h on V defined by

$$h(X, Y) = g_{\mathbf{C}}(X^{1,0}, Y^{0,1}) = \frac{1}{2}\Big(g(X, Y) + ig(X, JY)\Big) \tag{7.2}$$

is easily verified to be $\mathbf{C}$-linear in the first variable, $\mathbf{C}$-antilinear in the second, and to be conjugate symmetric. Thus, h is an Hermitian form on V in the usual sense. The imaginary part of h is denoted $\omega/2$, and is skew-symmetric.

Lemma 7.1 *If two of the tensors g, J, and ω are known, then the third is determined uniquely.*

Proof If J is known, then passing from g to ω is trivial: If X, $Y \in V$, then

$$\omega(X, Y) = g(X, JY), \tag{7.3}$$

and since g is J-Hermitian, $g(X, Y) = \omega(JX, Y)$.

Suppose that g is an inner product on V and that ω is a skew-symmetric two-tensor which in addition is non-degenerate. The latter means $\omega^n \neq 0$, or equivalently that for every $X \in V$, there is a $Y \in V$ with $\omega(X, Y) \neq 0$. There is a unique linear transformation $S : V \to V$ such that for all X and $Y \in V$, $g(X, SY) = \omega(X, Y)$. This mapping is invertible because ω is non-degenerate, and is skew-symmetric with respect to g because ω is skew-symmetric. The transformation $-S^2$ is (symmetric and) positive-definite with respect to g, hence has a square root T that commutes with S. Thus $J := ST^{-1}$ is a complex structure on V. ∎

When (V, J) is a complex vector space, the relation (7.3) generally gives a correspondance between real, symmetric, J-invariant 2-tensors and real, skew-symmetric $(1, 1)$-tensors (in the sense of complex bigrading).

This algebraic construction may be made pointwise on a complex manifold; the resulting structure is called an (*almost*-)*Hermitian* structure. However, the interaction between g and J is richer if at least one of the tensor fields is required to satisfy an integrability condition.

If g is integrable in the sense that the two-form ω is closed, then the pair (M,ω) is a *symplectic manifold*. Symplectic manifolds were originally studied by virtue of their appearance in Hamiltonian mechanics; the cotangent bundle of a configuration space—the so-called *phase space* of the physical system—carries a natural symplectic form, and the time evolution of the system is elegantly described in terms of geometry of the symplectic structure. Symplectic geometry experienced a renaissance in the mid-1980's that shows no signs of abating. Four-dimensional topology, algebraic geometry, quantum field theory, and string theory are just a few of the branches of mathematics and physics that have benefitted from the "new" symplectic geometry.

If J is integrable in the sense of having vanishing Nijenhuis tensor, then M has a holomorphic atlas for which J is the induced complex structure on TM. A triple (M,J,g) is an *Hermitian* manifold if the Riemannian metric g is Hermitian as an inner product in each tangent space. The two-form ω is the *fundamental form* of g, and has the following useful interpretation.

Lemma 7.2 *Let (M,J,g) be an Hermitian manifold of complex dimension n. Then the volume form* dvol_g *of g is $(\omega^n/n!)$.*

Proof With respect to a unitary coframe $\{\mathbf{e}^i\}_{i=1}^n$, the fundamental form is $\omega = \mathbf{e}^1 \wedge \bar{\mathbf{e}}^1 + \cdots + \mathbf{e}^n \wedge \bar{\mathbf{e}}^n$; taking the top exterior power proves the lemma. ∎

An Hermitian metric is *Kähler* if the fundamental two-form ω is closed. In this case, ω is called the *Kähler form* of g, and the de Rham class $\Omega = [\omega] \in H^2_d(M,\mathbf{R})$ is called the *Kähler* (or *fundamental*) *class* of g. When a complex structure is fixed, it is convenient to identify g and ω, and to speak of "a Kähler metric ω" or of "a metric g in Ω." A holomorphic manifold that admits a Kähler metric is *Kählerian* or *of Kähler type*.

A two-form η is *positive* if $\eta(JX,X) > 0$ for every non-zero tangent vector X, cf. equation (7.3) above. A de Rham cohomology class containing a positive representative is a *positive class*. The fundamental two-form ω of an Hermitian metric is positive, and a positive $(1,1)$-class on a compact holomorphic manifold is often called a Kähler class.

Every holomorphic manifold admits an Hermitian structure; indeed, an arbitrary Riemannian metric may be averaged as in (7.1). By contrast, there are topological obstructions to existence of a Kähler metric on a compact manifold. The most elementary of these is a simple consequence of Lemma 7.2. Since the volume of M with respect to g may be computed as

$$\int_M \frac{\omega^n}{n!} = \frac{1}{n!}\langle \Omega^n, [M]\rangle = \langle e^{\Omega}, [M]\rangle,$$

the de Rham class Ω^n is non-zero, so *a fortiori* Ω is not zero, nor are its k-fold exterior powers for $k \le n$. There are additional, much more subtle, necessary conditions for a compact holomorphic manifold to be Kählerian. Some of these will appear later as consequences of the Hodge Theorem.

Example 7.3 Every Hermitian metric on a Riemann surface is Kähler, simply because $d\omega$ is a 3-form, hence vanishes identically. The Hopf manifolds, which are diffeomorphic to $S^1 \times S^{2n-1}$, $n > 1$, are not Kählerian. Nor is the sphere S^6, leaving aside the question of whether or not S^6 has a holomorphic atlas. □

Example 7.4 The flat metric on $\mathbf{C}^n$ is Kähler. The Kähler form

$$\omega = \sqrt{-1}(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n)$$

is, indeed, exact. Since the flat metric is invariant under translation, every compact complex torus admits a Kähler metric. □

Example 7.5 Complex projective space $\mathbf{P}^n$ admits a $U(n+1)$-invariant Kähler metric, the *Fubini-Study* metric. Let Z be standard coordinates on $\mathbf{C}^{n+1}$, put $\rho = \|Z\|^2$, and set

$$\begin{aligned} 2\pi\widetilde{\omega} &= \sqrt{-1}\partial\bar{\partial}\log\rho = \sqrt{-1}\left[\frac{\partial\bar{\partial}\rho}{\rho} - \frac{\partial\rho\wedge\bar{\partial}\rho}{\rho^2}\right] \\ &= \sqrt{-1}\left[\frac{\|Z\|^2\sum dZ^j\wedge d\bar{Z}^j - (\sum \bar{Z}^j\,dZ^j)\wedge(\sum Z^j\,d\bar{Z}^j)}{\|Z\|^4}\right]. \end{aligned} \tag{7.4}$$

The form $\widetilde{\omega}$ is $U(n+1)$-invariant ("a function of $\|Z\|^2$ alone"), and is invariant under scalar multiplication by non-zero complex numbers (numerator and denominator are homogeneous of weight four). There is consequently a well-defined push-forward $2\pi\,\omega$ on $\mathbf{P}^n$. To see that ω is positive-definite, evaluate at $[1 : 0 : \cdots : 0]$ where definiteness is obvious, then use invariance under the unitary group $U(n+1)$. The de Rham class of ω generates the integral cohomology of $\mathbf{P}^n$. To see this, consider the embedding $i : [Z^0 : Z^1] \in \mathbf{P}^1 \longmapsto [Z^0 : Z^1 : 0 : \cdots : 0] \in \mathbf{P}^n$. In the chart U_0 with coordinate $z = Z^1/Z^0$,

$$i^*\omega = \frac{\sqrt{-1}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}, \quad \text{so} \quad \int_{\mathbf{P}^1}\omega = 1$$

by integrating in polar coordinates. □

A large—and extremely important—class of Kähler manifolds comes from the following observation.

Proposition 7.6 *Let (M, J, g) be a Kähler manifold, and let N be a complex submanifold. Then the restriction of g to N is Kähler.*

Proof By hypothesis, the restriction of J to N preserves the tangent bundle of N and is equal to the complex structure of N. The fundamental two-form of $g|_N$ is therefore, by equation (7.3), equal to $\omega|_N$, which is closed. □

Example 7.7 Every smooth algebraic variety admits a Kähler metric. In fact, the restriction of the Fubini-Study form to a subvariety $V \subset \mathbf{P}^n$ is an integral form, that is, represents a class in $H^2(V, \mathbf{Z})$.

A *compact* Kähler manifold whose Kähler class is rational is called a *Hodge manifold*. The *Kodaira Embedding Theorem* (Theorem 10.10 and Corollary 10.11 below) asserts that every Hodge manifold can be embedded in some projective space. Among compact complex tori, Hodge manifolds form a set of measure zero; such a torus admits a Hodge metric if and only if its lattice satisfies certain rationality conditions. These conditions are automatic in dimension one, but are non-trivial in dimension greater than one.

A similar assertion is true for smooth complex surfaces diffeomorphic to the Fermat quartic in $\mathbf{P}^3$, the so-called $K3$ surfaces (after Klein, Kummer, and Kodaira). A generic $K3$ surface is not projective algebraic, but there is a 19-dimensional family of quartic surfaces in $\mathbf{P}^3$. See also Example 10.8. □

There are a number of useful alternate characterizations of when a metric is Kähler, some of which are given in Proposition 8.11 below. As a general point of philosophy, Kähler manifolds exhibit a large degree of interplay between their real structures (differential-geometric, smooth topological) and complex-analytic structure (Dolbeault cohomology).

7.2 Connections in Vector Bundles

Let $E \to M$ be a complex vector bundle of rank k over a holomorphic manifold. A *connection* in E is a $\mathbf{C}$-linear map $D : A^0(E) \to A^1(E)$ satisfying the Leibniz rule $D(fs) = df\, s + f\, Ds$ for all smooth functions f and all smooth sections s. Every connection admits unique extensions $D : A^r(E) \to A^{r+1}(E)$ satisfying

$$D(\psi s) = d\psi\, s + (-1)^r \psi\, Ds \qquad \text{for } \psi \in A^r. \tag{7.5}$$

A connection in E gives a means of transporting frames along paths in M, thereby comparing or "connecting" fibres of E. If $\gamma : [0,1] \to M$ is a smooth path, and if $\mathbf{e}_0$ is a frame at $\gamma(0)$, then there is a unique section $\mathbf{e}_t$ of γ^*E, the *parallel transport* of $\mathbf{e}_0$ along γ, satisfying

$$D_{\dot\gamma(t)}\mathbf{e}_t = 0 \qquad \text{for all } t \in [0,1].$$

Indeed, this condition is a linear first-order system of ordinary differential equations on $[0,1]$, which therefore has a unique solution for every choice of initial conditions.

Locally, a connection is specified by a *connection matrix*. Precisely, to a local frame $\mathbf{e}$ of E is associated a $k \times k$ matrix $\theta = \theta(\mathbf{e})$ of one-forms satisfying $D\mathbf{e} = \theta\mathbf{e}$; this equation is interpreted as meaning $D\mathbf{e}_i = \sum_j \theta_i^j \mathbf{e}_j$. If $\mathbf{e}' = a\mathbf{e}$ is another local frame, with a a local $GL(k, \mathbf{C})$-valued function, and if $D\mathbf{e}' = \theta'\mathbf{e}'$, then the connection matrices are related by

$$\theta' a = a\theta + da, \qquad \text{or} \quad \theta' = a\theta a^{-1} + (da)a^{-1}. \tag{7.6}$$

The space of connections in E is an affine space, and every choice of connection D furnishes an isomorphism with the vector space $A^1(\operatorname{End} E)$; indeed, if D_1 and D_2 are connections in E, then their difference $D_1 - D_2$ is linear over A^0 by (7.6) or by the Leibniz rule, hence is given by wedging with the endomorphism-valued one-form $\theta_1 - \theta_2 \in A^1(\operatorname{End} E)$.

The *curvature operator* $R = D^2 : A^0(E) \to A^2(E)$ measures the extent to which parallel transport around a closed loop is not the identity. Remarkably, R is an *algebraic* operator in the sense that $R(fs) = f\,R(s)$ for all smooth f; the value of Rs at $x \in M$ depends only on the value of s at x. With respect to a local frame, there is a $k \times k$ matrix $\Theta = \Theta(\mathbf{e})$ of two-forms—the *curvature* matrix of D—such that $R(\mathbf{e}) = \Theta\mathbf{e}$. The following facts are easily checked.

- Under a change of frame $\mathbf{e}' = a\mathbf{e}$, the curvature matrix transforms by a similarity: $\Theta' = a\Theta a^{-1}$. In other words, the curvature operator is determined by an $\operatorname{End} E$-valued two-form $\Theta \in A^2(\operatorname{End} E)$. If E is a line bundle, then Θ is an ordinary two-form on M since in this case $\operatorname{End} E$ is canonically trivial, the identity endomorphism being a natural choice of section.

- With respect to an arbitrary frame, the *Cartan Structure equation*

$$\Theta = d\theta - \theta \wedge \theta \tag{7.7}$$

holds, the wedge product being taken as a matrix product with the entries wedged.

- Taking the exterior derivative of Θ and using (7.7) gives the *differential Bianchi identity* $DR = 0$, or

$$d\Theta + \Theta \wedge \theta - \theta \wedge \Theta = 0. \tag{7.8}$$

In words, the curvature tensor is parallel.

If E is a holomorphic vector bundle, then a connection D in E is *compatible* with the holomorphic structure if $D^{0,1} = \bar{\partial}_E$. The curvature of a compatible connection has no $(0,2)$-component. Conversely, in a complex vector bundle over a holomorphic manifold, a connection D whose $(0,1)$ part satisfies $(D^{0,1})^2 = 0$ determines a holomorphic structure; a local section s of E is declared to be holomorphic if $D^{0,1}s = 0$. This point of view is useful for constructing the moduli space of holomorphic structures on a fixed complex vector bundle.

An *Hermitian structure* in a complex vector bundle E is a smooth field of Hermitian inner products in the fibres of E; alternately, an Hermitian structure is a positive-definite section $h \in A^0(E^* \otimes \overline{E}^*)$. With respect to a local frame, an Hermitian structure is given by an Hermitian matrix-valued function H, with $H_{ij} = H(\mathbf{e}_i, \mathbf{e}_j)$, that transforms according to $H' = aH\bar{a}^t$. An Hermitian structure in TM is exactly an Hermitian metric on M.

A connection is *compatible* with an Hermitian structure h if

$$d\Big(h(s_1, s_2)\Big) = h(Ds_1, s_2) + h(s_1, Ds_2)$$

for all smooth, local sections s_1, s_2. Geometrically, this means that under parallel transport a unitary frame remains unitary. In terms of a local frame, the connection and curvature matrices satisfy

$$dH = \theta H + H\bar{\theta}, \qquad \Theta H + H\overline{\Theta}^t = 0,$$

i.e. the matrix ΘH is skew-Hermitian. In particular, with respect to a local unitary frame of E, the curvature matrix Θ is skew-Hermitian.

Proposition 7.8 *Let $(E, h) \to M$ be an Hermitian holomorphic vector bundle. Then there is a unique connection D in E that is compatible with the metric and holomorphic structure.*

Proof With respect to a local *holomorphic* frame $\mathbf{e}$, the connection form θ is of type $(1, 0)$. Comparing types in the equation

$$d\theta = \partial\theta + \bar{\partial}\theta = \theta H + H\bar{\theta}$$

gives $\theta = (\partial H)H^{-1}$. The curvature matrix is

$$\Theta = \bar{\partial}\theta = -(\partial\bar{\partial}H)H^{-1} + (\partial H)H^{-1} \wedge (\bar{\partial}H)H^{-1}.$$

It is straightforward to check that the locally defined matrix θ of one-forms defines a connection matrix (i.e. transforms correctly) which is compatible with h, and is clearly the only such connection matrix of type $(1, 0)$. ∎

The connection whose existence is asserted in Proposition 7.8 is called the *Chern* connection or the *canonical* connection of (E, h). The connection form is formally $\partial \log H$, the formality being that H is matrix-valued. If E is a line bundle, then this equation is literally true. The curvature form is related to the *Chern form* $\gamma_1(E, h)$ by

$$2\pi\gamma_1(E, h) = -\sqrt{-1}\partial\bar{\partial}\log H = \sqrt{-1}\Theta.$$

The Chern form represents the first Chern class of the line bundle E.

Example 7.9 The total space of the tautological bundle $\mathcal{O}_{\mathbf{P}^n}(-1)$ is a subbundle of $\mathbf{P}^n \times \mathbf{C}^{n+1}$, and has an induced Hermitian structure given by the usual Hermitian structure on $\mathbf{C}^{n+1}$. The curvature form is exactly minus the Fubini-Study form, see Example 7.5 and Exercise 7.1. □

A connection D in E induces connections in bundles associated to E. The *conjugate bundle*, whose transition functions are complex conjugates of the transition functions of E, has induced connection $D\bar{s} = \overline{Ds}$ for $s \in A^0(E)$. The dual bundle acquires a connection via the Leibniz rule. If the dual pairing is denoted by $\langle\ ,\ \rangle : A^0(E) \times A^0(E^*) \to A^0$, then for a section $\eta \in A^0(E)$, $D\eta$ is defined by requiring

$$d\langle s, \eta\rangle = \langle Ds, \eta\rangle + \langle s, D\eta\rangle \qquad \text{for every section } s \in A^0(E).$$

If D_E and D_F are connections in vector bundles E and F over M, then $D_E \oplus D_F$ is a connection in $E \oplus F$. An *adapted* frame is a collection of local sections $\{\mathbf{e}_i\}_{i=1}^{k+\ell}$ with $\{\mathbf{e}_i\}_{i=1}^{k}$ a frame for E and $\{\mathbf{e}_i\}_{i=k+1}^{\ell}$ a frame for F. With respect to an adapted frame, the connection and curvature forms of $D_{E\oplus F}$ split as

$$\theta_{E\oplus F} = \begin{bmatrix} \theta_E & 0 \\ 0 & \theta_F \end{bmatrix}, \qquad \Theta_{E\oplus F} = \begin{bmatrix} \Theta_E & 0 \\ 0 & \Theta_F \end{bmatrix}. \tag{7.9}$$

The tensor product $E \otimes F$ has an induced connection $D_E \otimes I_F + I_E \otimes D_F$. The exterior products $\bigwedge^p(E)$ acquire connections as subbundles of $E^{\otimes p}$. An important special case is the top exterior power $\det E = \bigwedge^k E$; the connection and curvature forms with respect to the frame $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k$ are

$$\operatorname{tr}\theta, \qquad \operatorname{tr}\Theta. \tag{7.10}$$

Let $S \subset E$ be a holomorphic subbundle of an Hermitian vector bundle $E \to M$, and let $Q = E/S$ be the quotient. There is a short exact sequence

$$0 \to S \to E \to Q \to 0$$

of vector bundles. Such a sequence does not usually split holomorphically. Of course, the orthogonal complement $S^\perp \subset E$ is isomorphic to Q as a smooth complex vector bundle, and via this isomorphism the bundle Q acquires an Hermitian structure and a canonical connection. In Riemannian geometry, the Levi-Civita connection of a submanifold is the tangential component of the Levi-Civita connection of the ambient space. The holomorphic analogue of this result is as follows.

Theorem 7.10 *Let (E, h) be an Hermitian holomorphic vector bundle, S a subbundle with the restricted metric, and $Q = E/S$ the quotient with the induced metric. There is a one-form A with values in* $\operatorname{Hom}(S, S^\perp)$ *such that the connection matrix and curvature operator of E are*

$$\theta_E = \begin{bmatrix} \theta_S & A \\ -A^* & \theta_Q \end{bmatrix} \qquad R_E = \begin{bmatrix} R_S - A^* \wedge A & -D^{1,0}A^* \\ D^{0,1}A & R_Q - A \wedge A^* \end{bmatrix}. \tag{7.11}$$

Proof (Sketch) There are several assertions to check, but all are straightforward. First, apply D_E to a local section s of S and decompose into S and $S^\perp$ components to deduce that the S component of $D_E|_S$ is D_S and the $S^\perp$ component is A^0-linear in s, i.e. is a one-form A with values in $\mathrm{Hom}(S, S^\perp)$. This one-form is called the *second fundamental form* of S in E, and measures extrinsic geometry of S via its embedding.

