Complex Analysis - Math 305
Presentation 11 - Morera’s Theorem
March 28, 2006

Morera’s Theorem is useful for proving functions are holomorphic without resorting to showing that \( f' \) exists or using the Cauchy-Riemann equations. We’ll prove it for a function on a disk, but it’s true for open sets in general.

Cauchy’s Theorem tells us that if \( f \) is holomorphic in an open set containing a simple closed curve \( C \) and the region bounded by \( C \), then \( \int_C f(z) \, dz = 0 \). Morera’s theorem is a converse of Cauchy’s Theorem. We will prove: if \( f \) continuous in a disk \( D = \{ z : |z - a| < r \} \) and

\[
\int_C f(z) \, dz = 0
\]

for every simple closed curve in \( D \), then \( f \) is holomorphic in \( D \).

Proof: Let \([a, z]\) denote the line segment joining \( a \) to \( z \in D \). Define

\[
F(z) = \int_{[a, z]} f(\zeta) \, d\zeta
\]

First we show that \( F'(z) \) exists for each \( z \in D \). Note

\[
F(z + h) = \int_{[a, z+h]} f(\zeta) \, d\zeta
\]

if \( h \) is small enough so that \( z + h \in D \). By the hypothesis on \( f \),

\[
\int_{[a, z]} f(\zeta) \, d\zeta + \int_{[z, z+h]} f(\zeta) \, d\zeta - \int_{[a, z+h]} f(\zeta) \, d\zeta = 0
\]

since the path from \( a \) to \( z \) to \( z + h \) and back to \( a \) forms a simple closed curve. Therefore,

\[
F(z) + \int_{[z, z+h]} f(\zeta) \, d\zeta = F(z + h)
\]

or

\[
F(z + h) - F(z) = \int_{[z, z+h]} f(\zeta) \, d\zeta
\]

Replace \( f(\zeta) \) by \( f(\zeta) - f(z) + f(z) \) in the integral to get

\[
F(z + h) - F(z) = \int_{[z, z+h]} f(\zeta) - f(z) \, d\zeta + \int_{[z, z+h]} f(z) \, d\zeta
\]

(1)

The second integral on the right of (1) is just \( f(z)h \), since \( f(z) \) is a constant. Since \( f \) is continuous, given \( \epsilon > 0 \), we can choose \( h \) so that

\[
|f(\zeta) - f(z)| \leq \epsilon
\]
for all \( \zeta \) on the line segment \([z, z + h]\). Therefore,

\[
|F(z + h) - F(z) - f(z)h| = |\int_{[z, z+h]} f(\zeta) - f(z) \, d\zeta| \leq \epsilon |h|
\]

using the fact that for any curve \( C \), and any function \( g \) with \(|g| < M\) on \( C \),

\[
\left| \int_C g(\zeta) \, d\zeta \right| \leq M \cdot \text{length of } C
\]

Dividing by \(|h|\) gives

\[
\left| \frac{F(z + h) - F(z)}{h} - f(z) \right| \leq \epsilon
\]

Since \( \epsilon \) was arbitrary, we conclude

\[
\lim_{h \to 0} \frac{F(z + h) - F(z)}{h} - f(z) = 0
\]

i.e., \( F'(z) \) exists and \( F'(z) = f(z) \).

Our goal was to show that \( f \) was holomorphic, i.e., that \( f'(z) \) exists for all \( z \in D \). We know that \( F'(z) \) exists; we also know that whenever \( F' \) exists, then all derivatives of \( F \) exist, by Cauchy’s formulas for derivatives. In particular, \( F'' \) exists, so \( f' \) exists. This completes the proof.