1 Inner Products and Norms on Real Vector Spaces

Recall that an inner product on a real vector space $V$ is a function from $V \times V$ to $\mathbb{R}$ such that for any $v, w, x \in V$ and scalars $c \in \mathbb{R}$, we have

- $\langle v, w \rangle = \langle w, v \rangle$
- $\langle cv, w \rangle = c \langle v, w \rangle$
- $\langle v, w + x \rangle = \langle v, w \rangle + \langle v, w \rangle$
- $\langle v, v \rangle > 0$ for $v \neq 0$

Given a vector space $V$ with an inner product, it is possible to define notions such as lengths and angles. The length or norm of an element $v \in V$ is given by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$  

It is not difficult to prove that inner products and lengths satisfy the Cauchy-Schwarz inequality:

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

From this it follows that it makes sense to define the angle between $v$ and $w$ to be the angle $\theta$ such that

$$\cos \theta = \frac{\langle v, w \rangle}{\|v\| \|w\|},$$

since the right hand side is between $-1$ and $1$. We then say that two vectors $v$ and $w$ are orthogonal if $\langle v, w \rangle = 0$. It also follows from the definition of an inner product, together with the Cauchy-Schwarz inequality, that for all $v, w \in V$ and scalars $c \in \mathbb{R}$, we have

- $\|cv\| = |c|\|v\|$
- $\|v\| > 0$ for $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$

These are actually the defining properties of a norm.

2 Orthogonal Bases

Suppose that we have a basis $\{v_1, v_2, \ldots, v_n\}$ for a vector space $V$ consisting of vectors that are mutually orthogonal. That is, suppose that $\langle v_j, v_k \rangle = 0$ for every $j \neq k$. Now suppose we are given a vector $w$ that we wish to write as a linear combination of the $v_k$’s. First write

$$w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$
for some unknown coefficients $c_1$ through $c_n$. Now fix $k$ and take the inner product of both sides with $v_k$ to get

$$\langle w, v_k \rangle = c_1 \langle v_1, v_k \rangle + c_2 \langle v_2, v_k \rangle + \cdots + c_n \langle v_n, v_k \rangle$$

Now because of the orthogonality of the basis vectors, every one of the inner products on the right hand side is zero, except for one, namely $\langle v_k, v_k \rangle$. Thus the equation reduces to

$$\langle w, v_k \rangle = c_k \langle v_k, v_k \rangle$$

and thus

$$c_k = \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle}.$$

We therefore have the general formula

$$w = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots + \frac{\langle w, v_n \rangle}{\langle v_n, v_n \rangle} v_n.$$

Thus if we have a basis consisting of orthogonal vectors, then it is very simple to express a given vector in terms of the basis vectors. Simply take inner products to find the coefficients.

### 3 The $L^2$ Inner Product

Now consider the vector space $V$ of real-valued continuous functions on an interval $[a, b]$. The $L^2$ inner product on $V$ is defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, dx$$

It is left as an exercise to verify that this satisfies the properties of an inner product. The $L^2$ norm of a function is then defined by

$$\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

The thing that makes Fourier series work so well is that the “basis” functions are all orthogonal with respect to the $L^2$ inner product.

**Example 1.** Consider the Fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right)$$

of a function $\phi$ on the interval $[0, L]$. Denoting by

$$s_n(x) = \sin \left( \frac{n\pi x}{L} \right),$$

we proved in class that

$$\langle s_m, s_n \rangle = \int_0^L \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad \text{for } m \neq n,$$

$$\langle s_n, s_n \rangle = \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = \frac{L}{2}$$
From this we can derive formulas for the coefficients $A_n$ as we did for the $c_k$ above:

$$A_n = \frac{\langle \phi, s_n \rangle}{\langle s_n, s_n \rangle} = \frac{2}{L} \int_0^L \phi(x) \sin \left( \frac{n\pi x}{L} \right) \, dx$$

**Example 2.** Similar calculations apply to the Fourier cosine series

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right)$$

of a function $\phi$ on the interval $[0, L]$. Here the basis functions are

$$c_n(x) = \cos \left( \frac{n\pi x}{L} \right),$$

together with the constant function 1 (which can be thought of as $\cos \left( 0 \frac{n\pi x}{L} \right)$). we have

$$\langle c_m, c_n \rangle = \int_0^L \cos \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad \text{for } m \neq n$$

$$\langle c_n, c_n \rangle = \int_0^L \cos^2 \left( \frac{n\pi x}{L} \right) \, dx = \frac{L}{2} \quad n = 1, 2, \ldots$$

$$\langle 1, 1 \rangle = \int_0^L 1 \, dx = L$$

from which we get

$$A_n = \frac{\langle \phi, c_n \rangle}{\langle c_n, c_n \rangle} = \frac{2}{L} \int_0^L \phi(x) \cos \left( \frac{n\pi x}{L} \right) \, dx$$

and

$$A_0 = 2 \frac{\langle \phi, 1 \rangle}{\langle 1, 1 \rangle} = \frac{2}{L} \int_0^L \phi(x) \, dx.$$

This explains the reason for the factor $\frac{1}{2}$ in front of $A_0$. It just ensures that the formulas for all the coefficients have the same form.