A similar argument shows that $D_E|_{S^\perp}$ has S somponent $-A^*$ (minus the adjoint) and $S^\perp$ component D_Q. The expression of the curvature operator in (7.11) is the vector bundle version of the Gauss-Codazzi equations. □

Corollary 7.11 *Let $0 \to S \to E \to Q \to 0$ be a short exact sequence of Hermitian holomorphic vector bundles over M. Then the sequence splits holomorphically if and only if the second fundamental form of S in E vanishes identically.*

Let $(E, h) \to (M, g)$ be an Hermitian holomorphic vector bundle over a Kähler manifold. The curvature Θ is a two-form with values in $\mathrm{End}(E)$. In a local coordinate system z, with local frame $\mathbf{e}_j$ and dual coframe $\mathbf{e}^i$, the curvature is

$$\Theta = \sum_{i,j=1}^{k} \Theta_i^j \mathbf{e}^i \otimes \mathbf{e}_j = \sum_{i,j=1}^{k} \left(\sum_{\alpha,\bar{\beta}=1}^{n} R_{i\alpha\bar{\beta}}^j dz^\alpha \wedge d\bar{z}^\beta \right) \mathbf{e}^i \otimes \mathbf{e}_j.$$

Letting $(g^{\bar{\beta}\alpha})$ be the inverse matrix of $(g_{\alpha\bar{\beta}})$, the *mean curvature* of h is the endomorphism

$$\operatorname{tr} \Theta = \sum_{i,j=1}^{k} \left(\sum_{\alpha,\bar{\beta}=1}^{n} g^{\bar{\beta}\alpha} R_{i\alpha\bar{\beta}}^j \right) \mathbf{e}^i \otimes \mathbf{e}_j.$$

The bundle (E, h) is *Einstein-Hermitian* if the mean curvature is a constant multiple of the identity endomorphism.

When $E = TM$, the trace of the curvature may be regarded (after lowering an index) as a symmetric 2-tensor, called the *Ricci tensor* of g. A Riemannian metric is *Einstein* if the Ricci tensor is a multiple of the metric tensor. An Einstein-Kähler metric may be regarded as a (very special) Einstein-Hermitian metric on TM; explicitly, in searching for an Einstein-Hermitian metric, a Kähler metric g on M is fixed, and a metric (E, h) is sought. Even if $E = TM$, h and g are generally different. The existence problem for Einstein-Kähler metrics is substantially more difficult than the corresponding question for Einstein-Hermitian metrics. Finally, an Einstein-Hermitian metric should not be confused with an Hermitian metric g on M that happens to be Einstein in addition.

Exercises

Hermitian Structures on Line Bundles over $\mathbf{P}^1$

Recall that the total space H^k of the line bundle $\mathcal{O}(k) \to \mathbf{P}^1$ is obtained from two copies of $\mathbf{C} \times \mathbf{C}$, with coordinates (z^0, ζ^0) and (z^1, ζ^1), by identifying

$$z^0 = \frac{1}{z^1}, \qquad \zeta^0 = \frac{\zeta^1}{(z^1)^k}.$$

Thus H^k is a family of lines parametrized by points of $\mathbf{P}^1$, the family is trivial in each coordinate system separately, and the fibres in the z^1 coordinates rotate k times with respect to the z^0 coordinate system.

An *Hermitian structure* is a smooth assignment of an Hermitian inner product to each fibre of $\mathcal{O}(k)$. In the present context, an Hermitian structure is specified by giving two positive, real-valued functions $h_0(z^0, \bar{z}^0)$ and $h_1(z^1, \bar{z}^1)$, such that

$$(*) \qquad h_0(z^0, \bar{z}^0) = (z^1 \bar{z}^1)^k \, h_1(z^1, \bar{z}^1)$$

on the set $\mathbf{C}^\times = \mathbf{P}^1 \backslash \{0, \infty\}$ where z^0 and z^1 are non-zero. By condition $(*)$, the function $h_i(z^i, \bar{z}^i)\zeta^i\bar{\zeta}^i$ is well-defined; it is called the *norm* (squared) function of the Hermitian line bundle and is denoted $\| \, \|^2 : H^k \to [0, \infty)$.

Exercise 7.1 Show that the functions $h_i(z^i, \bar{z}^i) = (1 + z^i\bar{z}^i)^{-k}$, $i = 0,\ 1$, define an Hermitian structure on $\mathcal{O}(k)$. Recall that the total space of $\mathcal{O}(-1)$ is a submanifold of $\mathbf{P}^1 \times \mathbf{C}^2$; let (w^0, w^1) be coordinates on $\mathbf{C}^2$. Show that the Hermitian structure $h_i(z^i, \bar{z}^i) = (1 + z^i\bar{z}^i)$ has norm function equal to $w^0\bar{w}^0 + w^1\bar{w}^1$. In other words, the Hermitian structure on the tautological bundle is induced from the usual Hermitian structure on $\mathbf{P}^1 \times \mathbf{C}^2$. ◇

Exercise 7.2 Let σ be a local holomorphic section of H^k that does not vanish on $U \subset \mathbf{P}^1$. Show that the two-form $-\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2$ is invariant under holomorphic gauge transformations. In particular, the *curvature form* $\gamma_1(h) = -\sqrt{-1}\partial\bar{\partial}\log h_i(z^i, \bar{z}^i)$ is a globally defined two-form on $\mathbf{P}^1$. Prove the *Gauss-Bonnet Theorem*: With the previous notation,

$$\int_{\mathbf{P}^1} \gamma_1(h) = 2\pi k.$$

As an elementary sub-exercise, verify the previous equation for the "standard" Hermitian structure given above. ◇

Exercise 7.3 Write stereographic projection in local coordinates, find the Hermitian structure on $T\mathbf{P}^1$ gotten by restricting the Euclidean metric $du^2 + dv^2 + dw^2$ on $\mathbf{R}^3$, and use the Gauss-Bonnet Theorem to show (yet again) that $T\mathbf{P}^1 = \mathcal{O}(2)$. ◇

Flat Vector Bundles and Monodromy Representations

Exercise 7.4 Let $p : (E, h) \to (M, g)$ be a holomorphic Hermitian vector bundle of rank k over an Hermitian manifold. Prove that there is an exact sequence (generally non-split)

$$0 \to p^*E \to TE \to p^*TM \to 0 \tag{7.12}$$

of holomorphic vector bundles over E. In particular, a non-zero section of E (as a vector bundle over M) gives rise to a non-zero vector field on E (as a holomorphic manifold). ◇

Exercise 7.5 Let $\theta = h^{-1}\partial h \in A^1(TE)$ be the Chern connection form of the Hermitian structure h. Prove that there is a unique Hermitian metric on the total space of E (regarded as a manifold) such that

i. The Hermitian structure of E is induced by the inclusion $p^*E \subset TE$.

ii. The kernel of θ, which is smoothly isomorphic to p^*TM by $(*)$, is orthogonal to p^*E and acquires the Hermitian structure p^*g on p^*TM.

Intuitively, this Hermitian metric on the total space of E is h in the vertical directions and g in the horizontal directions. ◇

Exercise 7.6 With the notation of Exercise 7.4, let $\mathrm{H} \subset TE$ denote the kernel of θ, i.e. H is the bundle of horizontal vectors in TE; as *smooth* vector bundles, $TE = p^*E \oplus \mathrm{H}$. Show that the curvature $\Theta = d\theta - \theta \wedge \theta$ vanishes identically if and only if H is involutive, if and only if the sequence (7.12) splits holomorphically. Intuitively, curvature of a connection may be regarded as an obstruction to involutivity of the horizontal distribution. ◇

Exercise 7.7 (Monodromy representations) Let $p : (E, h) \to M$ be an Hermitian holomorphic vector bundle over a connected holomorphic manifold, and assume the Chern connection of E has vanishing curvature. Let $\gamma : [0, 1] \to M$ be a loop based at $x \in M$, and let $\rho(\gamma) \in \operatorname{Aut} E_x$ denote parallel transport around γ. Prove that $\rho(\gamma)$ depends only on the homotopy class of γ in $\pi_1(M)$. It may be helpful to consider the leaves of the distribution H; the lift of γ lies in a fixed leaf. Fixing a frame at x identifies $\operatorname{Aut} E_x$ and $GL(k, \mathbf{C})$, and the map $\rho : \pi_1(M) \to GL(k, \mathbf{C})$ defined by lifting loops based at x is a group homomorphism, called the *monodromy representation* of $\pi_1(M)$. (In particular, each leaf is the total space of a principal bundle over M with structure group $\pi_1(M)$.)

Let $\widetilde{M}$ denote the universal cover of M. Prove that $E = \widetilde{M} \times_\rho \mathbf{C}^k$. In summary, flat vector bundles of rank k over M correspond (more or less naturally) to conjugacy classes of representations $\rho : \pi_1(M) \to GL(k, \mathbf{C})$, or to reductions of the structure group of E from $GL(k, \mathbf{C})$ to $\pi_1(M)$. ◇

Exercise 7.8 Prove that if M is simply-connected, then a flat vector bundle is holomorphically trivial. (This is essentially trivial from Exercise 7.7.) Prove that every topologically trivial line bundle over $\mathbf{P}^n$ (or $\mathbf{P}^1$, though the general case is not much more difficult) admits a flat Hermitian structure. Let x and y be distinct points in an elliptic curve C/Λ. Prove that the line bundle associated to the divisor $(x) - (y)$ is topologically trivial but does not admit a flat Hermitian structure. ◇

8

Hodge Theory and Applications

Suppose $\pi : E \to M$ is a holomorphic vector bundle over a compact manifold. There is a Dolbeault cohomology theory of $\bar{\partial}$-closed (p, q)-forms with values in E, and an isomorphism of complex vector spaces

$$H^{p,q}_{\bar{\partial}}(M, E) \simeq H^q\big(M, \Omega^p(E)\big).$$

Because these cohomology spaces are invariants of the holomorphic structure (as opposed to being topological invariants), it is not obvious whether or not they are finite-dimensional, see Exercise 4.1.

Suppose that E and M are endowed with metric structures. More precisely, assume (E, h) is an Hermitian holomorphic vector bundle (i.e. h is a metric in the fibres of E) and that (M, g) is an Hermitian manifold (i.e. g is a fibre metric in TM). It is then possible to introduce a norm on the space of (p, q)-forms with values in E. The Hodge Theorem (Theorem 8.3 below) asserts that each Dolbeault cohomology class contains a unique representative of smallest norm, and that the set of such representatives is indeed finite-dimensional when M is compact. In the event that (M, g) is a Kähler manifold and $E = \mathcal{O}$ (the trivial line bundle), the interplay between real and complex potential theory gives extra information about the de Rham cohomology of M *versus* the Dolbeault cohomology, see Theorem 8.17.

8.1 The Hodge Theorem

Let $\pi : (E, h) \to (M, g)$ be an Hermitian holomorphic vector bundle of rank k over a (connected) compact Hermitian manifold of dimension n. The metric g naturally induces an Hermitian structure in each of the bundles $T^*_{1,0}(M)^{\otimes p} \otimes T^*_{0,1}(M)^{\otimes q}$ of (p,q)-tensors, and in the subbundle $\bigwedge^{p,q}(M)$ of (p,q)-forms. Together with the Hermitian structure of E, these Hermitian structures determine an Hermitian structure $\langle\ ,\ \rangle$ in the bundle of E-valued (p,q)-forms.

It is instructive to write these Hermitian structures in terms of local frames. Let $\{\vartheta^\alpha,\ J\vartheta^\alpha\}$ be a local orthonormal coframe for g, i.e. the one-forms ϑ^α are real-valued, and

$$g = \sum_{\alpha=1}^{n} \Big(\vartheta^\alpha \otimes \vartheta^\alpha + J\vartheta^\alpha \otimes J\vartheta^\alpha\Big).$$

Setting $\theta^\alpha = \vartheta^\alpha + i\,J\vartheta^\alpha \in T^*_{1,0}M$, the sesquilinear extension of the metric g is $g + \sqrt{-1}\omega = \sum_\alpha \theta^\alpha \otimes \bar\theta^\alpha$, and $\|\theta^\alpha\|^2 = 2$. If I and J are multi-indices of length p and q, then the (p,q)-forms $\theta^I \wedge \bar\theta^J$ have (squared) length 2^{p+q}, and the set of such forms is an orthogonal frame for $\bigwedge^{p,q}(M)$. Concretely, if $\{\mathbf{e}_i\}_{i=1}^k$ is a local frame for E, and

$$\varphi = \sum_{\substack{|I|=p,|J|=q\\ 1\le i\le k}} \Big(\varphi^i_{I\bar J}\theta^I \wedge \bar\theta^J\Big) \otimes \mathbf{e}_i, \qquad \psi = \sum_{\substack{|I|=p,|J|=q\\ 1\le j\le k}} \Big(\psi^j_{I\bar J}\theta^I \wedge \bar\theta^J\Big) \otimes \mathbf{e}_j,$$

then the *pointwise inner product* $\langle\varphi,\psi\rangle \in A^0(M)$ is equal to

$$\langle\varphi,\psi\rangle = 2^{p+q} \sum_{\substack{|I|=p,|J|=q\\ 1\le i,\ j\le k}} \varphi^i_{I\bar J}\,\overline{\psi^j_{I\bar J}}\ h(\mathbf{e}_i,\mathbf{e}_j),$$

and the *global inner product* is defined to be

$$(\varphi,\psi) = \int_M \langle\varphi,\psi\rangle\,\mathrm{dvol}_g = \int_M \langle\varphi,\psi\rangle\,\frac{\omega^n}{n!}. \tag{8.1}$$

The global inner product is an Hermitian inner product on $A^{p,q}(E)$, the space of smooth (p,q)-forms with values in E, but of course $A^{p,q}(E)$ is not complete. It will presently be convenient to introduce a family of *Sobolev completions* of $A^{p,q}(E)$ which will include the L^2-completion as a special case.

Let $\bar\partial^* : A^{p,q}(E) \to A^{p,q-1}(E)$ be the formal adjoint of $\bar\partial$ with respect to the global inner product; thus $(\bar\partial^*\psi,\varphi) = (\psi,\bar\partial\varphi)$ for all $\varphi \in A^{p,q-1}(E)$, $\psi \in A^{p,q}(E)$. The desired "smallest norm" representative of a Dolbeault class is formally expressed very simply.

Proposition 8.1 *Let ψ be a $\bar\partial$-closed (p,q)-form. Then ψ has smallest L^2-norm among all forms $\psi + \bar\partial\eta$ if and only if $\bar\partial^*\psi = 0$.*

Proof Suppose $\|\psi\|^2$ is minimal. Then for every $\eta \in A^{p,q-1}(E)$,

$$0 \quad = \quad \frac{d}{dt}\Big|_{t=0} \|\psi + t\bar\partial\eta\|^2 \quad = \quad 2\mathrm{Re}\,(\psi,\bar\partial\eta)$$

$$0 = \frac{d}{dt}\Big|_{t=0} \|\psi + it\bar{\partial}\eta\|^2 = 2\mathrm{Im}\,(\psi, \bar{\partial}\eta).$$

Thus $(\bar{\partial}^*\psi, \eta) = (\psi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(E)$, so $\bar{\partial}^*\psi = 0$.

Conversely, if $\bar{\partial}^*\psi = 0$, then for every $\eta \in A^{p,q-1}(E)$,

$$\begin{aligned} \|\psi + \bar{\partial}\eta\|^2 &= \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\mathrm{Re}\,(\psi, \bar{\partial}\eta) \\ &= \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\mathrm{Re}\,(\bar{\partial}^*\psi, \eta) \;=\; \|\psi\|^2 + \|\bar{\partial}\eta\|^2 \;\geq\; \|\psi\|^2 \end{aligned}$$

with equality if and only if $\bar{\partial}\eta = 0$. ∎

Formally, Proposition 8.1 says the Dolbeault space $H^{p,q}_{\bar{\partial}}(M, E)$ is isomorphic to the solution space of $\bar{\partial}^*\psi = 0$ where ψ ranges over the space of $\bar{\partial}$-closed (p, q)-forms, or in other words that

$$H^{p,q}_{\bar{\partial}}(M, E) \simeq (\ker \bar{\partial}) \cap (\ker \bar{\partial}^*) \cap A^{p,q}(E).$$

There is a convenient reformulation of the latter in terms of a single second-order diffential operator, the *$\bar{\partial}$-Laplacian* $\Box^{p,q}_{\bar{\partial}} = \Box_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$.

Lemma 8.2 *As subspaces of $A^{p,q}(E)$, $\ker \Box_{\bar{\partial}} = (\ker \bar{\partial}) \cap (\ker \bar{\partial}^*)$.*

Proof This is immediate from $(\Box_{\bar{\partial}}\psi, \psi) = \|\bar{\partial}\psi\|^2 + \|\bar{\partial}^*\psi\|^2$. ∎

The kernel of $\Box^{p,q}_{\bar{\partial}}$ is the space of ($\bar{\partial}$-)*harmonic* (p, q)-forms, and is denoted $\mathbf{H}^{p,q}(M, E)$. The following result, which justifies the formal arguments made above, is the *Hodge Theorem*.

Theorem 8.3 *Let $\pi : (E, h) \to (M, g)$ be an Hermitian holomorphic vector bundle over a compact Hermitian manifold. Then the space of harmonic (p, q)-forms is finite-dimensional and is isomorphic to the Dolbeault space $H^{p,q}_{\bar{\partial}}(M, E)$. Let $\mathbf{H}^{p,q} : A^{p,q}(E) \to \mathbf{H}^{p,q}(M, E)$ denote orthogonal projection. The Laplacian $\Box^{p,q}_{\bar{\partial}}$ is invertible on the orthogonal complement of the space of harmonic forms, that is, there exists a unique operator $G^{p,q}$ of degree -2 such that*

$$\mathbf{H}^{p,q} + G^{p,q}\Box^{p,q}_{\bar{\partial}} = I \qquad (8.2)$$

on $A^{p,q}(M, E)$, $\mathbf{H}^{p,q}G^{p,q} = 0$, $[G^{p,q}, \bar{\partial}] = 0$, and $[G^{p,q}, \bar{\partial}^*] = 0$.

The quasi-inverse G of $\Box_{\bar{\partial}}$ is called *Green's operator*. Equation (8.2) is sometimes expressed by saying that every (p, q)-form ψ has a unique decomposition

$$\psi = \mathbf{H}\psi + \bar{\partial}(\bar{\partial}^*G\psi) + \bar{\partial}^*(\bar{\partial}G\psi) \qquad (8.3)$$

as the sum of a harmonic form, an exact form, and a co-exact form, or that the equation $\Box_{\bar{\partial}}\psi = \eta$ has a solution if and only if η is orthogonal to the space of harmonic (p, q)-forms, and in this case $\psi = G\eta$.

Proof The technical preliminaries for the proof of the Hodge Theorem are construction of a formal adjoint $\bar{\partial}^*$ and of appropriate completions of

$A^{p,q}(E)$, so that techniques of Hilbert space analysis may be brought to bear. Once the groundwork is laid, a weak solution is constructed (functional analysis), and is shown to be of class $\mathcal{C}^\infty$ (regularity theory). These steps are sketched below, mostly without proof.

