Complex Analysis - Math 305
Presentation 5 - Complex powers
Tuesday, February 7 2006

To define the function \( w = z^\alpha \) for complex values of \( \alpha \).

We know that

\[
z = e^{\log(z)}
\]

so we define

\[
z^\alpha = (e^{\log(z)})^\alpha = e^{\alpha \log(z)} \tag{1}
\]

Since \( \log(z) \) has infinitely many possible values, \( z^\alpha \) also has infinitely possible values, unless \( \alpha \) is real.

**Example 1:** Take \( \alpha = i \) in equation (1):

\[
z^i = e^{i \log(z)} = e^{i(\ln(|z|)+i \arg(z))} = e^{i \ln(|z|)-\arg(z)} = e^{-\arg(z)}e^{i \ln(|z|)}
\]

so

\[
|z^i| = e^{-\arg(z)}; \quad \arg(z^i) = \ln(|z|)
\]

Since \( \arg(z) \) has infinitely many values for a given \( z \), \( |z^i| \) has infinitely many values. However, \( \ln(|z|) \) has only one value, so \( \arg(z^i) \) has only one value. All the values of \( z^i \) therefore lie on a single ray extending from the origin. For example,

\[
|z^i| = e^{-(\frac{\pi}{2}+2\pi n)}, n \in \mathbb{Z}; \quad \arg(z^i) = \ln(|z|) = \ln(1) = 0
\]

All the possible values of \( z^i \) therefore lie on the positive real axis. As \( n \to \infty \) the sequence of these points approaches the origin; as \( n \to -\infty \) they tend to \( \infty \).

**Example 2:** Take \( \alpha = \frac{1}{3} \) in equation (1):

\[
z^{1/3} = e^{(1/3) \log(z)} = e^{(1/3)(\ln(|z|)+i \arg(z))} = e^{(1/3) \ln(|z|)}e^{i \arg(z)/3}
\]

so

\[
|z^{1/3}| = e^{(1/3) \ln(|z|)} = |e^{\ln(|z|)}|^{1/3} = |z|^{1/3}
\]

and

\[
\arg(z^{1/3}) = \arg(z)/3
\]

Now \( \arg(z) \) has infinitely many values obtained by adding \( 2\pi n \), but there are only three distinct values of \( \arg(z)/3 \) - if one choice of \( \arg(z) \) is \( \theta \), then

\[
\arg(z)/3 = \theta/3, \theta/3 + 2\pi/3, \theta/3 + 4\pi/3
\]

Adding more multiples of \( 2\pi \) to \( \arg(z) \) gives no new values for \( \arg(z)/3 \). The point is that we get exactly the same three values for \( z^{1/3} \) using (1) as we did when found roots previously.