**Example 3.** Finally, consider the full Fourier series

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) + B_n \sin \left( \frac{n\pi x}{L} \right)$$

of a function $\phi$ on the interval $[-L, L]$. The reason for using the entire interval $[-L, L]$ instead of $[0, L]$ is that the functions $s_n$ and $c_n$ are not all mutually orthogonal on $[0, L]$, but they are orthogonal on $[-L, L]$. That is,

$$\langle c_m, c_n \rangle = \int_{-L}^L \cos \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad \text{for } m \neq n$$

$$\langle s_m, s_n \rangle = \int_{-L}^L \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad \text{for } m \neq n$$

$$\langle c_m, s_n \rangle = \int_{-L}^L \cos \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad \text{for all } m, n$$
Since
\[ \langle c_n, c_n \rangle = \langle s_n, s_n \rangle = L \]
and
\[ \langle 1, 1 \rangle = 2L, \]
we then have
\[
A_n = \frac{\langle \phi, c_n \rangle}{\langle c_n, c_n \rangle} = \frac{1}{L} \int_{-L}^{L} \phi(x) \cos \left( \frac{n\pi x}{L} \right) \, dx
\]
\[
B_n = \frac{\langle \phi, s_n \rangle}{\langle s_n, s_n \rangle} = \frac{1}{L} \int_{-L}^{L} \phi(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]
and
\[ A_0 = \frac{1}{L} \int_{-L}^{L} \phi(x) \, dx. \]

4 Hermitian Inner Products

Let \( V \) be a complex vector space (the field of scalars is \( \mathbb{C} \)). Then a Hermitian inner product is function \( V \times V \) to \( \mathbb{C} \) such that
\[
\begin{align*}
\bullet & \quad \langle v, w \rangle = \langle w, v \rangle^* \\
\bullet & \quad \langle cv, w \rangle = c \langle v, w \rangle \\
\bullet & \quad \langle v, w + x \rangle = \langle v, w \rangle + \langle v, w \rangle \\
\bullet & \quad \langle v, v \rangle > 0 \text{ for } v \neq 0
\end{align*}
\]
The notions of orthogonality and length are defined in the same way as we did using inner products on real vector spaces. Also, given an orthogonal basis \( v_1, \ldots, v_n \) for \( V \) we can derive the coefficients \( c_k \) in the expansion
\[ w = c_1 v_1 + \cdots + c_n v_n \]
in the same way as before:
\[ c_k = \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle} \]
The \( L^2 \) inner product for complex-valued functions on an interval \([a, b]\) is given by
\[ \langle f, g \rangle = \int_{a}^{b} f(x) \overline{g(x)} \, dx \]
It is not hard to prove that it satisfies the four conditions above.
Example 4. The complex form of the full Fourier series of a function \( \phi \) on \([-L, L]\) is given by

\[
\phi(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L}
\]

The functions

\[
e_n(x) = e^{in\pi x/L}
\]

are orthogonal with respect to the \(L^2\) inner product on \([-L, L]\). To see this, we can use the fact that \(e^{i\pi} = -1\) to compute

\[
\langle e_m, e_n \rangle = \int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L} dx
\]

\[
= \int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L} dx
\]

\[
= \int_{-L}^{L} e^{i(m-n)\pi x/L} dx
\]

\[
= \frac{L}{i\pi (m-n)} e^{i(m-n)\pi x/L} \bigg|_{-L}^{L}
\]

\[
= \frac{L}{i\pi (m-n)} (e^{i(m-n)\pi} - e^{-i(m-n)\pi})
\]

\[
= \frac{L}{i\pi (m-n)} ((-1)^{m-n} - (-1)^{n-m})
\]

\[
= 0
\]

for \(m \neq n\) and

\[
\langle e_n, e_n \rangle = \int_{-L}^{L} e^{in\pi x/L} e^{-in\pi x/L} dx = \int_{-L}^{L} 1 dx = 2L
\]

Therefore

\[
C_n = \frac{\langle \phi, e_n \rangle}{\langle e_n, e_n \rangle} = \frac{1}{2L} \int_{-L}^{L} \phi(x) e^{-in\pi x/L} dx.
\]