The *Hodge star operator* $* : A^{p,q}(E) \to A^{n-p,n-q}(E^*)$ is defined by the requirement that

$$\langle \varphi, \psi \rangle \,\mathrm{dvol}_g = \varphi \wedge *\psi \qquad \text{for all } \varphi, \psi \in A^{p,q}(E). \tag{8.4}$$

In order to write the star operator in terms of a frame, let I^0, $J^0 \subset \{1, \dots, n\}$ be multi-indices complementary to I and J, and let $\varepsilon_{IJ} = \mathrm{sign}(I\,I^0)\mathrm{sign}(J\,J^0)$, where "sign" denotes the sign of a permutation of $\{1, \dots, n\}$. If $\{\mathbf{e}_i^*\}$ denotes the dual coframe of $\{\mathbf{e}_i\}$, then

$$\begin{aligned} *\varphi &= * \left(\sum_{\substack{|I|=p,\,|J|=q \\ 1 \le i \le k}} \left(\varphi^i_{I\bar{J}} \theta^I \wedge \bar{\theta}^J \right) \otimes \mathbf{e}_i \right) \\ &= 2^{p+q-n} \sum_{\substack{|I|=p,\,|J|=q \\ 1 \le i \le k}} \left(\varepsilon_{IJ} \overline{\varphi^i_{I\bar{J}}} \theta^{I^0} \wedge \bar{\theta}^{J^0} \right) \otimes \mathbf{e}_i^*. \end{aligned}$$

It is easy to verify that $** = (-1)^{p+q}$ acting on $A^{p,q}(E)$. It is sometimes desirable to introduce a complex-linear star operator, defined as above but without taking complex conjugates of the components of φ; in this event, the star operator as defined above is denoted $\bar{*}$.

Proposition 8.4 *The formal adjoint of $\bar{\partial}$ is $\bar{\partial}^* = - * \bar{\partial} *$.*

Proof Let $\varphi \in A^{p,q}(E)$ and $\psi \in A^{p,q-1}(E)$. By the definitions of the global inner product (8.1) and the Hodge star operator (8.4),

$$(\bar{\partial}\psi, \varphi) = \int_M \bar{\partial}\psi \wedge *\varphi = (-1)^{p+q} \int_M \psi \wedge \bar{\partial}(*\varphi) + \int_M \bar{\partial}(\psi \wedge *\varphi).$$

Because $\psi \wedge *\varphi$ is of type $(n, n-1)$, $\bar{\partial}(\psi \wedge *\varphi) = d(\psi \wedge *\varphi)$, so the last term vanishes by Stokes' Theorem. The remaining integral is

$$- \int_M \psi \wedge *(*\bar{\partial}* \,\varphi) = (\psi, - *\bar{\partial}* \,\varphi),$$

proving that $\bar{\partial}^* = - * \bar{\partial} *$ as claimed. ∎

Cover M by coordinate charts that are also trivializing neighborhoods for E. Each chart is modelled on $U \times \mathbf{C}^k$, with $U \subset \mathbf{C}^n \simeq \mathbf{R}^{2n}$ an open set, and M is covered by finitely many charts. (It is here that compactness of M is used in an essential way.) To define suitable norms on $A^{p,q}(E)$, first define norms on smooth, compactly supported sections of $\mathbf{R}^{2n} \times \mathbf{C}^k$, then use a partition of unity to express an element of $A^{p,q}(E)$ as a finite sum of smooth, compactly supported sections in coordinate charts.

It is sufficient to consider smooth, compactly supported functions $f : \mathbf{R}^{2n} \to \mathbf{C}$; the norm of a vector-valued function may then be computed with respect to a fixed norm on $\mathbf{C}^k$. Recall that the Fourier transform of f is defined to be

$$\widehat{f}(y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbf{R}^{2n}} f(x) e^{-ix\cdot y}\, dx.$$

For each real number s, define the *Sobolev s-norm* of f to be

$$\|f\|_s^2 = \int_{\mathbf{R}^{2n}} |\widehat{f}(y)|^2 \Big(1 + |y|^2\Big)^s\, dy, \tag{8.5}$$

and define the *Sobolev space* $\mathrm{H}_s(\mathbf{R}^{2n}) = \mathrm{H}_s$ to be the completion in the s-norm of the space of smooth, compactly supported functions on $\mathbf{R}^{2n}$. Let $\mathrm{H}_s^{p,q}(M, E) = \mathrm{H}_s(M, \bigwedge^{p,q} \otimes E)$ denote the completion of $A^{p,q}(E)$ with respect to the Sobolev s-norm as described above. While the norm depends on the choice of trivializing cover and subordinate partition of unity, it is easy to see that the space $\mathrm{H}_s(M, \bigwedge^{p,q} \otimes E)$ and its topology do not depend on these choices; as above, this is a consequence of compactness of M. There is an obvious nesting, namely if $r < s$ then $\mathrm{H}_r \supset \mathrm{H}_s$. Set

$$\mathrm{H}_\infty = \bigcap_{s\in\mathbf{R}} \mathrm{H}_s, \qquad \mathrm{H}_{-\infty} = \bigcup_{s\in\mathbf{R}} \mathrm{H}_s.$$

Intuitively, the rate at which $\widehat{f}(y)$ decays as $|y| \to \infty$ is related ("proportionally") to the smoothness of f. As $s \to \infty$, the term $(1+|y|^2)^s$ forces $\widehat{f}$ to decay rapidly as $|y| \to \infty$. More precisely, a measurable function has finite Sobolev s-norm if and only if the derivatives of order up to and including s are in $L^2(\mathbf{R}^{2n})$; in particular, $\mathcal{C}^s \subset \mathrm{H}_s$:

Proposition 8.5 *For $s \geq 0$, the Sobolev s-norm is equivalent to the norm*

$$\left(\sum_{|I|\leq s} \int_{\mathbf{R}^{2n}} |D^I f(x)|^2\, dx \right)^{1/2},$$

and H_{-s} is the topological dual of H_s.

The norm appearing in Proposition 8.5 comes from an inner product, so the Sobolev space H_s is a Hilbert space. The is following (partial) reverse inclusion is called the *Sobolev Lemma*.

Theorem 8.6 $\mathrm{H}_{s+n+1}(\mathbf{R}^{2n}) \subset \mathcal{C}^s(\mathbf{R}^{2n})$. *Globally,*

$$\mathrm{H}_{s+n+1}(M, \bigwedge^{p,q} \otimes E) \subset \mathcal{C}^s(M, \bigwedge^{p,q} \otimes E).$$

N.B. In $\mathbf{R}^m$, the "n" in Theorem 8.6 becomes $[m/2]$. The dimension n enters via finiteness of the integral

$$\int_{\mathbf{R}^{2n}} \Big(1 + |y|^2\Big)^s \, dy.$$

As a consequence of Theorem 8.6, $\mathrm{H}_\infty(M, E) = A^0(E)$. The next result, called the *Rellich Lemma*, is important for technical reasons; it is used below to extract a compact, self-adjoint operator on a fixed Hilbert space. The Hodge Theorem is then essentially the Spectral Theorem.

Theorem 8.7 *If $s > r$, then the inclusion $\mathrm{H}_s \subset \mathrm{H}_r$ of Hilbert spaces is a compact embedding. Globally, the inclusion*

$$\mathrm{H}_s(M, \textstyle\bigwedge^{p,q} \otimes E) \subset \mathrm{H}_r(M, \textstyle\bigwedge^{p,q} \otimes E)$$

is compact.

The Sobolev norms described above may be given more globally in terms of connections. Specifically, there are natural connections on E and TM (the respective Chern connections), and the Sobolev norms may be computed from covariant derivatives of tensor fields rather than by using a partition of unity. As above, Sobolev norms defined in this way are equivalent to Sobolev norms defined using a partition of unity because M is compact.

Define the *Dirichlet inner product* on $A^{p,q}(E)$ by

$$D(\varphi, \psi) = (\varphi, \psi + \Box_{\bar\partial}\psi); \qquad D(\varphi, \varphi) = \|\varphi\|^2 + \|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2.$$

By expressing $\Box_{\bar\partial}$ in local coordinates as the Euclidean Laplacian plus terms of lower degree (using a so-called *Weitzenböck formula*), it is possible to deduce an *a priori* estimate for the Dirichlet norm, called *Garding's Inequality*.

Theorem 8.8 *For each (p, q), there exists a $C > 0$, depending only on (M, g) and (E, h), such that*

$$\|\varphi\|_1^2 \leq C\, D(\varphi, \varphi) \qquad \textit{for all } \varphi \in A^{p,q}(E).$$

From Garding's Inequality, it follows that the operator $I + \Box_{\bar\partial}$ on $A^{p,q}(E)$ has a bounded weak inverse

$$T : \mathrm{H}_0^{p,q}(M, E) \to \mathrm{H}_1^{p,q}(M, E).$$

By the Rellich Lemma, the induced operator on $\mathrm{H}_0^{p,q}(M, E)$ (T followed by inclusion) is compact, and is clearly self-adjoint. By the Spectral Theorm, there is a decomposition of $\mathrm{H}_0^{p,q}(M, E)$ into finite-dimensional eigenspaces of T. Thus $I + \Box_{\bar\partial}$ is weakly invertible, hence $\Box_{\bar\partial}$ is weakly invertible on

$(\ker \Box_{\bar\partial})^\perp$. Explicitly, given a form $\eta \in \mathrm{H}_0^{p,q}(M,E)$, there is a form $\psi \in \mathrm{H}_0^{p,q}(M,E)$ such that for all $\varphi \in A^{p,q}(E)$,

$$(\psi, \Box_{\bar\partial}\varphi) = (\eta, \varphi).$$

It remains to show that eigenforms of $\Box_{\bar\partial}$, which are eigenforms of $I+\Box_{\bar\partial}$, are smooth. This follows from the following regularity result; intuitively, the inverse of the Laplacian "adds" two derivatives.

Theorem 8.9 *Suppose $\psi \in \mathrm{H}_0^{p,q}(M,E)$ is a weak solution of $\Box_{\bar\partial}\psi = \eta$ for some $\eta \in \mathrm{H}_s^{p,q}(M,E)$. Then $\psi \in \mathrm{H}_{s+2}^{p,q}(M,E)$.*

Smoothness of eigenforms ψ of $\Box_{\bar\partial}$ follows immediately by a "bootstrap" argument, for if $\Box_{\bar\partial}\psi = \lambda\psi$ for some $\lambda \in \mathbf{R}$, then by Theorem 8.9, ψ is in $\mathrm{H}_2^{p,q}(M,E)$. Again by Theorem 8.9 ψ is in $\mathrm{H}_4^{p,q}(M,E)$. Inductively, $\psi \in \mathrm{H}_\infty^{p,q}(M,E) = A^{p,q}(E)$. This completes the proof of the Hodge Theorem. ∎

As above, let $\pi : (E,h) \to (M,g)$ be an Hermitian holomorphic vector bundle over a compact Hermitian manifold. If $\psi \neq 0$ is a harmonic (p,q)-form with values in E, then $*\psi$ is a harmonic $(n-p,n-q)$-form with values in E^*, and the global inner product of these forms is $\|\psi\|^2 > 0$. Since these forms are natural representatives of their Dolbeault cohomology classes, the following duality, the *Kodaira-Serre Duality Theorem*, follows immediately.

Theorem 8.10 *With the above notation, the pairing*

$$H^q\big(M,\Omega^p(E)\big) \otimes H^{n-q}\big(M,\Omega^{n-p}(E^*)\big) \to H^n(M,\Omega^n) \simeq \mathbf{C}$$

is non-degenerate.

The top exterior power of the holomorphic cotangent bundle of M is called the *canonical bundle* K_M; the sheaf of germs of sections is Ω^n, the sheaf of germs of holomorphic n-forms. The Kodaira-Serre Duality Theorem for $p = 0$ becomes

$$H^q(M,E) \overset{\text{dual}}{\simeq} H^{n-q}(M, K_M \otimes E^*).$$

8.2 The Hodge Decomposition Theorem

On a compact Kähler manifold (M,J,g), there is compatibility between the Dolbeault decomposition and Hodge theory for the de Rham complex. Consequently, the cohomology of a compact Kähler manifold satisfies several non-trivial restrictions, see Theorem 8.17 and its corollaries.

Let (M,J,g) be an Hermitian manifold. Recall that there is a two-form ω defined by $\omega(X,Y) = g(X,JY)$ for tangent vectors X and Y. Since (M,g)

is in particular a Riemannian manifold, there is a unique *Levi-Civita* connection ∇ that is *symmetric* (or *torsion-free*) and *compatible* with g; respectively, these conditions mean that for all vector fields X, Y, and Z on M,

$$\begin{aligned} [X,Y] &= \nabla_X Y - \nabla_Y X, \\ Z\Big(g(X,Y)\Big) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned} \tag{8.6}$$

The former is sometimes expressed by saying that the *torsion tensor* $T(X,Y)$ vanishes, while the latter is expressed succinctly as $\nabla g = 0$. On the other hand, TM is a holomorphic, Hermitian vector bundle, so there is a unique *Chern* connection D which is compatible with the metric and also *complex*, namely for which $D^{0,1} = \bar{\partial}$, or (equivalently) for which $DJ = 0$. On a general Hermitian manifold these connections are unrelated. The Kähler condition is equivalent to equality of these connections.

At each point p of a Riemannian manifold, there exist *geodesic normal coordinates*, in which the metric satisfies $g_{ij}(p) = \delta_{ij}$ and the connection satisfies $\nabla_{\partial/\partial x^i}|_p = \partial/\partial x^i$. On an Hermitian manifold, it is generally impossible to find local *holomorphic* coordinates with this property. The Kähler condition is equivalent to existence of a local holomorphic coordinate system in which the metric approximates the Euclidean metric to second order. Each of these properties has an expression in terms of components of tensor fields, Christoffel symbols, and their derivatives, but for present purposes the following conditions suffice.

Proposition 8.11 *Let (M, J, g) be an Hermitian manifold. The following are equivalent.*

i. *The metric is Kähler, i.e. $d\omega = 0$.*

ii. *The Levi-Civita connection is complex.*

iii. *The Chern connection is torsion-free.*

iv. *For each $p \in M$, there exist local holomorphic coordinates centered at p such that $g_{ij} = \delta_{ij} + O(|z|^2)$.*

v. *For each $p \in M$, there is a neighborhood U of p and a smooth, real-valued function $f : U \to \mathbf{R}$, such that $\omega = \sqrt{-1}\partial\bar{\partial} f$ on U.*

The function f appearing in property v. is called a *Kähler potential* (function). Each of the properties above is essentially local, so there is no need to assume M is compact. By contrast, the most striking global results about Kähler manifolds depend on existence of adjoints to various operators, and therefore require compactness.

Proof Each of ii. and iii. is equivalent to equality of the Levi-Civita and Chern connections, so these conditions are equivalent. Closedness of ω may be expressed in terms of derivatives of components of g as either of the conditions

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i} \qquad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}. \tag{8.7}$$

The connection form $\theta = g^{-1}\partial g$ of the Chern connection, and the torsion tensor T, are expressed in a coordinate coframe as

$$\theta^i_j = \sum_{k=1}^n g^{i\bar{k}}\partial g_{j\bar{k}}, \qquad T = \theta \wedge dz = \sum_{i,j,k,\ell=1}^n \left(g^{i\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z^\ell} dz^\ell \wedge dz^j \right) \frac{\partial}{\partial z^i},$$

from which equivalence of i. and iii. follows.

Condition iv. implies $d\omega = 0$ at p for each $p \in M$. Conversely, if $d\omega = 0$, that is, if (8.7) holds locally, then it is straightforward to check that the change of coordinates

$$w^k = z^k - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial g_{j\bar{k}}}{\partial z^i} w^i w^j$$

kills the linear terms in the Taylor expansion of $g_{i\bar{j}}$ about p.

Condition v. implies condition i. immediately. Conversely, assume $d\omega = 0$ in a simply-connected coordinate neighborhood U. By the Poincaré lemma, there is a real one-form η with $d\eta = \omega$ on U. The $(1,0)$ and $(0,1)$-components of η are conjugate since η is real, and since ω is of type $(1,1)$, $\partial\eta^{1,0} = 0$, $\bar{\partial}\eta^{0,1} = 0$, and $\omega = \bar{\partial}\eta^{1,0} + \partial\eta^{0,1}$. By the Dolbeault-Grothendieck lemma, there exists a function φ on U with $\partial\varphi = \eta^{1,0}$ and $\bar{\partial}\bar{\varphi} = \eta^{0,1}$. The function $f = 2\operatorname{Im}\varphi = i(\varphi - \bar{\varphi})$ is the desired potential function. ∎

On a compact Kählerian manifold, non-trivial holomorphic forms are d-closed and non-exact (Proposition 8.13), and holomorphic forms are harmonic with respect to an arbitrary Kähler metric (Proposition 8.14). Each of these is a consequence of a simple type-decomposition argument.

Lemma 8.12 *Let (M, J, g) be a compact Kähler manifold. If $\eta \in \mathcal{Z}^{p,0}_{\bar{\partial}}$ is an exact holomorphic p-form, then $\eta = 0$.*

Proof For every $(p,0)$-form η, $\displaystyle\int_M \eta \wedge \bar{\eta} \wedge \omega^{n-p} = (\eta, *\eta) = \|\eta\|^2$. If $\eta = d\psi$ is d-exact, then

$$\|\eta\|^2 = \int_M d\psi \wedge \bar{\eta} \wedge \omega^{n-p} = \int_M d\Big(\psi \wedge \bar{\eta} \wedge \omega^{n-p}\Big) = 0$$

by Stokes' Theorem (both $\bar{\eta}$ and ω are d-closed), so $\eta = 0$. ∎

Proposition 8.13 *Let (M, J, g) be a compact Kähler manifold. Then there is an injection $H^0(M, \Omega^p_M) \hookrightarrow H^p_d(M, \mathbf{C})$, that is, holomorphic forms are closed and non-exact.*

Proof Let $\eta \neq 0$ be a holomorphic p-form. Then $d\eta = \partial\eta$ is an exact, holomorphic $(p+1)$-form, hence zero by Lemma 8.12. □

Proposition 8.14 *Let (M, J, g) be a compact Kähler manifold, and let η be a holomorphic p-form. Then η is harmonic.*

Proof If η is a holomorphic p-form, then $\bar\partial\eta = 0$, while $*\eta$ is of type $(n-p, n)$, so for type reasons $\bar\partial^*\eta = -*\bar\partial{*}\eta = 0$. □

Let (M, J, g) be a compact Kähler manifold with Kähler form ω. Define the operator $d^c = \sqrt{-1}(\bar\partial - \partial)$. Then d and d^c are real operators, as is

$$dd^c = 2\sqrt{-1}\partial\bar\partial. \tag{8.8}$$

Let $d^* = -*d*$ and $(d^c)^*$ denote the formal adjoints, and let $\Pi^{p,q} : A^*(M) \to A^{p,q}(M)$ denote the projection operator, so that

$$\bigoplus_{p+q=r} \Pi^{p,q} = I : A^r(M) \to A^r(M).$$

Define $\mathrm{L} : A^{p,q}(M) \to A^{p+1,q+1}(M)$ by $\mathrm{L}(\eta) = \omega \wedge \eta$, and let $\Lambda = \mathrm{L}^*$ be the adjoint with respect to the global inner product. On $(1,1)$-forms, Λ is the trace with respect to ω. The formula $[\mathrm{L}, d] = 0$ is easily proven; if η is an r-form, then

$$[\mathrm{L}, d]\,\eta = \mathrm{L}(d\eta) - d(\mathrm{L}\eta) = \omega \wedge d\eta - d(\omega \wedge \eta) = 0$$

since $d\omega = 0$. Taking adjoints, $[\Lambda, d^*] = 0$. The following commutator formulas are messy to establish, even in flat Euclidean space. However, once verified there, they may be deduced to hold on an arbitrary compact Kähler manifold by virtue of Proposition 8.11 iv.

