**College of the Holy Cross**, Spring Semester, 2011
**Department of Mathematics and Computer Science**
**Problem of the Week #1: Solved by Chris Schaller**

For the purposes of this problem, a real number $x$ *mimics an integer to $k$ decimals* if there exists a decimal representation of $x$ having at least $k$ consecutive zeros to the right of the decimal point, but does not admit a decimal representation having at least $k+1$ zeros. For example,

$$1.01 = 1.009\overline{9}$$

mimics an integer to two decimals, as does 1.00314159, while 1.01000000001 mimics an integer to one decimal.

When you take the square root of an integer $N$, the result is either an integer (for example, $\sqrt{4} = 2.0\overline{0}$, which mimics an integer to an infinite number of decimals) or is irrational (for example, $\sqrt{2} = 1.4142135623\ldots$, *ad infinitum* without repetition).

**Problem 1** Find the smallest positive integer $N$ whose square root mimics an integer to one million decimals.
**Solution** We're looking for an integer $N$ such that $\sqrt{N}$ mimics an integer to one million decimals. If $n$ denotes the integer "close to" $\sqrt{N}$, then we want

$$n < \sqrt{N} < n + 10^{-1{,}000{,}000}.$$

Squaring,

$$n^2 < N < n^2 + 2n \times 10^{-1{,}000{,}000} + 10^{-2{,}000{,}000}.$$

To ensure $N$ is "as small as possible", we should take $N = n^2 + 1$. Consequently, we first seek the smallest integer $n$ such that

$$n^2 + 1 < n^2 + 2n \times 10^{-1{,}000{,}000} + 10^{-2{,}000{,}000},$$

or

$$1 < 2n \times 10^{-1{,}000{,}000} + 10^{-2{,}000{,}000}.$$

Manipulating inequalities, we find

$$\frac{1 - 10^{-2{,}000{,}000}}{2 \times 10^{-1{,}000{,}000}} = 0.5 \times 10^{1{,}000{,}000} - 0.5 \times 10^{-1{,}000{,}000} < n.$$

The smallest such $n$ is $0.5 \times 10^{1{,}000{,}000} = 5 \times 10^{999{,}999}$, so finally we have

$$N = n^2 + 1 = 2.5 \times 10^{1{,}999{,}999} + 1 = 25,\underbrace{000,000\ldots000,001}_{1,999,997 \text{ zeroes}}$$