Saddle Points and Domination

Consider the game with matrix

\[
\begin{array}{c|cc}
 & B \\
\hline
A & (1) & (2) \\
(1) & 1 & -1 \\
(2) & 2 & 4 \\
\end{array}
\]

You should have run into difficulties when trying to find optimal strategies for the two players by the method of equalizing payoffs. However, we can easily see what the players should do in this game. First of all, A should never play strategy (1). No matter which strategy B chooses, the payoff for A is greater if A plays (2). So A’s optimal strategy must be (0,1) - play (2) all the time. How about B? Once B sees that A should play (2) all the time, B’s best response is to play (1), to get a payoff of -2 rather than -4. Therefore, B’s optimal strategy is (1,0) - play (1) all the time. The value \( V \) of the game must be 2, according to the definition of value in the minimax theorem: if A plays her optimal strategy, A wins at least 2, while if B plays his optimal strategy, B “wins” at least -2.

The game just considered is an example of a game with a saddle point. A saddle point is an entry of the payoff matrix that is the minimum of its row and the maximum of its column. If a game has a saddle point, say in row \( i \) and column \( j \), then the optimal strategy for player A is to play her \( i \)th strategy and the optimal strategy for B is to play his \( j \)th strategy. The value of the game is the payoff in row \( i \) and column \( j \). For example, consider the game with matrix

\[
\begin{array}{c|ccc}
 & B \\
\hline
A & (1) & (2) & (3) \\
(1) & -2 & 4 & 0 \\
(2) & 2 & 3 & 1 \\
(3) & 3 & -2 & -1 \\
\end{array}
\]

There is a saddle point in row , column . The value must be 1, and the optimal strategies (0,1,0) for A and (0,0,1) for B, because by playing these strategies, A wins at least 1 because 1 is the minimum of the second row, while B wins at most -1 because 1 is the maximum of the third column.

There is a very simple technique for finding saddle points: below each column write the column max and beside each row write the row min. If any of these numbers match, you have a saddle point. Try this with
Zero-sum games with saddle points are very boring. Each player only plays one strategy and the outcome is a foregone conclusion. However, the general idea of immediately eliminating certain strategies, that we used for the first $2 \times 2$ matrix today, is a useful one. We say that A’s pure strategy (i) dominates pure strategy (j) if for each pure strategy that B plays, the payoff to A for playing (i) is at least as big as the payoff to A for playing (j). In this case, A can effectively eliminate (j) from consideration. Similarly, B’s pure strategy (i) dominates pure strategy (j) if for each pure strategy that A plays, the payoff to B for playing (i) is at least as big as the payoff to B for playing (j). In this case, B can effectively eliminate (j) from consideration. (Remember that B’s payoffs are the negative of the numbers in the matrix). Consider

\[
\begin{array}{cccc}
A & & & \\
\hline
(1) & 3 & 5 & 2 & 4 \\
(2) & 2 & 1 & 0 & 3 \\
(3) & 2 & 2 & 1 & 2 \\
\end{array}
\]

Notice that the payoffs for B using strategy (2) are better than his payoffs using strategy (1), no matter what A does (remember that B’s payoffs are the negatives of the numbers in the table.) If A plays (1), B wins -5 by playing (1) but wins -4 by playing (2). If A plays (2), B wins -4 by playing (1) but wins -3 by playing (2), and so on. For B, strategy (2) dominates strategy (1), and we can remove strategy (2) from consideration. This leaves the matrix as:

\[
\begin{array}{cccc}
A & & & \\
\hline
(1) & 4 & 1 & 0 \\
(2) & 3 & 2 & -1 \\
(3) & 0 & -1 & 4 & 3 \\
(4) & 1 & 2 & 1 & 2 \\
\end{array}
\]

Next we notice that for A, strategy (3) dominates strategy (4): for each strategy that B plays, A’s payoff for playing (3) is better than her payoff for playing (4). So we eliminate A’s strategy (4) to get:

\[
\begin{array}{cccc}
A & & & \\
\hline
(1) & 4 & 1 & 0 \\
(2) & 3 & 2 & -1 \\
(3) & -1 & 4 & 3 \\
\end{array}
\]
Next, B’s (4) dominates B’s (3), reducing us to

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<td>A</td>
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<tr>
<td>(1)</td>
<td>4</td>
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<td>(2)</td>
<td>3</td>
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<td>(3)</td>
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Finally, A’s (1) dominates A’s (2), leaving the matrix as

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<tr>
<td>A</td>
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<tr>
<td>(1)</td>
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<td>(3)</td>
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It’s easy to check that there are no longer any dominant strategies for either player, and that there is no saddle point. We can find the optimal strategies for each player and the value as we did for the Holmes-Moriarty game, by equalizing average payoffs. The result is this: A’s optimal strategy is \((\frac{1}{2}, 0, \frac{1}{2}, 0)\) while B’s optimal strategy is \((0, \frac{3}{8}, 0, \frac{5}{8})\). Note that we include probabilities for all four possible strategies in listing the optimal strategies, assigning a zero probability to strategies that have been eliminated. The value of the game is the expected average payoff to A, which works out to 1.5.

To summarize: when trying to find optimal strategies in any zero-sum game, first check (using the procedure outlined above) for saddle points. If there are saddle points, optimal strategies can be taken to be the pure strategies associated with the saddle point and the value is the payoff at that saddle point. If there are no saddle points, then iteratively eliminate dominated strategies to reduce the size of the game; if the game can be reduced to a two-strategy game for both players, we know how to solve it.

Problems:

3.1 Find all saddle points for the game with the following payoff matrix:

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<tbody>
<tr>
<td>A</td>
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<tr>
<td>(1)</td>
<td>1</td>
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<tr>
<td>(2)</td>
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<td>(3)</td>
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<tr>
<td>(1)</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(2)</td>
<td>3</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(3)</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>-1</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>
3.2 Suppose that a game has two saddle points, at the positions with payoffs \( a \) and \( b \) as indicated below. Explain why \( a \) must equal \( b \). (Hint: consider the relations of \( a, b \) to the payoffs \( c, d \).)

\[
\begin{array}{c|cccc}
\hline
& B & & & \\
A & (1) & (2) & (3) & (4) \\
(1) & & & a & \\
(2) & & & c & \\
(3) & d & & b & \\
(4) & & & & \\
\hline
\end{array}
\]

3.3 For the game with the following matrix, eliminate all dominated strategies to reduce to a \( 2 \times 2 \) game, then find the optimal strategies for both players and the value.

\[
\begin{array}{c|cccc}
\hline
& B & & & \\
A & (1) & (2) & (3) & (4) \\
(1) & 3 & -6 & 2 & -4 \\
(2) & 2 & 4 & 0 & 7 \\
(3) & -4 & 3 & -5 & 4 \\
\hline
\end{array}
\]

3.4 Show using dominance that in Colonel Blotto (the matrix is shown below) when army A has 4 troops and army B has three troops, army B should \textit{always} send all its troops to one city, while army A should \textit{never} send all its troops to a single city. Can you guess optimal strategies and the value?

\[
\begin{array}{c|cccc}
\hline
& B & & & \\
A & (1) & (2) & (3) & (0) \\
(4) & 1 & 0 & 0 & 0 \\
(3) & 1 & 1 & 0 & 0 \\
(2) & 0 & 1 & 1 & 0 \\
(1) & 0 & 0 & 1 & 1 \\
(0) & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]