Proposition 8.15 *On a compact Kähler manifold, $[\mathrm{L}, d^*] = d^c$, $[\Lambda, d] = -(d^c)^*$, and*

$$[\mathrm{L}, \Lambda] = \bigoplus_{p,q} (p+q-n)\Pi^{p,q}.$$

The type decomposition of the second formula in Proposition 8.15 is

$$[\Lambda, \bar\partial] = -\sqrt{-1}\bar\partial^*, \qquad [\Lambda, \partial] = -\sqrt{-1}\partial^*. \tag{8.9}$$

Define the *de Rham Laplacian* $\Box_d$ to be the second-order operator $dd^* + d^*d$. On a Riemannian manifold, there is a Hodge theorem analogous to the $\bar\partial$-Hodge theorem, to the effect that every de Rham class contains a unique harmonic representative. Proposition 8.15 has the following important consequence, which gives the compatibility between the de Rham and Dolbeault harmonic forms.

Theorem 8.16 *On a compact Kähler manifold,* $[\mathrm{L}, \Box_d] = 0$*,* $[\Lambda, \Box_d] = 0$*, and*

$$\Box_d = \Box_{\bar\partial} + \Box_\partial, \qquad \Box_{\bar\partial} = \Box_\partial. \tag{8.10}$$

In particular, $\Box_d$ *maps* $A^{p,q}(M)$ *to itself.*

In words, Theorem 8.16 says that on a compact Kähler manifold, $\bar\partial$-harmonic forms are d-harmonic, and that $\Box_d$ preserves bidegree. It therefore makes sense to *define* the de Rham (p,q)-harmonic space to be the kernel of $\Box_d$ acting on $A^{p,q}(M)$, and this is nothing but the space of $\bar\partial$-harmonic (p,q)-forms. The importance is that while Dolbeault cohomology—and the space of Dolbeault harmonic forms—is defined in terms of the complex-analytic structure of M, the de Rham spaces depend only on the smooth structure of M. But each of these spaces is isomorphic, via the corresponding Hodge theory, to a space of harmonic forms, and by Theorem 8.16 these harmonic spaces coincide. This result is known as the *Hodge Decomposition Theorem* for compact Kähler manifolds:

Theorem 8.17 *Let* (M, J, g) *be a compact Kähler manifold. Then there are isomorphisms*

$$H^r_d(M, \mathbf{C}) \simeq \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M), \qquad H^{p,q}_{\bar\partial}(M) \simeq_{\mathbf{C}} \overline{H^{q,p}_{\bar\partial}(M)}.$$

In particular, $h^{p,q} = h^{q,p} = h^{n-p,n-q}$*.*

Corollary 8.18 *Let* M *be a compact Kählerian manifold. Then the odd Betti numbers* b_{2k+1} *of* M *are even.*

Proof By the de Rham theorem, $b_{2k+1} = \dim_{\mathbf{C}} H^{2k+1}(M, \mathbf{C})$, which is equal to

$$\sum_{p+q=2k+1} h^{p,q} = 2\sum_{p=0}^{k} h^{p,2k+1-p}$$

since $h^{p,q} = h^{q,p}$ by the Hodge Decomposition Theorem. ∎

Corollary 8.19 $H^q(\mathbf{P}^n, \Omega^p) = H^{p,q}_{\bar\partial}(\mathbf{P}^n) = \mathbf{C}$ *if* $0 \le p = q \le n$*, and is* 0 *otherwise.*

Proof The de Rham cohomology (with complex coefficients) is $\mathbf{C}$ in even dimensions $\le 2n$ and is zero otherwise. By the Hodge Decomposition Theorem, $1 = b_{2p} = \sum_{i=0}^{2p} h^{i,2p-i}$. Thus $h^{p,p} = 1$ for $0 \le p \le n$, and all other Hodge numbers vanish. ∎

A very pretty consequence of the Hodge Decomposition Theorem is the so-called *Hard Lefschetz Theorem*. The modern proof, due to Chern, is an extremely elegant application of representation theory for $\mathfrak{sl}(2, \mathbf{C})$. Let M

be a compact Kähler manifold of complex dimension n, and introduce the weighted projection operator

$$h : A^*(M) \to A^*(M), \qquad h = \sum_{r=0}^{2n} (n - r)\Pi^r.$$

By Theorem 8.16, each of the operators L, Λ, and h commutes with $\Box_d = 2\Box_{\bar\partial}$, so that each operator acts on the space $\mathbf{H}_d^*(M, \mathbf{C})$ of de Rham harmonic forms. Chern made the following observation.

Proposition 8.20 $[\Lambda, \mathrm{L}] = h$, $[h, \mathrm{L}] = -2\mathrm{L}$, *and* $[h, \Lambda] = 2\Lambda$.

In words, the association

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftrightarrow \Lambda \qquad Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftrightarrow \mathrm{L} \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leftrightarrow h$$

defines a representation of the Lie algebra $\mathfrak{sl}(2, \mathbf{C})$ on the finite-dimensional complex vector space $\mathbf{H}_d^*(M, \mathbf{C})$. The irreducible submodules give the Lefschetz decomposition of the cohomology of M. A *primitive* element in an $\mathfrak{sl}(2, \mathbf{C})$-module is an eigenvector v of H such that $Xv = 0$.

Proposition 8.21 *If V is a finite-dimensional $\mathfrak{sl}(2, \mathbf{C})$-module, then primitive elements exist. If V is an irreducible $(N+1)$-dimensional representation space for $\mathfrak{sl}(2, \mathbf{C})$ and if $v \in V$ is primitive, then V is generated as a vector space by the elements $\{v, Yv, Y^2v, \dots\}$. The eigenvalues of H are the integers $-N$, $-N+2$, $\dots$, N, the λ-eigenspaces V_λ are one-dimensional, and V is the direct sum of the eigenspaces of H. The operators H, X, and Y act by*

$$H(V_\lambda) = V_\lambda, \qquad X(V_\lambda) = V_{\lambda+2}, \qquad Y(V_\lambda) = V_{\lambda-2},$$

with the convention that $V_\lambda = 0$ if λ is not an eigenvalue of H.

Irreducible $(N+1)$-dimensional representations of $\mathfrak{sl}(2, \mathbf{C})$ exist and are unique up to equivalence. In fact, the Nth symmetric power of the standard representation of $\mathfrak{sl}(2, \mathbf{C})$ on $\mathbf{C}^2$ is an irreducible representation of dimension $(N+1)$. Generally, the space of *primitive* elements PV of an $\mathfrak{sl}(2, \mathbf{C})$-module V is $\ker X$, and V is a direct sum

$$V = PV \oplus Y\,PV \oplus Y^2\,PV \oplus \cdots$$

Let (M, J, g) be a compact Kähler manifold, and define the *primitive cohomology* to be

$$P^{n-k}(M) = \ker(\mathrm{L}^{k+1}) = (\ker\Lambda) \cap \mathbf{H}_d^{n-k} = \omega\text{-}\textit{traceless } (n-k)\text{-forms}.$$

Applying Proposition 8.21 gives the Hard Lefschetz Theorem:

Theorem 8.22 *If (M, J, g) is a compact Kähler manifold of complex dimension n, then for $0 \leq k \leq n$, the map $\mathrm{L}^k : \mathbf{H}_d^{n-k} \to \mathbf{H}_d^{n+k}$ is an isomorphism, and*

$$\mathbf{H}_d^r(M) = \bigoplus_{k \in \mathbf{Z}} \mathrm{L}^k\, P^{r-2k}(M).$$

If $M \subset \mathbf{P}^N$ (equipped with the restriction of the Fubini-Study metric), then the map L corresponds via Poincaré duality to intersecting with a hyperplane class, and L^k corresponds to intersecting with a linear subspace of dimension $N - k$. Primitive classes in M correspond to cycles that do not intersect a hyperplane, and which therefore lie in an affine coordinate chart. Such cycles cannot be represented by embedded complex submanifolds, of course. If $M \subset \mathbf{P}^2$ is a smooth plane curve, then every real one-dimensional cycle is primitive, and holomorphic one-forms represent primitive cohomology classes.

9

Chern Classes

A vector bundle of rank k is a locally trivial family of k-dimensional vector spaces parametrized by points of a manifold M. Characteristic classes quantify global non-triviality of the family. The prototypical result expresses the first Chern class of a line bundle, defined to be the image of L under the connecting homomorphism $c_1 : H^1(M, \mathcal{O}^\times) \to H^2(M, \mathbf{Z})$, in terms of curvature, and as the Poincaré dual of the divisor of a meromorphic section.

Proposition 9.1 *Let $(L, h) \to M$ be an Hermitian holomorphic line bundle over a compact manifold. Then $c_1(L)$ is represented by the closed $(1,1)$-form*

$$\frac{1}{2\pi}\gamma_1(L, h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h. \tag{9.1}$$

If s is a global meromorphic section of L, so that $L = [(s)]$, then $c_1(L)$ is Poincaré dual to $(s) = (s)_0 - (s)_\infty$.

In equation (9.1), h may be interpreted as the norm of a local holomorphic section; it is easy to see the form $\gamma_1(L, h)$ does not depend on the choice of local section. More is in fact true: Every smooth form ρ representing $2\pi c_1(L)$ is the curvature form of an Hermitian structure conformal to h. The proof of this fact is an easy application of the Hodge theorem.

Chern classes may be introduced in several ways, from points of view ranging from functorial to differential-geometric to obstruction-theoretic. Because Chern classes are topological objects, there is no reason to work in the holomorphic category. The present treatment is to specify axioms for Chern classes, then to define certain closed forms using curvature forms of a connection in E, verifying that these forms satisfy the axioms. After this a few other definitions are given whose equivalence is asserted without proof.

Axioms for Chern Classes Let $\pi : E \to M$ be a smooth complex vector bundle over a smooth CW complex. For most purposes, M may be taken to be a smooth compact manifold, though occasionally it will be necessary to consider infinite-dimensional Grassmannian manifolds.

i. There is an element $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^*(M, \mathbf{R})$, called the *total Chern class* of E, with $c_j(E) \in H^{2j}(M, \mathbf{R})$.

ii. (Naturality) For every smooth map $f : N \to M$, $c(f^*E) = f^*\big(c(E)\big) \in H^*(N, \mathbf{R})$.

iii. (Whitney Sum Formula) If L_i, $1 \leq i \leq k$ are line bundles, then $c(\bigoplus_i L_i) = \prod_i c(L_i)$.

iv. (Normalization) If $L \to \mathbf{P}^1$ is the tautological bundle, and if ω denotes the positive generator of $H^2(\mathbf{P}^1, \mathbf{Z})$, then $c(L) = 1 - \omega$.

These axioms characterize Chern classes. It may be helpful to remark that if $E \to M$ is a vector bundle, then there exists a space N and a map $f : N \to M$ such that $f^*E \to N$ splits into a sum of line bundles, and such that the map $f^* : H^*(M, \mathbf{R}) \to H^*(N, \mathbf{R})$ is injective. This is the basis of the so-called *Splitting Principle*, to the effect that a universal relation among Chern classes (such as the Whitney Sum Formula) which holds for line bundles holds for arbitrary vector bundles. Construction of a “splitting space” N is an inductive variant on the construction of the tautological bundle $\tau_E \subset \pi^*E \to \mathbf{P}(E)$.

Example 9.2 Let $\omega \in H^2(\mathbf{P}^n, \mathbf{Z})$ denote the positive generator. There is an exact sequence of vector bundles $0 \to \mathbf{C} \to \mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathbf{C}^{n+1} \to TP^n \to 0$ over $\mathbf{P}^n$. From the Splitting Principle and the Whitney Sum Formula,

$$c(TP^n) = c\Big(\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathbf{C}^{n+1}\Big) = (1 + \omega)^{n+1} \in H^*(\mathbf{P}^n, \mathbf{Z}).$$

For example, $c(T\mathbf{P}^1) = 1+2\omega$ and $c(T\mathbf{P}^2) = 1+3\omega+3\omega^2$. □

9.1 Chern Forms of a Connection

Curvature is an infinitesimal measure of the failure of E to be “geometrically trivial.” Certain invariant functions of the curvature forms of a connection give closed $2j$-forms on M that quantify the “twisting” of E. If E is trivialized over an open set U, then it is possible to choose a connection whose curvature vanishes identically in U. If this neighborhood is “enlarged as much as possible,” then the global non-triviality of E will be concentrated along $M \setminus U$. Geometrically, the curvature is concentrating along certain cycles in M.

Let $\mathfrak{gl}(k, \mathbf{C}) = \mathbf{C}^{k \times k}$ denote the Lie algebra of the general linear group. A symmetric polynomial $f \in S^j(\mathfrak{gl}(k, \mathbf{C}))$ is *invariant* (or ad-invariant) if $f(a^{-1}X_1 a, \ldots, a^{-1}X_j a) = f(X_1, \ldots, X_j)$ for all $a \in GL(k, \mathbf{C})$ and $X_i \in \mathfrak{gl}(k, \mathbf{C})$ Infinitesimally this means

$$f([Y, X_1], X_2, \ldots, X_j) + \cdots + f(X_1, \ldots, X_{j-1}, [Y, X_j]) = 0$$

for all X_i, $Y \in \mathfrak{gl}(k, \mathbf{C})$.

The elementary symmetric functions $f_j \in S^j(\mathfrak{gl}(k, \mathbf{C}))$ are defined by

$$\det\left(I - \frac{1}{2\pi i}X\right) = \sum_{j=0}^{k} f_j(X) = 1 - \frac{1}{2\pi i}\operatorname{tr} X + \cdots + \left(\frac{-1}{2\pi i}\right)^k \det X.$$

Their importance is indicated by the following well-known result from algebra.

Proposition 9.3 *The elementary symmetric functions $\{f_j\}$ generate the algebra of invariant symmetric polynomials on $\mathfrak{gl}(k, \mathbf{C})$.*

Let $D : A^0(E) \to A^1(E)$ be a connection, for example, the Chern connection of an Hermitian structure, and let $D^2 \in A^2(\operatorname{End} E)$ be the curvature operator. With respect to a local frame for E, there exists a curvature matrix Ω of two-forms, and under a change of frame Ω transforms by a similarity. Consequently it makes sense to define $\gamma_j(E, D)$, the *jth Chern form of E* with respect to the connection D, by

$$\det\left(I - \frac{1}{2\pi i}\Omega\right) = \sum_{j=0}^{k} \gamma_j(E, D). \tag{9.2}$$

In other words, $\gamma_j(E, D) = f_j(\Omega, \ldots, \Omega)$.

Theorem 9.4 *The forms $\gamma_j(E, D)$ are closed, real $2j$-forms. The de Rham class represented by $\gamma_j(E, D)$ does not depend on the connection D, and the classes $c_j(E) = [\gamma_j(E, D)]$ satisfy Axioms* i.–iv.

Proof By the Bianchi Identity, $D\Omega = 0$. It follows that the Chern forms are closed, since

$$d\,\gamma_j(E, D) = D\,\gamma_j(E, D) = f_j(D\Omega, \Omega, \ldots, \Omega) + \cdots + f_j(\Omega, \ldots, \Omega, D\Omega) = 0.$$

To see that the cohomology class is independent of D, choose connections D_0 and D_1, and let $D_t = (1 - t)D_0 + tD_1$ be the path of connections interpolating them. With respect to a local frame, there is a matrix-valued one-form $\alpha = \theta_1 - \theta_0$ that transforms by a similarity under a change of frame. The connection form of D_t is $\theta_t = \theta_0 + t\alpha$, the curvature form is $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$, and

$$\frac{d}{dt}\Omega_t = d\alpha - \alpha \wedge \theta_t - \theta_t \wedge \alpha = D_t\alpha.$$

Since f_j is ad-invariant, and α and Ω transform by a similarity under change of frame, the expression $f_j(\alpha, \Omega_t, \dots, \Omega_t)$ is a globally defined $(2j-1)$-form on M. Set

$$\phi = j\int_0^1 f_j(\alpha, \Omega_t, \dots, \Omega_t)\, dt \in A^{2j-1}(M).$$

A short calculation gives $jd[f_j(\alpha, \Omega_t, \dots, \Omega_t)] = (d/dt)f_j(\Omega_t, \dots, \Omega_t)$. It follows that

$$\begin{aligned} d\phi &= j\int_0^1 d\Big(f_j(\alpha, \Omega_t, \dots, \Omega_t)\Big)\, dt \\ &= \int_0^1 \frac{d}{dt} f_j(\Omega_t, \dots, \Omega_t)\, dt = \gamma_j(E, D_1) - \gamma_j(E, D_0), \end{aligned}$$

proving that the de Rham class of $\gamma_j(E, D)$ is independent of D. It remains to verify Axioms i.–iv. This is straightforward, since connections and curvature are natural with respect pullbacks by smooth maps, and since the connection in a sum of line bundles splits into a diagonal matrix of one-forms. □

Corollary 9.5 *Let $E \to M$ be a holomorphic vector bundle, and let E^* be the dual bundle. The Chern classes of E^* are related to those of E by $c_j(E^*) = (-1)^j c_j(E)$.*

Proof If E is endowed with an Hermitian structure h, then there is a conjugate-linear vector bundle isomorphism given by the map $v \in E \longmapsto h(\cdot, v) \in E^*$. Thus E^* is isomorphic as a smooth complex vector bundle to the dual bundle $\overline{E}$, so $c_j(E^*) = c_j(\overline{E})$. If D is a connection in E, then $\overline{D}$ is a connection in $\overline{E}$; since the curvature matrix Ω is pure imaginary, the curvature matrix of $\overline{D}$ differs by a sign from the curvature matrix of D. By equation (9.2) the Chern forms satisfy $\gamma_j(\overline{E}, \overline{D}) = (-1)^j \gamma_j(E, D)$. □

9.2 Alternate Definitions

Let $p : E \to M$ be a vector bundle of rank k over a compact manifold. The *tautological bundle* τ_E is defined as follows. Let $E^\times$ denote the complement of the zero section, and let $\mathbf{C}^\times$ act by scalar multiplication in the fibres of $E^\times$. The quotient space is denoted $\mathbf{P}(E)$ and is called the *projectivization* of E. There is an induced map $p : \mathbf{P}(E) \to M$ whose fibres are $(k-1)$-dimensional projective spaces. Consider the bundle $p^*E \to \mathbf{P}(E)$. For $x \in$

M, each point of $\mathbf{P}(E_x)$ represents a line through the origin in the fibre p^*E_x, and the tautological bundle is defined to be the line subbundle of p^*E whose fibre at a point of $\mathbf{P}(E)$ is the line represented by that point. It is not difficult to verify that the total spaces of $E^\times$ and $\tau_E^\times$ are biholomorphic; indeed, the restriction of $p : p^*E \to E$ to $\tau_E^\times$ is a biholomorphism. Along the zero section, p collapses the fibres of $\mathbf{P}(E)$. The total space of τ_E is said to be obtained from the total space of E by *blowing up* the zero section.

Let E^* denote the dual bundle, and let $\zeta = -c_1(\tau_{E^*}) \in H^2(\mathbf{P}(E^*), \mathbf{R})$; since τ_{E^*} is a line bundle, the first Chern class has been defined. By the Leray-Hirsch Theorem from topology, the cohomology ring $H^*(\mathbf{P}(E), \mathbf{R})$ is generated, as a $p^*H^*(M, \mathbf{R})$-module, by ζ subject to a single relation

$$\sum_{j=0}^{k}(-1)^j c_j \zeta^{k-j} = \zeta^k - c_1\zeta^{k-1} + \cdots + (-1)^k c_k = 0 \tag{9.3}$$

for some $c_j \in H^{2j}(M, \mathbf{R})$.

Theorem 9.6 *The classes c_j in* (9.3) *are the Chern classes of E.*

Another functorial definition arises via classifying spaces. Let $G_k(\mathbf{C}^n) = G_{k,n}$ denote the Grassmannian manifold of k-dimensional linear subspaces of $\mathbf{C}^n$. There are standard inclusions $\mathbf{C}^n \hookrightarrow \mathbf{C}^n \times 0 \subset \mathbf{C}^{n+1}$, which induce holomorphic embeddings $G_{k,n} \subset G_{k,n+1}$. The union over n, with the direct limit topology (in which a set is closed if and only if its intersection with each $G_{k,n}$ is closed), is called the infinite Grassmann manifold G_k.

