For a simple closed curve $C$ bounding the region $U_C$, Green’s Theorem says:

$$\int_C ax + by = \int \int_{U_C} \left( b_x - a_y \right) dA$$

where $a, b$ are continuously differentiable functions of $x, y$ on $\overline{U_C}$. The goal is to prove this when $U_C$ is a rectangle, $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1$.

Start by rewriting the double integral as follows: first, because integrating the $x$-partial derivative $b_x$ with respect to $x$ gives the original function $b$, we get

$$\int \int_{U_C} b_x dA = \int_{y_0}^{y_1} \left( \int_{x_0}^{x_1} b_x \, dx \right) \, dy = \int_{y_0}^{y_1} \left. b(x,y) \right|_{x_0}^{x_1} \, dy = \int_{y_0}^{y_1} b(x_1,y) - b(x_0,y) \, dy$$

Next, simplify $\int a_y dA$ in a similar $y$, by integrating first with respect to $y$:

$$\int \int_{U_C} a_y dA = \int_{x_0}^{x_1} \left( \int_{y_0}^{y_1} a_y \, dy \right) \, dx = \int_{x_0}^{x_1} \left. a(x,y) \right|_{y_0}^{y_1} \, dx = \int_{x_0}^{x_1} a(x,y_1) - a(x,y_0) \, dx$$

Therefore the right-hand side of Green’s Theorem is

$$\int_{y_0}^{y_1} b(x_1,y) - \int_{y_0}^{y_1} b(x_0,y) \, dy - \int_{x_0}^{x_1} a(x,y_1) + \int_{x_0}^{x_1} a(x,y_0) \, dx \quad (1)$$

To calculate the left-hand side of Green’s formula, note that the boundary $C$ of the rectangle consists of four segments, which we can parameterize (tracing out $C$ counterclockwise) as follows:

- $C_1: x = t, x_0 \leq t \leq x_1, y = y_0$;
- $C_2: x = x_1, y_0 \leq y \leq y_1$;
- $C_3: x = -t + x_1 + x_0, y = y_1$;
- $C_4: x = x_0, y = -t + y_1 + y_0$.

With this parametrization,

$$\int_{C_1} ax + by = \int_{x_0}^{x_1} a(t,y_0) \, dt$$

This is exactly the last term in (1), with $t$ replacing $x$. Similarly

$$\int_{C_2} ax + by = \int_{y_0}^{y_1} b(x_1,t) \, dt$$

which is the first term in (1); $\int_{C_3} ax + by$ is the third term in (1), and $\int_{C_4} ax + by$ is the second term in (1). This completes the proof.