In our first example of using a “pivotal quantity” to derive confidence intervals this day, I wasn’t very clear about exactly how you derive the formulas for the endpoints. This sheet should clarify the algebra involved.

We considered the case of sampling from a $N(\mu, \sigma^2)$ population to estimate the target parameter $\mu$ (the population mean). From our work in section 7.2 of the text, we know that with $n$ independent samples, the quantity

$$Q = \frac{\bar{Y} - \mu}{s/\sqrt{n}},$$

(where $s = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$, the sample standard deviation) satisfies

- $Q$ contains $\mu$ as the only unknown, and
- $Q$ has a $t(n-1)$ distribution.

Hence we can use $Q$ as a pivotal quantity to derive confidence intervals. Here are complete derivations of each of the three cases (being more careful about signs than I was in class):

**Two-sided, $(1 - \alpha) \times 100\%$ confidence interval**

Let $t_{\alpha/2}$ be the $\alpha/2$-percentile point for the $t$-distribution: that is $P(T > t_{\alpha/2}) = \alpha/2$. Then by symmetry:

$$1 - \alpha = P\left(-t_{\alpha/2}(n-1) < \frac{\bar{Y} - \mu}{s/\sqrt{n}} < t_{\alpha/2}(n-1)\right)$$

$$= P\left(-t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} < \bar{Y} - \mu < t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}}\right)$$

$$= P\left(-\bar{Y} - t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} < -\mu < -\bar{Y} + t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}}\right)$$

$$= P\left(\bar{Y} + t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} > \mu > \bar{Y} - t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}}\right)$$

$$= P\left(\bar{Y} - t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}} < \mu < \bar{Y} + t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}}\right)$$

(Note: at the next-to-last step, we multiply through by $-1$, which reverses all inequalities, then we switch back to write the terms increasing left to right as usual).

*(turn over)*
**One-sided, \((1-\alpha) \times 100\%\) lower confidence bound**

*Note:* The formula I gave for this in class *was correct*, but the way I suggested to derive it wasn’t (d’oh!). We start from the statement (with \(t_\alpha\) instead of \(t_{\alpha/2}\)):

\[
1 - \alpha = P \left( \frac{\bar{Y} - \mu}{s/\sqrt{n}} < t_\alpha (n-1) \right) \\
= P \left( \bar{Y} - \mu < t_\alpha (n-1) \frac{s}{\sqrt{n}} \right) \\
= P \left( -\mu < -\bar{Y} + t_\alpha (n-1) \frac{s}{\sqrt{n}} \right) \\
= P \left( \mu > \bar{Y} - t_\alpha (n-1) \frac{s}{\sqrt{n}} \right)
\]

**One-sided, \((1-\alpha) \times 100\%\) upper confidence bound**

Starting from

\[
1 - \alpha = P \left( -t_\alpha (n-1) < \frac{\bar{Y} - \mu}{s/\sqrt{n}} \right)
\]

and proceeding as for the lower confidence bound,

\[
1 - \alpha = P \left( \mu < \bar{Y} + t_\alpha (n-1) \frac{s}{\sqrt{n}} \right)
\]