For each $n \in \mathbf{N}$, there is a *universal* or *tautological* rank k vector bundle $U_{k,n} \subset G_{k,n} \times \mathbf{C}^n$, whose fibre at $x \in G_{k,n}$ is the k-dimensional subspace of $\mathbf{C}^n$ represented by the point x. There are bundle maps $U_{k,n} \subset U_{k,n+1}$ compatible with the embeddings of base spaces, and the direct limit is the universal bundle $U_k \to G_k$.

Consider the *standard flag* $0 = \mathbf{C}^0 \subset \mathbf{C}^1 \subset \mathbf{C}^2 \subset \cdots \subset \mathbf{C}^n$. Every k-plane $W \subset \mathbf{C}^n$ gives rise to a non-decreasing sequence $\dim(W \cap \mathbf{C}^i)$ whose consecutive terms differ by at most one. Let $1 \le m_1 < m_2 < \cdots < m_k \le n$ denote "the dimensions in which jumps occur," that is

$$\dim(W \cap C^{m_i}) = i, \qquad \dim(W \cap C^{m_i - 1}) = i - 1.$$

Proposition 9.7 *Each sequence $1 \le m_1 < m_2 < \cdots < m_k \le n$ gives rise to an open cell of (complex) dimension $\sum_i (m_i - i)$ consisting of k-planes in $\mathbf{C}^n$ having the prescribed incidence on the standard flag. The closure of such a cell is a subvariety of $G_{k,n}$, called a* Schubert cycle. *The number of Schubert cycles of dimension r is equal to the number of partitions of r into at most k integers, each of which is at most $n - k$.*

In the infinite Grassmannian $\mathbf{G}_k$, there is a Schubert cycle of (complex) dimension j corresponding to each partition of j into at most k integers.

Theorem 9.8 *If $j \leq k$, then the jth Chern class of the universal bundle is the cohomology class which takes the value 1 on the Schubert cycle corresponding to the partition $(1, \ldots, 1)$ and vanishes on all other Schubert cycles.*

The Chern classes of the universal bundle generate the integral cohomology ring $H^*(G_k)$, and there are no relations among them. In particular, $H^*(G_k, \mathbf{Z})$ has no torsion and no non-zero elements of odd dimension.

The importance of the universal bundle $U_k \to G_k$ is that the Grassmannian is the classifying space for the unitary group $U(k)$. More concretely, there is a one-to-one correspondance between equivalence classes of complex rank k vector bundles over a CW complex M and homotopy classes of maps $\phi : M \to G_k$. The correspondance associates the bundle $E = \phi^* U_k$ to a "classifying map" ϕ. The Chern classes of E may now be defined as the pullbacks of the universal Chern classes under ϕ.

There is a simple intuitive principle that describes the Poincaré duals of the Chern classes of E: The $(k - j + 1)$st Chern class is Poincaré dual to the cycle on which j generic smooth sections of E are linearly dependent. In particular, the top Chern class $c_k(E)$ is the Euler class, which is Poincaré dual to the zero set of a generic section. This principle has a precise obstruction-theoretic statement. To describe this, it is convenient to associate to E the *bundle of j-frames* $V_j(E)$, whose fibre over $x \in M$ is the Stiefel manifold of j-frames in E_x.

Theorem 9.9 *Let $E \to M$ be a complex vector bundle of rank k over a CW complex. Then there is a section of $V_j(E)$ over the $(2k - 2j + 1)$-skeleton of M, and $c_{k-j+1}(E)$ is the primary obstruction to extending over the $(2k - 2j + 2)$-skeleton.*

Example 9.10 The top Chern class is the Euler class; applying this fact to the tangent bundle of a complex manifold gives the generalized Gauss-Bonnet Theorem

$$\chi(M) = \int_M c_n(M) = \left(\frac{i}{2\pi}\right)^n \int_M \det \Omega. \tag{9.4}$$

One way to see this geometrically is to embed $M \hookrightarrow G_{n,N}$ for N sufficiently large by a classifying map for TM. The top Chern class $c_n(M)$ is Poincaré dual to the Schubert cycle consisting of all n-planes in $\mathbf{C}^N$ that lie in the hyperplane C^{N-1}. Let $v = \mathbf{e}_N$ be the Nth standard basis vector, and define a smooth vector field on M by orthogonal projection to $T_x M \subset \mathbf{C}^N$. This vector field vanishes exactly when $T_x M \subset \mathbf{C}^{N-1}$, so the Euler number of M is equal to the intersection of M and the Poincaré dual of $c_n(M)$. □

Example 9.11 By investigating the homotopy groups of the unitary groups, Bott showed that if $E \to S^{2n}$ is a complex vector bundle over the $2n$-sphere, then $(n-1)!$ divides $\chi(S^{2n}, E)$. Applying this result to the tangent bundle $E = TS^{2n}$, for which $\chi = 2$, implies that if the sphere S^{2n} admits a complex structure, then $n \leq 3$. □

The *Chern character* is a ring homomorphism from the K-theory of M to the rational cohomology $H^*(M, \mathbf{Q})$. It is defined for vector bundles in terms of a formal factorization of the total Chern class. Let x be an indeterminate, and write $\sum_j c_j(E)x^j = \prod_j (1 + \xi_j x)$. The Chern character of E is defined to be $\sum_j \exp \xi_j$. In terms of a curvature matrix,

$$\mathrm{ch}(E) = \mathrm{tr}\, \exp\left(\frac{-1}{2\pi i}\Omega\right).$$

Theorem 9.12 *If E and F are vector bundles over M, then*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \qquad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

If the total Chern class of $E = TM$ is factored as above, then the *Todd class* of M is defined by

$$\mathrm{td}(M) = \prod_{j=0}^{k} \frac{\xi_j}{1 - \exp \xi_j} \in H^*(M, \mathbf{Q}).$$

In terms of the Chern classes of E and M, the Chern character and Todd class are given by

$$\begin{aligned} \mathrm{ch}(E) &= k + c_1(E) + \frac{1}{2}\Big(c_1(E)^2 - 2c_2(E)\Big) + \cdots \\ \mathrm{td}(M) &= 1 + \frac{1}{2}c_1(M) + \frac{1}{12}\Big(c_1(M)^2 + c_2(M)\Big) + \cdots \end{aligned} \tag{9.5}$$

Let $E \to M$ be a holomorphic vector bundle over a compact manifold. The *Euler characteristic of M with respect to E* is

$$\chi(M, E) = \sum_{i=0}^{n} (-1)^i\, h^i(M, E).$$

The *holomorphic Euler characteristic of M* is defined to be $\chi(M, \mathcal{O}_M)$. The Riemann-Roch-Hirzebruch Theorem expresses the Euler characteristic $\chi(M, E)$ as the integral over M of a universal polynomial in the curvature forms of connections on E and TM:

Theorem 9.13 *The Euler characteristic is given by*

$$\chi(M, E) = \int_M \mathrm{ch}(E)\,\mathrm{td}(M).$$

Example 9.14 Let M be a curve of genus g, and let ω be the positive generator of $H^2(M, \mathbf{Z})$. Recall that if $E \to M$ is a holomorphic vector bundle of rank k, then the degree of E is

$$d = \deg E = \int_M c_1(E).$$

Using equation (9.5), the Todd class of M and Chern character of E are given by

$$\mathrm{td}(M) = 1 + (1-g)\omega \qquad \mathrm{ch}(E) = k + d\omega.$$

By Theorem 9.13, $\chi(M, E) = d + k(1-g)$. If E is a line bundle, then $h^0(M,E) - h^1(M,E) = \chi(M,E) = d+1-g$. Consequently, if the degree of E is at least g, then E has a non-trivial holomorphic section. □

Example 9.15 Let $M = \mathbf{P}^2$, and let $\omega \in H^2(M, \mathbf{Z})$ be the positive generator. By Example 9.2, the total Chern class of $T\mathbf{P}^2$ is $1 + 3\omega + 3\omega^2$, so the Todd class is $\mathrm{td}(\mathbf{P}^2) = 1 + (3/2)\omega + 3\omega^2$ by equation (9.5). For each integer k, consider the rank-two vector bundles

$$T(k) := T\mathbf{P}^2 \otimes \mathcal{O}_{\mathbf{P}^2}(k), \qquad T^*(k) := T^*\mathbf{P}^2 \otimes \mathcal{O}_{\mathbf{P}^2}(k).$$

By Theorem 9.12, the Chern character of $T(k)$ is

$$\begin{aligned} \mathrm{ch}\, T(k) &= \Big(\mathrm{ch}\, T\mathbf{P}^2\Big)\Big(\mathrm{ch}\, \mathcal{O}_{\mathbf{P}^2}(k)\Big) \\ &= \Big(2 + 3\omega + (3/2)\omega^2\Big)\Big(1 + k\omega + (1/2)\omega^2\Big) \\ &= 2 + (2k+3)\omega + \Big(k^2 + 3k + (3/2)\Big)\omega^2. \end{aligned}$$

A similar calculation shows $\mathrm{ch}\, T^*(k) = 2 + (2k-3)\omega + \big(k^2 - 3k + (3/2)\big)\omega^2$. Theorem 9.13 applied to $T^*(k)$ gives $\chi(T^*(k)) = k^2 - 1 = (k+1)(k-1)$. When $k = 0$, this is easily verified, since $T^*\mathbf{P}^2 = \Omega^1$ is the bundle of holomorphic one-forms on $\mathbf{P}^2$. The Dolbeault isomorphism gives $h^i(T^*\mathbf{P}^2) = h^{1,i}(\mathbf{P}^2)$, so by direct calculation

$$\chi(T^*\mathbf{P}^2) = h^{1,0}(\mathbf{P}^2) - h^{1,1}(\mathbf{P}^2) + h^{1,2}(\mathbf{P}^2) = 0 - 1 + 0 = -1$$

as expected. If $k > 0$, then $h^0(T^*(-k)) = 0$; indeed, if the bundle $T^*(-k)$ had a non-trivial holomorphic section, then tensoring with a section of $\mathcal{O}_{\mathbf{P}^2}(k)$ would give a non-trivial section of $T^*\mathbf{P}^2$, that is, a non-trivial holomorphic one-form on $\mathbf{P}^2$.

Applying Theorem 9.13 to $T(k)$ gives $\chi(T(k)) = k^2+6k+8 = (k+3)^2-1$. For $k = 0$,

$$h^0(T\mathbf{P}^2) - h^1(T\mathbf{P}^2) + h^2(T\mathbf{P}^2) = \chi(T\mathbf{P}^2) = 8.$$

Since $h^0(T\mathbf{P}^2)$ is the dimension of the space of holomorphic vector fields on $\mathbf{P}^2$, and the automorphism group of $\mathbf{P}^2$ is the eight-dimensional Lie group

$PGL(3, \mathbf{C})$, $h^0(T\mathbf{P}^2) = 8$. The canonical bundle $K_{\mathbf{P}^2} = \bigwedge^2 T^*\mathbf{P}^2$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2}(-3)$, so Kodaira-Serre duality gives $h^2(T\mathbf{P}^2) = h^0(T^*(-3))$ which is zero as noted above. Consequently, $h^1(T\mathbf{P}^2) = 0$. The vector space $H^1(TM)$ is the space of infinitesimal deformations of the holomorphic structure of M. The previous calculation shows that $\mathbf{P}^2$ has no non-trivial infinitesimal deformations, that is, the holomorphic structure of $\mathbf{P}^2$ is *infinitesimally rigid*. Much more is known:

Theorem 9.16 *Let M be a complex surface having the homotopy type of $\mathbf{P}^2$. Then M is biholomorphic to $\mathbf{P}^2$.*

In more detail, a complex structure J on the underlying four-manifold $\mathbf{P}^2$ may be regarded as a section of the quotient $GL(4, \mathbf{R})/GL(2, \mathbf{C})$ of the real frame bundle of $T\mathbf{P}^2$ by the bundle of complex frames. The theorem asserts that if J is an integrable complex structure homotopic to the standard complex structure J_0, then J is biholomorphic to J_0.

Surprisingly, it is not known whether or not there exists a holomorphic structure of "general type" on the underlying real four-manifold $\mathbf{P}^2$. □

10
Vanishing Theorems and Applications

Line bundles are of fundamental importance in the study of holomorphic manifolds for two principal reasons. First, their sections play the role of global holomorphic functions, which are lacking on compact manifolds. Second, line bundles have a cohomological interpretation, which helps make their classification amenable to homological methods. The latter is of course mostly of importance because of the former; results about vanishing of cohomology can be used to extract information about global sections of line bundles. Interesting geometric theorems may in turn be expressible in terms of sections of line bundles.

10.1 Ampleness and Positivity

Let $p : L \to M$ be a holomorphic line bundle over a (connected) compact manifold of (complex) dimension n. Assume further that for each point $x \in M$ there is a holomorphic section of L that is non-zero at x; in this event, L is said to be *generated by global sections*. Associated functorially to L is a holomorphic map from M to a projective space, defined as follows. Let $V = H^0(M, L)$ be the space of global sections of L. The map $i_L : M \to \mathbf{P}(V^*)$ sends each point $x \in M$ to the set of sections of L that vanish at x. Because L is generated by global sections, the set of sections vanishing at $x \in M$ is a hyperplane in V; indeed, with respect to a trivialization of L, evaluation at x is a linear functional $V \to \mathbf{C}$. A hyperplane in V determines a linear functional on V up to a multiplied constant, which is exactly a point of the projective space $\mathbf{P}(V^*)$. The map $i_L : M \to \mathbf{P}(V^*)$ is holomorphic, but is not generally an embedding. Holomorphicity is easily seen (equation (10.1) below) by choosing a basis $\{s_i\}_{i=0}^N$ of sections of L, which defines an isomorphism $V \simeq \mathbf{C}^{N+1}$. With respect to a trivialization, the map i_L is given by sending $x \in M$ to "evaluation of $[s_0 : \cdots : s_N]$ at x," which is an element of $\mathbf{P}(\mathbf{C}^{N+1*})$. For simplicity, it is customary to write

$$x \in M \longmapsto i_L(x) = [s_0(x) : \cdots : s_N(x)] \in \mathbf{P}(\mathbf{C}^{N+1}), \tag{10.1}$$

from which holomorphicity of i_L is clear.

A line bundle $p : L \to M$ is *very ample* if the map i_L is an embedding, i.e. is one-to-one and has differential of rank n everywhere. In this event, an unravelling of definitions shows that L itself is the pullback of the hyperplane bundle $[H] = \mathcal{O}_{\mathbf{P}^N}(1)$ by the map i_L. If some tensor power L^k is very ample, then L is *ample*. A divisor D is (*very*) *ample* if the associated line bundle $[D]$ is (very) ample.

There is a precise differential-geometric analogue of ampleness, called positivity. A $(1,1)$-form η is *positive* if $\eta(Z, \bar{Z}) > 0$ for every $(1,0)$ tangent vector Z. A positive $(1,1)$-form is exactly a fundamental form of a Kähler metric. A $(1,1)$-class $\Omega \in H^{1,1}_{\bar\partial}(M, \mathbf{R})$ is *positive* if there is a positive $(1,1)$-form representing Ω; a positive $(1,1)$-class is also called a *Kähler class*, since such classes are exactly those represented by Kähler forms. A holomorphic line bundle $p : L \to M$ over a complex manifold is *positive* if there is an Hermitian structure in L whose first Chern form $\gamma_1(L, h)$ is positive. Equivalently, $p : L \to M$ is positive if $c_1(L)$ is a Kähler class. Finally, a divisor is positive if the Poincaré dual $(1,1)$-class is positive. By an obvious sign change, it makes sense to speak of negative $(1,1)$-forms and classes, line bundles, and divisors.

A very ample line bundle—and hence an ample line bundle—is positive; indeed, the hyperplane bundle $[H] = \mathcal{O}_{\mathbf{P}^n}(1)$ is positive, so if L is very ample, then $L = i_L^*[H]$ is positive. The converse assertion is the *Kodaira Embedding Theorem*, see Theorem 10.10 below.

Remark 10.1 *There is another "positivity" condition a divisor may possess: A divisor D is* numerically effective *(or "nef") if the intersection product $D \cdot C$ is non-negative for every smooth curve $C \subset M$. A positive divisor is nef, as is easily seen by Poincaré duality. A nef divisor is not generally positive (i.e. ample), however. There is a numerical condition, called the* Nakai-Moishezon Criterion*, under which a divisor is ample. Specifically, if $D^k \cdot V > 0$ for every smooth, k-dimensional subvariety of M ($k = 1, \ldots, n$ arbitrary), then D is ample. It is not difficult to show this condition is necessary.*

Let $p : L \to M$ be a holomorphic line bundle, and let $f : M \to M$ be a biholomorphism (or an "automorphism"). If the action of f lifts to a bundle map $\widetilde{f} : L \to L$, then there is an induced projective automorphism covering f. It can be shown that if $p : L \to M$ is very ample, then every automorphism of M lifts to L. Since $i_L : M \to \mathbf{P}^N$ is an embedding, every automorphism of M is induced by an automorphism of the ambient projective space.

10.2 The Kodaira-Nakano Vanishing Theorem

Let $\pi : L \to M$ be an Hermitian holomorphic line bundle over a compact Kähler manifold with Kähler form ω, let D denote the Chern connection of (L, h), and let $D = D^{1,0} + \bar{\partial}_L$ be the type decomposition. There is a map

$$\mathrm{L} : A^{p,q}(L) \to A^{p+1,q+1}(L), \qquad \eta \longmapsto \omega \wedge \eta$$

whose adjoint with respect to the global inner product is denoted Λ. In analogy with equation (8.9) above, there is a commutator relation

$$[\Lambda, \bar{\partial}_L] = -\sqrt{-1}(D^{1,0})^*. \tag{10.2}$$

The main technical result of this section, the *Kodaira-Nakano Vanishing Theorem*, follows from a short calculation using this equation. The result for $p = 0$ is due to Kodaira, while Akizuki and Nakano proved the version stated here. (The content of Remark 10.3 below is called *Nakano's Inequality.*)

Theorem 10.2 *Let $\pi : L \to M$ be a positive line bundle over a compact complex manifold. Then*

$$H^q\big(M, \Omega^p(L)\big) = 0 \quad \textit{for } p + q > n = \dim_{\mathbf{C}} M.$$

Proof Choose an Hermitian structure h in L so that the Chern form $\gamma_1(L, h) = \sqrt{-1}\Omega$ is a Kähler form. Then

$$D^2\eta = R\eta = \Omega \wedge \eta = -\sqrt{-1}\omega \wedge \eta = -\sqrt{-1}\mathrm{L}\eta;$$

in words, the curvature operator R has a dual interpretation as the second covariant derivative D^2 and as the algebraic operator $-\sqrt{-1}\mathrm{L}$. Comparing these interpretations and using the various commutation identities will prove the theorem.

Let $\eta \in \mathbf{H}^{p,q}(L)$ be a harmonic (p, q)-form with values in L. It suffices to show that if $p + q > n$, then $\eta = 0$. By Proposition 8.15, $[\mathrm{L}, \Lambda] = n - (p + q)$ acting on (p, q)-forms with values in L. The curvature operator is of type $(1, 1)$, so $D^2 = \bar{\partial}_L D^{1,0} + D^{1,0}\bar{\partial}_L = -\sqrt{-1}\mathrm{L}$. In particular, since η is harmonic, $\bar{\partial}_L^*\eta = 0$ and $D^2\eta = \bar{\partial}_L D^{1,0}\eta$, so

$$\mathrm{L}\Lambda\, \eta = \sqrt{-1}\left(\bar{\partial}_L D^{1,0} + D^{1,0}\bar{\partial}_L\right)\Lambda\, \eta, \qquad \mathrm{L}\, \eta = \sqrt{-1}\, \bar{\partial}_L D^{1,0}\, \eta. \tag{10.3}$$

Simple algebra and integration by parts gives

$$\begin{aligned}(n-p-q)\,\|\eta\|^2 &= ([\Lambda,\mathrm{L}]\,\eta,\eta) \;=\; (\Lambda\mathrm{L}\,\eta,\eta)-(\mathrm{L}\Lambda\,\eta,\eta)\\ &= \sqrt{-1}(\Lambda\,\bar\partial_L D^{1,0}\,\eta,\eta)-\sqrt{-1}(D^{1,0}\bar\partial_L\,\Lambda\,\eta,\eta) \qquad \text{by (10.3)}\\ &= \|D^{1,0}\eta\|^2+\|(D^{1,0})^*\eta\|^2\end{aligned}$$

since $\Lambda\bar\partial_L=\bar\partial_L\Lambda-\sqrt{-1}(D^{1,0})^*$ by equation (10.2). This is non-negative, so $\eta=0$ provided $p+q>n$, as claimed. ∎

Remark 10.3 *Examination of the terms involved shows that, as operators on forms,*

$$\begin{aligned}[\Lambda,\mathrm{L}] &= -\sqrt{-1}\,\big[\bar\partial_L,[\Lambda,D^{1,0}]\big]+D^{1,0}(D^{1,0})^*+(D^{1,0})^*D^{1,0}\\ &= -\sqrt{-1}\,\big[\bar\partial_L,[\Lambda,D^{1,0}]\big]+\Box_{D^{1,0}}.\end{aligned}$$

The bracket term does not contribute to $([\Lambda,\mathrm{L}]\,\eta,\eta)$ *since* $\bar\partial_L\eta=0$ *and* $\bar\partial_L^*\eta=0$.

Corollary 10.4 *If* $1\le q\le n-1$ *and* $k\in\mathbf{Z}$*, then* $H^q\big(\mathbf{P}^n,\mathcal{O}_{\mathbf{P}^n}(k)\big)=0$.

10.3 Cohomology of Projective Manifolds

An extremely important and useful consequence of Theorem 10.2 is the *Lefschetz Hyperplane Theorem*, which roughly asserts that the "interesting" cohomology of a projective manifold is in the middle dimension. The proof is a typical application of a vanishing theorem: A geometric problem is phrased in terms of exact sequences of sheaves, then isomorphisms are extracted from the long exact cohomology sequence by cohomology vanishing.

Theorem 10.5 *Let* $M\subset\mathbf{P}^N$ *be a smooth algebraic subvariety of complex dimension* $n+1$*,* $V\subset M$ *a smooth hypersurface, e.g. a hyperplane section of* M*. Then the inclusion map* $i:V\to M$ *induces an isomorphism* $i^*:H^{p,q}_{\bar\partial}(M,\mathbf{C})\to H^{p,q}_{\bar\partial}(V,\mathbf{C})$ *for* $p+q\le n-1$*, and an injection* $i^*:H^{p,q}_{\bar\partial}(M,\mathbf{C})\to H^{p,q}_{\bar\partial}(V,\mathbf{C})$ *for* $p+q=n$.

In particular, the map $i^*:H^k(M,\mathbf{Q})\to H^k(V,\mathbf{Q})$ is an isomorphism for $k\le n-1$ and is injective for $k=n$. Note that the real dimension of V is $2n$. Even the special case $M=\mathbf{P}^{n+1}$ gives interesting information, cf. Corollary 10.6 below.

Proof By the Dolbeault theorem, it suffices to show that the map $H^q(M,\Omega^p_M)\to H^q(V,\Omega^p_V)$ induced by restriction is an isomorphism for $p+q\le n-1$ and is injective for $p+q=n$.

Consider the following three sheaves on M:

$$\begin{aligned} \Omega^p_M &= \text{germs of holomorphic } p\text{-forms on } M \\ \Omega^p_M\big|_V &= \text{restriction of } \Omega^p_M \text{ to } V\text{, extended by zero} \\ \Omega^p_V &= \text{germs of holomorphic } p\text{-forms on } V\text{, extended by zero} \end{aligned}$$

Intuitively, the sheaf $\Omega^p_M\big|_V$ contains information in directions not tangent to V, even though the only non-zero stalks are over points of V. There are sheaf morphisms $r : \Omega^p_M \to \Omega^p_M\big|_V$ and $i : \Omega^p_M\big|_V \to \Omega^p_V$. Let $s \in H^0(M, [V])$ be given by local defining functions for V. The kernel of r is the sheaf of germs of p-forms vanishing along V, which is the sheaf $\Omega^p_M(-V) = \Omega^p_M \otimes [-V]$, so there is an exact sequence

$$0 \longrightarrow \Omega^p_M(-V) \xrightarrow{\wedge s} \Omega^p_M \xrightarrow{r} \Omega^p_M\big|_V \longrightarrow 0 \qquad (10.4)$$

The restriction map i fits into the short exact sequence

$$0 \longrightarrow \Omega^{p-1}_V(-V) \xrightarrow{\wedge ds} \Omega^p_M\big|_V \xrightarrow{i} \Omega^p_V \longrightarrow 0. \qquad (10.5)$$

The 1-form ds defines a global section of the bundle $\nu^*_{V/M} \otimes [V]$, showing explicitly that the map i "loses exactly the non-tangential information about germs of p-forms along V."

By Kodaira-Serre duality and the Kodaira-Nakano vanishing theorem,

$$H^q\left(M, \Omega^p_M(-V)\right) = 0 \quad \text{and} \quad H^q\left(V, \Omega^{p-1}_V(-V)\right) = 0 \quad \text{for } p+q < n.$$

Applying this vanishing to the long exact sequences associated to the sheaf sequences in equations (10.4) and (10.5) gives maps

$$H^q(M, \Omega^p_M) \xrightarrow{r} H^q(M, \Omega^p_M\big|_V) \xrightarrow{i} H^q(V, \Omega^p_V)$$

which are isomorphisms if $p + q \le n - 1$ and are injective for $p + q = n$. ∎

Corollary 10.6 *Let $V^n \subset \mathbf{P}^N$ be a complete intersection of complex dimension n. Then the Dolbeault cohomology of V matches that of $\mathbf{P}^N$ except possibly in the middle dimension. Precisely, $h^{p,p}(V) = 1$ for $0 \le p \le n$, $2p \ne n$, and all other Hodge numbers $h^{p,q}(V)$ with $p + q \ne n$ vanish. In particular, a complete intersection of positive dimension is connected, and is simply-connected if of dimension at least two.*

Proof By induction on the codimension of V, Theorem 10.5 immediately implies $h^{p,p}(V) = 1$ for $0 \le 2p \le n-1$, and all other Hodge numbers $h^{p,q}(V)$ with $p+q \le n-1$ vanish. Kodaira-Serre duality gives the information about Hodge numbers $h^{p,q}$ with $p + q > n$. ∎

The next result gives topological information from which it is possible to calculate Hodge numbers of some interesting varieties.

Proposition 10.7 *If $V = V_{d,n}$ is a smooth hypersurface of degree d in $\mathbf{P}^{n+1}$, then the topological Euler characteristic is equal to*

$$\chi(V_{d,n}) = d\sum_{j=0}^{n}\binom{n+2}{j+2}(-d)^j = \frac{1}{d}\big((1-d)^{n+2} - 1 + (n+2)d\big). \quad (10.6)$$

Proof The set of degree d polynomials defining a smooth hypersurface is the complement of proper algebraic subvarieties and hence is connected. Two such polynomials therefore have diffeomorphic zero loci, so $\chi(V_{d,n})$ depends only on d and n. Let $H \subset \mathbf{P}^{n+1}$ be a hyperplane such that $H \cap V$ is smooth. Projection of V to H is a d-sheeted branched cover, whose branch locus is a smooth, degree d variety of dimension $n-1$. Triangulate the branch locus $H \cap V$, and extend to a triangulation of $H \simeq \mathbf{P}^n$. Pulling back gives a triangulation of V, and calculating the Euler characteristic from this triangulation gives $\chi(V) = d\,\chi(H) - (d-1)\chi(H \cap V)$, or

$$\chi(V_{d,n}) = d(n+1) - (d-1)\chi(V_{d,n-1}).$$

Since $\chi(V_{d,0}) = d$, equation (10.6) follows by induction on n. ∎

Example 10.8 (K3 surfaces) Let V be a smooth quartic hypersurface in $\mathbf{P}^3$. By the second adjunction formula, the canonical bundle of V is trivial, and by the Lefschetz hyperplane theorem, $h^{1,0}(V) = 0$. A complex surface satisfying these properties is called a K3 surface.

By Proposition 10.7, the Euler characteristic of V is 24. From Theorem 10.5, and Poincaré duality, $b_0(V) = b_4(V) = 1$ and $b_1(V) = b_3(V) = 0$, so $b_2(V) = 22$. Most of the Hodge numbers may be found immediately from Corollary 10.6: $h^{0,0}(V) = h^{2,2}(V) = 1$, and $h^{p,q}(V) = 0$ if $p+q = 1$ or 3. To calculate the remaining Hodge numbers, note that since the canonical bundle of V is trivial, $h^{2,0}(V) = h^{0,2}(V) = 1$, leaving $h^{1,1}(V) = 20$. The significance of $h^{1,1}$ is seen by the Dolbeault theorem and Kodaira-Serre duality. Since the bundle of holomorphic 1-forms is dual to the holomorphic tangent bundle, and the canonical bundle is trivial,

$$H^1(V, \Omega^1_V) \simeq H^1(M, TM).$$

The space $H^1(M, TM)$ is the space of infinitesimal deformations of the complex structure, so this calculation shows there is a 20-dimensional family of complex surfaces deformation equivalent to a smooth quartic in $\mathbf{P}^3$. An easy count shows that only a 19-dimensional family can be accounted for by quartic hypersurfaces themselves, and indeed a "generic" deformation of a quartic hypersurface is not algebraic.

A great deal about K3 surfaces is known. Among other things, every K3 surface is Kählerian, every smooth K3 surface is diffeomorphic to the Fermat quartic in $\mathbf{P}^3$, and there is a branched covering from the "Teichmüller space"—which is diffeomorphic to a 20-dimensional ball—to the moduli space of K3 surfaces. K3 surfaces enjoy a number of striking differential-geometric properties, some of which are discussed in more detail later, though one can be mentioned here: A smooth K3 surface admits three integrable almost-complex structures and an Hermitian metric which is Kähler with respect to all three. These almost-complex structures satisfy the same algebraic relations that the unit quaternions do, so there is actually an S^2 of integrable almost-complex structures.

K3 surfaces are of interest to algebraic geometers because of their role in the classification of complex surfaces, to four-manifold topologists because of their curious intersection form, to number theorists because of their relationship to elliptic curves and higher-dimensional Abelian varieties, and to theoretical physicists working on supersymmetric field theories because of their differential-geometric properties. □

Example 10.9 (Calabi-Yau 3-folds) Consider a smooth quintic hypersurface $V \subset \mathbf{P}^4$. Again, the canonical bundle is trivial, and by Proposition 10.7 $\chi(V) = -200$. By Corollary 10.6, $h^{p,p}(V) = 1$ for $0 \leq p \leq 3$, all other Hodge numbers $h^{p,q}(V)$ with $p + q \neq 3$ vanish, and $h^{0,3}(V) = h^{3,0}(V) = 1$ because the canonical bundle is trivial. Thus $h^{1,2}(V) = h^{2,1}(V) = 101$, and by Kodaira-Serre duality this may be interpreted as the dimension of the space of infinitesimal deformations of the complex structure. The space of quintic polynomials in five variables has dimension $\binom{9}{5} = 126$, while the automorphism group of $\mathbf{P}^4$ is 24-dimensional. Dividing out by non-zero scalars accounts for the last parameter: $101 = 126 - 24 - 1$. Thus every small deformation of a smooth quintic hypersurface is a smooth quintic hypersurface.

These manifolds are examples of what are usually called *Calabi-Yau three-folds*, for reasons that are explained later. Generally, a Calabi-Yau three-fold is a three-dimensional Kählerian manifold with no holomorphic one-forms ($h^{0,1} = 0$) and with trivial canonical bundle ($h^{0,3} = 0$). Calabi-Yau three-folds are three-dimensional analogues of K3 surfaces, but are much less well understood. Even Calabi-Yau three-folds which arise as complete intersections in weighted projective spaces are not classified (up to deformation type), though it is known that the number of deformation types is less than 8,000. □

10.4 The Kodaira Embedding Theorem

As mentioned earlier, every positive divisor is ample. This is the content of the Kodaira Embedding Theorem. An alternate statement (Corollary 10.11) gives an intrinsic criterion for a compact Kählerian manifold to be projective algebraic.

Theorem 10.10 *Let $p : L \to M$ be a positive line bundle over a compact* *complex manifold. Then there exists a positive integer k_0 such that if $k \geq k_0$, then $L^k \to M$ is very ample.*

Proof (Brief sketch) Proving the Kodaira Embedding Theorem amounts to establishing the following conditions for $k \gg 1$, i.e. that there is a k_0 such that if $k \geq k_0$, then

- The bundle L^k is generated by global sections, i.e. for every $x \in M$ there is a section $s \in H^0(M, L^k)$ with $s(x) \neq 0$;
- Sections of L^k separate points, that is, if x and y are distinct points of M, then there is a section s with $s(x) = 0$ and $s(y) \neq 0$;
- Sections of L^k separate tangent vectors, in the sense that if Z and W are vectors tangent to M at x, then there is a section s such that $ds(x)Z = 0$ and $ds(x)W \neq 0$.

Each of these conditions has an interpretation in terms of morphisms of ideal sheaves and sheaves of germs of sections of L^k. Various morphisms are shown to be surjective by using the Kodaira-Nakano Vanishing Theorem. ∎

Corollary 10.11 *Let (M, J, g) be a compact Kähler manifold whose Kähler form is rational, that is, lies is $H^{1,1}_{\bar{\partial}}(M, \mathbf{C}) \cap H^2(M, \mathbf{Q})$. Then there is an integer $N > 0$ and an embedding $i : M \to \mathbf{P}^N$ such that the pullback of the Fubini-Study form on $\mathbf{P}^N$ is an integral multiple of the Kähler form of g.*

Proof The long exact sequence associated to the exponential sheaf sequence on M contains the terms

$$H^1(M, \mathcal{O}_M^{\times}) \xrightarrow{c_1} H^2(M, \mathbf{Z}) \longrightarrow H^2(M, \mathcal{O}_M).$$

Because the Kähler form ω_0 of g is rational, there is a positive integer m such that the $(1,1)$-form $\omega = m\omega_0$ is integral. For type reasons, the image of ω in $H^2(M, \mathcal{O}_M)$ is zero. By exactness, there is a holomorphic line bundle $p : L \to M$ with $c_1(L) = \omega > 0$. By Theorem 10.10, L is ample, so there is an integer $k > 0$ such that L^k is very ample and the restriction of the Fubini-Study metric to the image of M is $k\omega = km\omega_0$. ∎

Example 10.12 A metric g whose Kähler form is rational is called a *Hodge metric*, and a manifold admitting a Hodge metric is called a *Hodge manifold.* Not every Kählerian manifold is a Hodge manifold; an example is provided by the two-dimensional compact complex torus whose lattice in $\mathbf{C}^2$ is generated by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{-2} \\ \sqrt{-3} \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{-5} \\ \sqrt{-7} \end{bmatrix}.$$

In fact, this torus admits no non-constant meromorphic functions. □

Theorem 10.10 has a number of geometrically interesting consequences. Some of these are listed below; the proofs are for the most part immediately apparent.

Corollary 10.13 *Let M_1 and M_2 be projective algebraic manifolds. Then $M_1 \times M_2$ is projective algebraic.*

Corollary 10.14 *Let M be a compact Kählerian manifold with $b_2(M) = 1$. Then M is projective algebraic. In particular, if M is a compact complex curve or a compact, irreducible Hermitian symmetric space, then M is projective.*

Corollary 10.15 *If $\pi : \widetilde{M} \to M$ is a finite unbranched covering of compact complex manifolds, then M is projective if and only if $\widetilde{M}$ is.*

The next two results are immediate consequences of the proof of Theorem 10.10, but are not immediate from the sketch given above.

Corollary 10.16 *If M is projective algebraic, and if $\pi : \widetilde{M} \to M$ is a blow-up of M at one point, then $\widetilde{M}$ is projective algebraic.*

Corollary 10.17 *Let $L \to M$ be an ample line bundle. For every line bundle $L' \to M$, there is an integer k_0 such that if $k \geq k_0$, then $L^k \otimes L'$ is very ample.*

10.5 The Hodge Conjecture

There is a generalization of the Kodaira-Nakano vanishing theorem which is reminiscent of Corollary 10.17, and which has a useful GAGA-type consequence, see Corollary 10.19 below. While Theorem 10.18 is immediate from Corollary 10.17, a complete proof of the former is provided.

Theorem 10.18 *Let $p : L \to M$ be a positive line bundle over a compact complex manifold, and let $L' \to M$ be an arbitrary line bundle. Then there exists a $k_0 \in \mathbf{N}$ such that*

$$H^q(M, L^k \otimes L') = 0 \qquad \text{for } k \geq k_0, \quad q > 0.$$

Proof By hypothesis, there is an Hermitian structure in L whose first Chern form ω_0 is positive. Endow $L' \otimes K_M^{-1}$ with an Hermitian structure h, and consider the first Chern form $\gamma = \gamma_1(L' \otimes K_M^{-1}, h)$. For each $x \in M$, there is an integer k_x such that the $(1,1)$-form $k_x\omega_0 + \gamma > 0$ on some neighborhood of x. By compactness of M, there is a covering of M by finitely many such neighborhoods; take k_0 to be the maximum of the k_x's. If $k \geq k_0$, then $k\omega_0 + \gamma > 0$ everywhere on M, that is, $L^k \otimes L' \otimes K_M^{-1}$ is a positive line bundle. Interpreting K_M as the bundle associated to the sheaf Ω^n_M and using Theorem 10.10 with $p = n$ shows that if $q > 0$, then $H^q(M, L^k \otimes L') = H^q\big(M, \Omega^n_M(L^k \otimes L' \otimes K_M^{-1})\big) = 0$. ∎

Corollary 10.19 *Let $M \subset \mathbf{P}^N$ be a smooth submanifold. Then every holomorphic line bundle $L \to M$ is of the form $[D]$ for some divisor. Equivalently, L has a meromorphic section.*

Proof The strategy is to let D_0 be a smooth hyperplane section of M; if s is a global holomorphic section of $L(kD_0)$, and if t is a global section of $[kD_0]$, then t^{-1} is a meromorphic section of $[kD_0]^* = [-kD_0]$, so $s \otimes t^{-1}$ is a global meromorphic section of L.

The result is vacuous if M is a point. Assume the assertion is true for manifolds of dimension $n-1$. Choose a smooth hyperplane section $D_0 = M \cap H$, and choose k so that $H^1\big(M, L(kD_0)\big) = 0$ (by Theorem 10.18) and so that $L(kD_0)|_V$ has a holomorphic section (induction hypothesis).

Let $s \in H^0(M, [D_0])$ be a global section vanishing along D_0; tensoring with s and restriction from M to V fit into a short exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_M\big(L((k-1)D_0)\big) \xrightarrow{\otimes s} \mathcal{O}_M\big(L(kD_0)\big) \xrightarrow{r} \mathcal{O}_V\big(L(kD_0)|_V\big) \longrightarrow 0.$$

The long exact sequence contains the terms

$$H^0\big(M, L(kD_0)\big) \to H^0\big(V, L(kD_0)|_V\big) \to 0,$$

from which it follows that $L(kD_0)$ has a non-trivial holomorphic section. ∎

Remark 10.20 *Corollary 10.19 follows from the Kodaira embedding theorem as well. Let $D_0 = M \cap H$ be a smooth hyperplane section of M. The line bundle $[D_0]$ is positive, hence ample by Theorem 10.10, and by Corollary 10.17 there is an integer k such that $L(kD_0)$ is very ample. Let s be a global holomorphic section of $L(kD_0)$, and let t be a global section of $[kD_0]$. Then t^{-1} is a meromorphic section of $[kD_0]^* = [-kD_0]$, so $s \otimes t^{-1}$ is a global meromorphic section of L.*

Let (M, J, g) be a compact Kähler manifold. By Hodge theory, the de Rham cohomology (or singular cohomology) has a filtration

$$\begin{aligned} H^r(M, \mathbf{R}) &= \bigoplus_{p \leq q} \Big(\Big[H^{p,q}_{\bar\partial}(M, \mathbf{C}) \oplus H^{q,p}_{\bar\partial}(M, \mathbf{C})\Big] \cap H^r(M, \mathbf{R})\Big) \\ &=: \bigoplus_{\substack{p \leq q \\ p+q=r}} H^{p,q}(M, \mathbf{R}). \end{aligned}$$

A natural question is to determine which cycles are Poincaré dual to classes in $H^{p,q}(M, \mathbf{R})$. A homology class is *analytic* if there is a representative which is a rational linear combination of fundamental classes of analytic subvarieties of M. The Poincaré dual of an analytic class is of type (p, p).

Hodge Conjecture If $M \subset \mathbf{P}^N$ is smooth, then every class in $H^{p,p}(M, \mathbf{Q})$ is Poincaré dual to an analytic class.

The only case in which the Hodge Conjecture is known completely is for $p = 1$, by the *Lefschetz Theorem on* $(1, 1)$*-classes*:

Theorem 10.21 *Let $M \subset \mathbf{P}^N$ be a smooth projective variety. Then every $(1, 1)$-class η in $H^{1,1}(M, \mathbf{R}) \cap H^2(M, \mathbf{Z})$ is Poincaré dual to a divisor.*

Proof (Brief sketch) The map $i_* : H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O})$ induced by inclusion $i : \mathbf{Z} \hookrightarrow \mathcal{O}_M$ factors as

$$H^2(M, \mathbf{Z}) \longrightarrow H^2_d(M, \mathbf{C}) \xrightarrow{\Pi^{0,2}} H^{0,2}_{\bar\partial}(M, \mathbf{C}).$$

Consequently, if η is an integral $(1, 1)$-class, then $i_*\eta = 0$, which implies η is in the image of the Chern class map $c_1 : H^1(M, \mathcal{O}^\times) \to H^2(M, \mathbf{Z})$. By Corollary 10.17, η is Poincaré dual to a divisor. □

Exercises

Exercise 10.1 Prove that if $E \to M$ is a vector bundle over a compact manifold, then $c_1(\det E) = c_1(E) \in H^2(M, \mathbf{Z})$. ◇

Exercise 10.2 Let $V \subset \mathbf{P}^n$ be a complete intersection of multi-degree $(d_1, \ldots, d_k)$. Calculate the first Chern class of TV. ◇

Exercise 10.3 Let (M, J, g) be a compact Kähler manifold, and let ψ be a d-exact real (p, p)-form. Prove that there exists a real $(p - 1, p - 1)$-form η with $\sqrt{-1}\partial\bar\partial\eta = \psi$.
Suggestion: First write $\psi = d(\varphi + \bar\varphi)$, then consider the type decomposition upon expanding this out. Use the Hodge Theorem to write φ as the sum of a harmonic form and a $\bar\partial$-exact form $\partial\phi$, then express η in terms of ϕ. ◇

Exercise 10.4 Let $p : L \to M$ be a holomorphic line bundle over a compact Kähler manifold, and let $\rho \in 2\pi c_1(L)$ be a smooth $(1, 1)$-form. Prove there is an Hermitian structure in L whose first Chern form is ρ.
Suggestion: Let $\rho_0 = \gamma_1(L, h_0)$ be a curvature form; write $\rho - \rho_0 = \sqrt{-1}\partial\bar\partial\phi$, then search for a smooth function $u : M \to \mathbf{R}$ such that the curvature of $e^{2u}h_0$ is ρ. ◇

11

Curvature and Holomorphic Vector Fields

Let (M, J, g) be an Hermitian manifold, and let D be the Levi-Civita connection of the Riemannian metric g. The curvature tensor R maps local real vector fields X, Y, and Z to the local vector field

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

It is often convenient to use the four-tensor R' defined by

$$R'(X, Y, W, Z) = g\Big(R(X, Y)Z, W\Big). \tag{11.1}$$

The following properties of R' are well-known facts from differential geometry.

Proposition 11.1 *The tensor R' is skew-symmetric in X and Y, and in W and Z, and is symmetric under exchange of (X, Y) and (W, Z). Further, R' satisfies the "first Bianchi identity"*

$$\mathfrak{S}_{Y,W,Z} R'(X, \cdot, \cdot, \cdot) = 0. \qquad \text{(cyclic sum)}$$

If in addition g is Kähler with respect to J, then $R'(JX, JY, W, Z) = R'(X, Y, W, Z)$.

The skew-symmetry properties of R' encapsulate the fact that the curvature operator $D^2 : A^0(M) \to A^2(\operatorname{End} TM)$ is a two-form with values in $\operatorname{End} TM$, and that this endomorphism is skew-symmetric. Alternately, R' may be regarded as a symmetric endomorphism of the bundle of two-forms on M.

The *sectional curvature* of a real two-plane $P \subset T_xM$ is the value $R'(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2)$ of the curvature tensor on an orthonormal basis of P. Geometrically, the sectional curvature is the Gaussian curvature at x of the surface in M obtained by exponentiating P. The sectional curvature function K is defined on the Grassmannian bundle of real two-planes in TM. If P is a complex line, i.e. a J-invariant real two-plane, then the sectional curvature is equal to $R'(\mathbf{e}, J\mathbf{e}, \mathbf{e}, J\mathbf{e})$. The restriction of the sectional curvature function to the bundle of complex lines is called the *holomorphic sectional curvature* K_{hol}.

If the sectional curvature function $K : G_2(TM) \to \mathbf{R}$ is constant, then the curvature tensor has an explicit algebraic expression in terms of the metric g; in particular, for each $c \in \mathbf{R}$, there is a local model space with constant sectional curvature c. A similar fact is true when g is a Kähler metric with constant holomorphic sectional curvature. If g is (geodesically) complete—meaning every geodesic extends to have domain $\mathbf{R}$—and simply-connected, then spaces of constant curvature are classified.

Theorem 11.2 *Let (M, J, g) be a complete, simply-connected Kähler manifold of dimension n and constant holomorphic sectional curvature c. If $c > 0$, then g is isometric to a multiple of the Fubini-Study metric on $\mathbf{P}^n$; if $c = 0$, then g is isometric to the flat metric on $\mathbf{C}^n$; if $c < 0$, then g is isometric to a multiple of the "Bergmann metric," whose Kähler form is given by $\omega = \sqrt{-1}\partial\bar{\partial}\log(1 - \|z\|^2)$ on the unit ball in $\mathbf{C}^n$.*

At each point x of a standard model space, the holomorphic sectional curvature achieves the maximum of the (absolute value of) the sectional curvature, and the sectional curvatures are "1/4-pinched" in the sense that

$$(1/4)|K_{\mathrm{hol}}(x)| \le |K(x)| \le |K_{\mathrm{hol}}(x)|.$$

In fact, there is a simple formula for the sectional curvature of a two-plane P in terms of the angle between P and JP. The integral Fubini-Study metric on $\mathbf{P}^n$ has holomorphic sectional curvature 4π.

11.1 Ricci Curvature

The *Ricci tensor* of g is the symmetric, real two-tensor r whose value on vector fields X and Y is defined to be the trace of the endomorphism $V \mapsto R(V, X)Y$; symbolically,

$$\mathrm{r}(X, Y) = \mathrm{tr}\left(V \longmapsto R(V, X)Y\right).$$

The Ricci tensor may be expressed more concretely in terms of the tensor R' and a unitary frame $\{\mathbf{e}_i\}_{i=1}^n$ for T_xM as

$$\mathrm{r}(X, Y) = \sum_{i=1}^{n} R'(\mathbf{e}_i, X, \mathbf{e}_i, Y).$$

Because r is symmetric, it is determined by the real, J-invariant quadratic form $Q(X) = \mathrm{r}(X, X)$; in fact, the Ricci curvature is determined by restricting further to the unit tangent bundle, that is, to the sphere bundle of unit vectors in TM.

As a real, symmetric, J-invariant two-tensor, r is associated to a real $(1,1)$-form ρ, called the *Ricci form* of g, and defined by

$$\rho(X, Y) = \mathrm{r}(X, JY). \tag{11.2}$$

The profound differences between Riemannian and Kählerian geometry are in large part due to existence of the Ricci form, and to the following properties it enjoys.

Proposition 11.3 *The Ricci form is closed, and represents $2\pi c_1(M)$. If the curvature R is viewed as a symmetric endomorphism of $\bigwedge^{1,1} TM$, then $\rho = R(\omega)$. The Ricci form is $\sqrt{-1}$ times the curvature of the canonical bundle of M with the Hermitian structure induced by g; concretely, if the Kähler form of g is given in local coordinates by $\omega = \sqrt{-1} \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, then the Ricci form is equal to*

$$\rho = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}). \tag{11.3}$$

The geometric significance of Proposition 11.3 is that on a compact Kählerian manifold the Ricci form has a cohomological interpretation, that the Ricci form varies within a fixed de Rham class as the Kähler metric varies arbitrarily, and that the Ricci form depends only on the complex structure J and the volume form of the metric. The following useful fact may be proven in a fashion very similar to the proof of part v. of Proposition 8.11.

Proposition 11.4 *Let Ω be a Kähler class, and let ω_0 be a smooth representative. For every representative ω, there is a smooth, real-valued function, unique up to added constants, such that $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial} f$.*

Intuitively, smooth forms in a fixed Kähler class are parametrized by smooth functions, and the set of Kähler forms in a fixed Kähler class is (identified with) a convex open set in an infinite-dimensional vector space. The importance is that various geometric existence questions—such as existence of an Einstein metric, which reduces to an overdetermined nonlinear system of PDE's in the Riemannian situation—reduce to a single PDE in the Kählerian situation.

11.2 Holomorphic Vector Fields

It is assumed throughout this section that (M, J, g) is a fixed compact (connected) Kähler manifold. Every holomorphic vector field $Z \in H^0(M, T^{1,0}M)$

corresponds to an infinitesimal automorphism of J, that is, to a real vector field $X = \mathrm{Re}\, Z$ such that $L_X J = 0$.

Theorem 11.5 *The group* $\mathrm{Aut}(M)$ *of automorphisms of* M *is a complex Lie group whose Lie algebra* $\mathfrak{H}$ *is exactly the space of holomorphic vector fields on* M*.*

Since the space of holomorphic vector fields on a compact manifold is finite-dimensional, the automorphism group of a compact complex manifold is finite-dimensional. A "generic" compact manifold has no automorphisms at all, and under various hypotheses strong restrictions can be put on the number of automorphisms. The following theorem for Riemann surfaces is due to Hurwitz.

Theorem 11.6 *Let* M *be a compact curve of genus* $g \geq 2$*. Then* $|\mathrm{Aut}(M)| \leq 84(g-1)$*.*

In words, a compact Riemann surface of genus at least two has only finitely many automorphisms, and the number of automorphisms is bounded by the genus. A generic Riemann surface of genus at least two has no automorphisms, and while the bound in Theorem 11.6 is sharp for infinitely many genera $g \geq 3$, it also fails to be sharp for infinitely many genera. Indeed, a result of Accola asserts that for infinitely many genera $g \geq 3$, the order of the automorphism group of a compact Riemann surface of genus g is bounded above by $8(g+1)$.

Example 11.7 Let $M \subset \mathbf{P}^2$ be the (singular) curve with equation $x^2z^4(x-z) - y^7 = 0$. Then M has genus three and 168 automorphisms. There is a smooth planar model of this curve, called the *Klein quartic curve.* The automorphism group of the Klein quartic curve is the simple group of order 168. □

To study the space of holomorphic vector fields, it is convenient to dualize to the space of $(0,1)$-forms and use Hodge theory. If X is a vector field, then the dual one-form $X^\flat$ is defined by $X^\flat(Y) = g(X,Y)$. If X is a real vector field, then $X^\flat$ is a real one-form, while if X is of type $(1,0)$ (e.g. if X is holomorphic) then $X^\flat$ is of type $(0,1)$. The inverse map from one-forms to vector fields is denoted $\sharp$, and these maps are read "flat" and "sharp" respectively. The terminology comes from tensor calculus, since flat lowers indices and sharp raises them. Specifically, if $g_{\alpha\bar\beta}$ are the components of g in a local holomorphic coordinate system, and if $g^{\bar\beta\alpha}$ is the inverse matrix, then

$$X^\flat = \Big(\sum_{\alpha=1}^n X^\alpha \frac{\partial}{\partial z^\alpha}\Big)^\flat = \sum_{\alpha,\beta=1}^n g_{\alpha\bar\beta} X^\alpha d\bar z^\beta, \qquad \eta^\sharp = \Big(\sum_{\alpha=1}^n \eta_{\bar\beta}\, d\bar z^\beta\Big)^\sharp = \sum_{\alpha,\beta=1}^n g^{\bar\beta\alpha}\eta_{\bar\beta} \frac{\partial}{\partial z^\alpha}.$$

If $\phi : M \to \mathbf{C}$ is a smooth function, then the (*complex*) *gradient* of ϕ with respect to g is the smooth $(1,0)$ vector field

$$\operatorname{grad}_g \phi = (\bar{\partial}\phi)^\sharp =: \bar{\partial}^\sharp \phi.$$

The *Lichnerowicz ideal* is defined to be $\mathfrak{H}_1 = \{Z \in \mathfrak{H} : \eta(Z) = 0 \text{ for all } \eta \in \mathcal{Z}^{1,0}_{\bar{\partial}}\}$, namely, the space of holomorphic vector fields which annihilate the space of holomorphic one-forms under the natural pairing. Of course, the pairing of a holomorphic one-form with a holomorphic vector field is a global holomorphic function, hence is constant. The ideal $\mathfrak{H}_1 \subset \mathfrak{H}$ actually contains the derived subalgebra $[\mathfrak{H}, \mathfrak{H}]$. This is trivial from the equation

$$2d\eta(Z, W) = Z\,\eta(W) - W\,\eta(Z) - \eta([Z, W]),$$

since all terms except the last are automatically zero for all holomorphic one-forms η and all holomorphic vector fields Z and W.

Theorem 11.8 *Let Z be a holomorphic vector field, and let $\zeta = Z^\flat$ be the associated one-form. Then*

- *The Hodge decomposition of ζ is $\zeta = \mathbf{H}\zeta + \bar{\partial}\phi$, with ϕ unique up to an added constant.*
- *$\zeta = \bar{\partial}\phi$ if and only if $Z \in \mathfrak{H}_1$, i.e. $\eta(Z) = 0$ for every holomorphic one-form η.*
- *$X = \operatorname{Re} Z$ is Killing if and only if $\operatorname{Re}\phi$ is constant.*

Proof Because g is Kähler, there exist local holomorphic normal coordinates; in such a coordinate system, it is obvious that Z is holomorphic at $x \in M$ if and only if $\bar{\partial}\zeta = 0$ at x. The Hodge decomposition of ζ is

$$\zeta = \mathbf{H}\zeta + \bar{\partial}(\bar{\partial}^* G\zeta) + \bar{\partial}^*(G\bar{\partial}\zeta);$$

The last term vanishes because Z is holomorphic. Set $\phi = \bar{\partial}^* G\zeta$; the complex-valued function ϕ is unique up to an added harmonic function, i.e. an added constant since M is compact.

To prove the second assertion, let η be a holomorphic one-form. By Proposition 8.14, η is harmonic, and since $\eta(Z) = \langle \eta, \zeta \rangle$ (pointwise inner product) is constant, the global inner product (η, ζ) is equal to $\eta(Z)$ times the volume of M. Thus, if $\mathbf{H}\zeta = 0$, then $(\eta, \zeta) = 0$. Conversely, suppose $\eta(Z) = 0$ for every holomorphic one-form η. The $(1,0)$-form $\overline{\mathbf{H}\zeta}$ is harmonic, hence holomorphic for type reasons. But by hypothesis, $\|\mathbf{H}\zeta\|^2 = (\eta, \zeta) = 0$.

The last assertion is a consequence of the Weitzenböck formula for the Laplace operator acting on one-forms and is not given here. ∎

The term $\mathbf{H}\zeta$ is dual to a nowhere-vanishing, autoparallel holomorphic vector field, called the *autoparallel part* of Z, and the term $\bar{\partial}\phi$ is dual to the *gradient part* of Z. The (Abelian) Lie algebra of autoparallel vector fields is denoted $\mathfrak{a}$. If $h^{0,1}(M, \mathbf{C}) = 0$, e.g. if M is simply-connected, then $\mathbf{H}\zeta = 0$ and every holomorphic vector field is a gradient field. The (finite-dimensional) space of smooth, complex-valued functions ϕ with $\mathrm{grad}_g \phi$ holomorphic is called the space of *g-holomorphy potentials*. While the space of holomorphy potentials depends on the choice of metric, the dimension does not.

Corollary 11.9 *If Z is a holomorphic vector field which vanishes somewhere, then Z is a gradient field. If $X = \mathrm{Re}\, Z$ is Killing, and if ξ is the real one-form dual to X, then the following are equivalent:*

i. *The zero set of Z (or of X) is non-empty.*

ii. *$\zeta = \bar{\partial}\phi$ with ϕ pure imaginary.*

iii. *$\xi = J\, du$ for some real-valued function u.*

A smooth function ϕ is a holomorphy potential if and only if $\bar{\partial}\,\bar{\partial}^{\sharp}\phi = 0$. It is convenient to introduce the fourth-order scalar *Lichnerowicz operator* $L = (\bar{\partial}\,\bar{\partial}^{\sharp})^{*}(\bar{\partial}\,\bar{\partial}^{\sharp})$, whose kernel is the space of g-holomorphy potentials. The Weitzenböck formula for L is found by standard differential-geometric techniques, namely integration by parts and application of Ricci identities:

Proposition 11.10 *Let ∇ denote the Levi-Civita connection of g, r the Ricci tensor, and $s = \mathrm{tr}\,\mathrm{r}$ the scalar curvature function. Then for every smooth function ϕ,*

$$2L\phi = \Box_d \phi + \langle \mathrm{r}, \nabla\nabla\phi \rangle + \partial s(\mathrm{grad}_g \phi). \tag{11.4}$$

In particular, the Lichnerowicz operator is a real operator if and only if the scalar curvature function s is constant, a fact which will be of great importance later.

12
Extremal Kähler Metrics

Without further mention, all manifolds are assumed to be connected. An "extremal" Kähler metric (in the sense of Calabi) is a critical point for the L^2-norm of the scalar curvature among metrics in a fixed Kähler class. Precisely, let (M, Ω) be a compact, real-polarized Kählerian manifold of complex dimension n. On the space Ω^+ of Kähler forms representing Ω consider the *Calabi energy* functional

$$\mathfrak{E}_\Omega(\omega) = \int_M \sigma_\omega^2 \,\frac{\omega^n}{n!}. \tag{12.1}$$

While the integral on the right is defined independently of the Kähler class, the Calabi energy functional depends upon the Kähler class Ω because its domain is Ω^+. A critical metric for the Calabi energy is an *extremal Kähler metric*. Theorem 12.14 below asserts that critical metrics are in fact minima, so the name is justified.

For brevity, a choice of Kähler class is sometimes called a *real polarization*, and a *real polarized manifold* is a pair (M, Ω) consisting of a compact, connected holomorphic manifold together with a Kähler class. This generalizes the terminology of algebraic geometry, where a "polarization" is a Hodge (*rational* Kähler) class, but note that a real polarization does not correspond to a line bundle in the way an ordinary polarization does. On a real-polarized manifold, the volume and total scalar curvature are determined; the integrals

$$V_\Omega = \int_M \frac{\omega^n}{n!}, \qquad S_\Omega = \int_M \sigma_\omega \,\frac{\omega^n}{n!} = \int_M \rho_\omega \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{12.2}$$

do not depend on the choice of $\omega \in \Omega^+$, by Stokes' theorem and Chern-Weil theory respectively. A metric of constant scalar curvature (on a compact manifold) is extremal by the Schwarz inequality:

$$\mathfrak{E}_\Omega(\omega) \geq \frac{S_\Omega^2}{V_\Omega}, \tag{12.3}$$

with equality if and only if ω has constant scalar curvature. However, there exist examples (due to Calabi) of extremal metrics of non-constant scalar curvature, suggestiong that (12.3) can be sharpened. Theorem 12.14 gives such a sharpening, for which the inequality is saturated exactly for extremal metrics.

A Riemannian metric g is *Einstein* if the Ricci tensor is proportional to the metric tensor, that is, if there is a real constant λ—the *Einstein constant*—such that $\mathrm{r} = \lambda g$. An Einstein metric has constant scalar curvature. If g is Kähler with respect to a complex structure J, then the triple (M, J, g) is an *Einstein-Kähler metric.*[1] The following simple result shows the stringency of the Einstein-Kähler condition.

Proposition 12.1 *Let (M, J, g) be a compact Einstein-Kähler manifold with Einstein constant λ.*

- *If $\lambda < 0$, then the canonical bundle K_M is ample.*
- *If $\lambda = 0$, then the canonical bundle is trivial.*
- *If $\lambda > 0$, then the anticanonical bundle is ample.*

Proof The Ricci form ρ is $\sqrt{-1}$ times the curvature form of the canonical bundle K_M; In other words, with respect to a local coordinate system,

$$\rho_\omega = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{\alpha\bar{\beta}}), \quad \text{where } \omega = \sqrt{-1}g_{\alpha\bar{\beta}}\,dz^\alpha \wedge d\bar{z}^\beta. \tag{12.4}$$

Consequently the Ricci form of an arbitrary Kähler metric represents the de Rham class $2\pi c_1(M, J)$. But a Kähler metric is Einstein if and only if the Ricci form is proportional to the Kähler form:

$$\rho = \lambda\omega, \tag{12.5}$$

and ω is a positive $(1, 1)$-form. Thus if (M, J) admits an Einstein-Kähler metric, then the first Chern class is either positive, negative, or zero. Because the first Chern class is integral, the first and third assertions follow from the Kodaira embedding theorem. If (M, J) admits a Ricci-flat Kähler metric (that is, $\lambda = 0$), then by (12.4), the volume density $\det(g_{\alpha\bar{\beta}})$ is constant in each holomorphic coordinate system, which means the local holomorphic section $dz^1 \wedge \cdots \wedge dz^n$ of K_M has constant norm, which can be

[1] Or a *Kähler-Einstein metric*, if the metric structure is being emphasized.

normalized to 1 by linear change of variables. Covering M by coordinate neighborhoods shows that the canonical bundle admits a non-vanishing holomorphic section, hence is trivial. ∎

In particular, a general Kählerian manifold does not admit an Einstein-Kähler metric; it is necessary that the first Chern class either be the zero class, or else be a positive or a negative class. Furthermore, the sign of the Einstein constant is determined by the complex structure; a single holomorphic manifold cannot admit two Einstein-Kähler metrics unless their curvatures are both zero, both positive, or both negative. A manifold (M, J) with $-K_M$ ample is called a *Fano* manifold, while a *simply-connected* manifold with K_M trivial is often called a *Calabi-Yau manifold.* A manifold not satisfying the conclusion of Proposition 12.1 has *indefinite first Chern class.*

Remark 12.2 *If the holomorphic structure is not fixed, then a single smooth manifold* can *admit a pair of Einstein-Kähler metrics of opposite sign. To emphasize, these metrics are Kähler with respect to* different *holomorphic structures, hence do not live on the same holomorphic manifold. There are known examples in complex dimension four; it is not currently known if this sort of example arises among complex surfaces. However, it* is *known (Catanese-Lebrun, and Kotschick) that there exist complex surfaces M_+ and M_- whose underlying smooth four-manifolds are homeomorphic* but not diffeomorphic*, and such that $M_\pm$ admits an Einstein-Kähler metric of the corresponding sign! Suitable products of these give the complex four-fold examples mentioned previously.*

Some examples may illustrate the restrictiveness of Proposition 12.1.

Example 12.3 A complete intersection in $\mathbf{P}^N$ always has *definite* first Chern class, and many of them are known to admit an Einstein-Kähler metric. The same is true for the blow-up of $\mathbf{P}^2$ at three *non-collinear* points. By contrast, the blow-up of $\mathbf{P}^2$ at three *collinear* points, the blow-up of $\mathbf{P}^2$ at nine arbitrary points, and (if $n \geq 3$) the blow-up of $\mathbf{P}^n$ at two points all have indefinite first Chern class. Blowing up an indefinite manifold always gives an indefinite manifold. □

Under scaling of the metric, the Ricci tensor and form do not change. Thus a Kählerian manifold (M, J) admits an Einstein-Kähler metric if and only if it admits such a metric with $\rho = \varepsilon\omega$, $\varepsilon = -1$, 0, or 1. This restriction is assumed from now on.

The converses of the first two conditions in Proposition 12.1 are true (Theorems 12.4 and 12.5 below), while the converse of the third is known to be false. There are additional necessary conditions in order that a Fano manifold admit an Einstein-Kähler metric, and additional hypotheses under which these conditions are sufficient for existence. It has recently been conjectured by Tian that a certain "stability" condition on (M, J) is sufficient to guarantee existence. Einstein-Kähler metrics, Kähler metrics of constant scalar curvature, and extremal Kähler metrics are closely related, and not merely by the trivial inclusions among them indicated above. The details are discussed below.

12.1 The Calabi Conjectures

The so-called *First Calabi Conjecture* (see Theorem 12.4) is concerned with specification of the Ricci curvature of a compact Kählerian manifold, while the *Second Calabi Conjecture* (see Theorem 12.5) is the converse of the first assertion in Proposition 12.1. The First Calabi Conjecture was established by Yau in 1977, while the second was proven independently by Yau and Aubin; the uniqueness assertion was known to Calabi in 1954.

Theorem 12.4 *Let (M, J) be a compact Kählerian manifold, let ω_0 be a positive $(1, 1)$-form on (M, J), and let $\rho \in 2\pi c_1(M)$ be an arbitrary smooth form. Then there is a unique Kähler metric g on (M, J) with $[\omega] = [\omega_0]$ and having Ricci form ρ.*

In words, given a representative of $2\pi c_1(M)$, each Kähler class on (M, J) contains a unique representative with the specified Ricci form. In particular, if $c_1(M) = 0$, then every Kähler class contains a unique Einstein-Kähler metric with $\rho = 0$.

Theorem 12.5 *Let (M, J) be a compact complex manifold with ample canonical bundle. Then there is a unique Einstein-Kähler metric g with $\rho = -\omega$.*

Example 12.6 By the adjunction formula, if $M^n \subset \mathbf{P}^N$ is a complete intersection of multidegree $d = d_1 + \cdots + d_{N-n}$, and if $N + 1 = d$, then M admits a Ricci-flat Kähler metric in each Kähler class, while if $N + 1 < d$, then M admits a unique (up to scaling) negative Einstein-Kähler metric. In particular, each smooth quartic surface in $\mathbf{P}^3$ and each smooth quintic three-fold in $\mathbf{P}^4$ admits a Ricci-flat Kähler metric; for this reason, quintic three-folds are called *Calabi-Yau* manifolds. The space of Kähler classes on a quartic surface is known to be three-dimensional by the Hirzebruch Signature Theorem, so there is a three-dimensional family of Ricci-flat Kähler metrics on each smooth quartic. As shown in Example 10.8, the space of Kähler classes on a quintic threefold in $\mathbf{P}^4$ is one-dimensional, so each quintic in $\mathbf{P}^4$ has—up to scaling—a unique Ricci-flat Kähler metric. □

Very little is known regarding existence if $2 < d < N + 1$. However, contrary to the natural guess, the restriction of the Fubini-Study metric to M is *never* Einstein if $2 < d$. In fact, only the "obvious" complete intersections inherit an Einstein metric from the Fubini-Study metric; this is a result of Kobayashi and Ochiai:

Theorem 12.7 *Let $M^n \subset \mathbf{P}^N$ be a smooth projective variety, and assume the restriction of the Fubini-Study metric to M is Einstein. Then M is contained in a linear subspace $\mathbf{P}^{n+1}$ and is either a linear subspace or a quadric hypersurface in $\mathbf{P}^{n+1}$.*

The proofs of Theorems 12.4 and 12.5 are entirely hard analysis. However, it is easy to reduce each theorem to a question of existence for a complex Monge-Ampère equation. Recall that if ω_0 is given, then every Kähler form cohomologous to ω_0 is given by $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ for a smooth, real-valued function φ unique up to added constants. Let ρ_0 be the Ricci form of ω_0, and let ρ be the prospective Ricci form of the unknown metric. By Proposition 11.3,

$$\rho - \rho_0 = -\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right);$$

this makes sense since the quotient of (non-vanishing) (n, n)-forms is a globally defined smooth, real-valued function. Yau showed that (with respect to suitable Hölder topologies) the map

$$\varphi \longmapsto \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)$$

is a diffeomorphism from the set of $\mathcal{C}^{2+\alpha}$ functions with $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ to the space of $\mathcal{C}^\alpha$ functions on M. This involved proving the map is open—a straightforward application of the Inverse Function Theorem for Banach spaces—together with an argument that the map is proper, which involved difficult estimates. As remarked above, that fact that this map is injective was proven by Calabi in 1954.

To express Theorem 12.5 in terms of a Monge-Ampère equation, fix $\omega_0 \in -2\pi c_1(M)$ (which by hypothesis is a Kähler class), and define the smooth, real-valued function f_0 by

$$\rho_0 + \omega_0 = \sqrt{-1}\partial\bar{\partial}f_0.$$

The function f_0 is called the *Ricci potential* of ω_0. With notation as above, it is easy to show that ω_φ is Einstein-Kähler with Einstein constant λ if and only if

$$\log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right) + \lambda\varphi = f_0.$$

Aubin's proof of Theorem 12.5 used the so-called *continuity method*; Aubin introduced a family of equations

$$(*)_t \qquad \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right) + t\lambda\varphi = f_0.$$

By Theorem 12.4, equation $(*)_0$ has a solution. By the Inverse Function Theorem, if $(*)_{t_0}$ has a solution, then there is an $\varepsilon > 0$ such that $(*)_t$ has a solution for $t < t_0 + \varepsilon$. It remains to show that the set of $t \in [0,1]$ for which $(*)_t$ has a solution is closed. It is only at this point that the assumption $\lambda = -1$ must be introduced. Proving closedness involves *a priori* estimates on φ and is the analogue of properness in Yau's proof. More explicitly, the idea is to show that if $(*)_t$ has a solution φ_t for $t < t_0$, then in a suitable Hölder norm, the functions φ_t remain bounded as $t \nearrow t_0$. The same general technique is used to prove existence of positive Einstein-Kähler metrics, though the details are quite different for geometric reasons. The problem is to connect geometric hypotheses about the complex manifold (M, J) to analytic properties, namely, existence of *a priori* estimates for $(*)_t$ with $\lambda = 1$.

At present, there are no explicit examples of non-positive Einstein-Kähler metrics on (compact) simply-connected manifolds. However, even existence of Kähler metrics with specified Ricci curvature gives useful information. A nice application is the following result of Kobayashi.

Proposition 12.8 *Let (M, J) be a Fano manifold. Then M is simply-connected.*

Proof (Sketch) By Theorem 12.4, there is a Kähler metric g with positive Ricci curvature. (Kobayashi's original article contained this as a hypothesis, though added the present weaker hypothesis modulo the First Calabi Conjecture.) There is a vanishing theorem, due to Bochner and analogous to the Kodaira-Nakano Theorem, that a Kähler manifold with positive Ricci tensor admits no non-trivial holomorphic p-forms if $p > 0$. (The proof is a simple application of the Weitzenböck formula for the Laplacian acting on $(p, 0)$-forms.) Consequently, the holomorphic Euler characteristic

$$\chi(M, \mathcal{O}_M) = \sum_{p=0}^{n} (-1)^p h^0(M, \Omega_M^p) = \sum_{p=0}^{n} (-1)^p h^{p,0}(M)$$

is equal to 1.

Let $\pi : \widetilde{M} \to M$ be the universal covering space. By Myers' Theorem (which holds for Riemannian manifolds with positive Ricci tensor), $\widetilde{M}$ is compact, so π is a finite-sheeted cover with d sheets. The pullback of g is a Kähler metric with positive Ricci curvature, so $\chi(\widetilde{M}, \mathcal{O}_{\widetilde{M}}) = 1$. By the Riemann-Roch-Hirzebruch Theorem (Theorem 9.13 above), the Euler characteristic $\chi(M, \mathcal{O}_M)$ is equal to the integral over M of the Todd class of M (since the Chern character of $\mathcal{O}$ is equal to 1), which is a universal polynomial in the curvature of ω. The same formula holds locally for the Todd class of $\widetilde{M}$, so $1 = \chi(\widetilde{M}, \mathcal{O}_{\widetilde{M}}) = d$, the number of sheets of the cover π. ∎

Another useful result, due to Yau, characterizes metrics of constant negative holomorphic sectional curvature among compact Einstein-Kähler metrics. The idea of the proof is to calculate Chern forms with respect to an

Einstein-Kähler metric, then use the fact that there is a unique simply-connected model space with constant negative holomorphic sectional curvature.

Proposition 12.9 *Let (M^n, J) be a compact Kählerian manifold with $c_1(M) < 0$. Then*

$$(-1)^n \Big(2(n+1)c_2(M) \cup c_1(M)^{n-2} - nc_1(M)^n \Big)[M] \geq 0, \tag{12.6}$$

with equality if and only if M is holomorphically covered by the unit ball in $\mathbf{C}^n$.

12.2 Positive Einstein-Kähler Metrics

In contrast to the existence problem for non-positive Einstein-Kähler metrics, the basic restriction (i.e. that M be Fano) is not sufficient to guarantee existence of a positive Einstein metric. The first non-trivial obstruction was found by Matsushima in 1954.

Theorem 12.10 *Let (M, J, g) be a compact Einstein-Kähler manifold with positive curvature. Then the Lie algebra $\mathfrak{H}$ is the complexification of the compact Lie algebra $\mathfrak{k}$ of Killing vector fields.*

A Lie group whose Lie algebra is the complexification of a compact subalgebra (i.e. the Lie algebra of a compact subgroup) is said to be *reductive*. Not every Fano manifold has reductive Lie algebra of holomorphic vector fields; such manifolds cannot admit an Einstein-Kähler metric. If M is obtained from $\mathbf{P}^n$ by blowing up a linear subspace of dimension ℓ, $0 \leq \ell \leq n-2$, then M is Fano but admits no Einstein-Kähler metric.

Theorem 12.10 was generalized by Lichnerowicz in 1957 to Kähler metrics with constant scalar curvature.

Theorem 12.11 *Let (M, J, g) be a Kähler manifold with constant positive scalar curvature. Then the Lie algebra $\mathfrak{H}$ of holomorphic vector fields decomposes as the sum*

$$\mathfrak{H} = \mathfrak{a} \oplus \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$$

of the space of autoparallel vector fields, Killing fields, and J(Killing fields).

Proof By equation (11.4), which expresses the Lichnerowicz operator in terms of the Laplace operator, Ricci tensor, and scalar curvature, the scalar curvature s is constant if and only if $L = (\bar{\partial}\bar{\partial}^\sharp)^*(\bar{\partial}\bar{\partial}^\sharp)$ is a real operator. If ϕ is a holomorphy potential, then $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ are also holomorphy potentials since L is real. By Theorem 11.8, if $\phi \in \ker L$ is pure imaginary, then $\operatorname{grad}_g\phi$ is a Killing field. The theorem follows immediately. $\square$

Thus, the blow-up of $\mathbf{P}^n$ along a linear subspace admits no Kähler metric of constant scalar curvature, regardless of Kähler class.

A second obstruction to existence of positive Einstein-Kähler metrics was discovered in 1983 by Futaki, who generalized an integral invariant found in 1974 by Kazdan and Warner (who had been attempting to determine which smooth functions on S^2 are Gaussian curvature functions of some Riemannian metric). The Futaki invariant was generalized in various ways between 1984–5 by Calabi, Futaki, and Bando.

Let (M, J) be a compact Kählerian manifold, and let $\mathfrak{H}$ be the space of holomorphic vector fields on M. For each Kähler form ω, let f_ω denote the normalized Ricci potential, that is, the unique smooth, real-valued function with

$$\rho_\omega - \mathbf{H}\rho_\omega = \sqrt{-1}\partial\bar{\partial} f_\omega, \qquad \int_M f_\omega \frac{\omega^n}{n!} = 0.$$

Let $\Omega = [\omega]$ denote the Kähler class of g, and consider the linear functional

$$\mathcal{F}_\Omega : X \in \mathfrak{H} \longmapsto \sqrt{-1} \int_M (X f_\omega) \frac{\omega^n}{n!}. \tag{12.7}$$

The following summarizes the properties of the so-called *Futaki character.*

Theorem 12.12 *The functional $\mathcal{F}_\Omega$ defined by equation* (12.7) *is a Lie algebra character, and depends only on the de Rham class $\Omega = [\omega]$. In particular, if there is a Kähler metric of constant scalar curvature whose Kähler form represents Ω, then the character $\mathcal{F}_\Omega$ vanishes identically.*

Example 12.13 Let M^n be obtained by blowing up $\mathbf{P}^n$ along skew linear subspaces of complementary dimension (neither of which is a hyperplane). Then M is Fano, and the set of Kähler classes for which $\mathcal{F}_\Omega$ vanishes identically is a real-algebraic hypersurface in the Kähler cone $H_+^{1,1}(M, \mathbf{R})$. The *Futaki invariant* $\mathcal{F}_{c_1(M)}$ vanishes identically if and only if $n = 2k + 1$ and the subspaces blown up are both of dimension k; in fact, M admits an Einstein-Kähler metric exactly under this condition. In particular, there exist Fano manifolds satisfying Matsushima's obstruction but having no Einstein-Kähler metric.

It is not at present known whether or not there exists a compact manifold M with non-reductive automorphism group but having vanishing Futaki character for every Kähler class, nor whether or not there exists a compact manifold having reductive automorphism group but non-vanishing Futaki character for every Kähler class. □

Theorem 12.14 *Let (M, Ω) be a real-polarized manifold, and let X_Ω be the extremal Kähler vector field associated to a fixed maximal compact group of automorphisms. Then the number $\mathcal{F}_\Omega(X_\Omega)$ does not depend on the choice of maximally compact subgroup, and*

$$\mathfrak{E}_\Omega(\omega) \geq \frac{S_\Omega^2}{V_\Omega} + \mathcal{F}_\Omega(X_\Omega) \qquad \textit{for all } \omega \in \Omega+, \tag{12.8}$$

with equality if and only if ω *is critical for the Calabi energy of* Ω.

In particular, every critical metric is a minimizer (so all have the same energy), and the "critical energy" depends smoothly on the Kähler class.