Fourier Transforms

Given a function \( f : \mathbb{R} \to \mathbb{R} \), its **Fourier transform** is the function

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx
\]

and its **inverse Fourier transform** is the function

\[
\check{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, d\xi
\]

Thought of as an operator, the Fourier transform is denoted by \( \mathcal{F} \) and the inverse Fourier transform by \( \mathcal{F}^{-1} \). That is, \( \mathcal{F}(f) = \hat{f} \) and \( \mathcal{F}^{-1}(\hat{f}) = \check{f} \). It should be noted that it is not at all obvious that the second formula really is the inverse of the first. Before proving this, we will look at some of the basic properties of the Fourier transform. It is helpful to first work within a special class of functions called the Schwartz class.

**Schwartz Class**

**Definition 1.** A function \( f \) is said to be **rapidly decreasing** if for every integer \( N \geq 0 \) there exists a constant \( C_N \) such that

\[
|x|^N |f(x)| \leq C_N
\]

for all \( x \in \mathbb{R} \).

**Definition 2.** The Schwartz class \( \mathcal{S} \) is the set of all functions \( f \in C^\infty(\mathbb{R}) \) such that \( f \) and all of its derivatives are rapidly decreasing.

It is easy to see that the Schwartz class is closed under differentiation and multiplication by polynomials. Also, since functions in \( \mathcal{S} \) are bounded and decay faster than any polynomial as \( |x| \to \infty \), it follows that Schwartz class functions are integrable, and therefore it makes sense to take their Fourier transform.

**Example 1.** For \( a > 0 \), \( f(x) = e^{-a|x|^2} \) is in \( \mathcal{S} \). Its Fourier transform is given by

\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ix\xi} \, dx
\]

\[
= \int_{-\infty}^{\infty} e^{-a[(x+i\xi/2a)^2+\xi^2/4a^2]} \, dx
\]

\[
= e^{-\xi^2/4a} \int_{-\infty}^{\infty} e^{-a(x+i\xi/2a)^2} \, dx
\]

In the remaining integral, we substitute \( z = x + i\xi/2a \) to get

\[
\int_{-\infty}^{\infty} e^{-a(x+i\xi/2a)^2} \, dx = \int_{-\infty+i\xi/2a}^{\infty+i\xi/2a} e^{-az^2} \, dz
\]
This integral is actually a line integral over the line \( z = x + i\xi/2a \) in the complex plane. Using methods from complex analysis, one can show that it is equivalent to the integral over the real axis:

\[
\int_{-\infty}^{\infty} e^{-ax^2} \, dz.
\]

Finally, making the change of variable \( z = y/\sqrt{a} \) this becomes

\[
\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\frac{\pi}{a}},
\]

so

\[
\hat{f}(\xi) = \sqrt{\frac{\pi}{a}} e^{-\xi^2/4a}.
\]

The following theorem is the most important algebraic property of Fourier transforms.

**Theorem 1.** If \( f \in \mathcal{S} \) then \( \hat{f} \in \mathcal{S} \) and

\[
\frac{\hat{f}}{dx} = i\xi \hat{f}
\]

\[
\hat{x}f = i \frac{\hat{f}}{d\xi}
\]

for \( 1 \leq k \leq n \).

So differentiation of \( f \) corresponds to multiplication of \( \hat{f} \) by a polynomial, and conversely multiplication of \( f \) by a polynomial corresponds to differentiation of \( \hat{f} \).

**Proof.** By the integrability of Schwartz class functions, the following calculations are justified. Integration by parts in \( x \) gives

\[
\frac{\hat{f}}{dx}(\xi) = \int_{\mathbb{R}} \frac{df}{dx} e^{-ix\xi} \, dx = \int_{\mathbb{R}} i\xi f(x) e^{-ix\xi} \, dx = i\xi \hat{f}(\xi)
\]

while differentiation with respect to \( \xi \) gives

\[
\frac{df}{d\xi}(\xi) = \int_{\mathbb{R}} -ix f(x) e^{-ix\xi} \, dx = -i\hat{x}f(\xi)
\]

This proves the two formulas.

To prove that \( f \in \mathcal{S} \) implies \( \hat{f} \in \mathcal{S} \), first notice that \( f \in \mathcal{S} \) implies \( \hat{f} \) is bounded, since \( f \) is integrable. Next, since

\[
\xi \hat{f} = -i \hat{f}_x
\]

and since \( f_x \in \mathcal{S} \) it follows that \( \xi \hat{f} \) is also bounded. By induction it follows that

\[
|\xi|^N |\hat{f}(\xi)|
\]

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Theorem 2. \textit{(Fourier Inversion Theorem)} Let $f \in S$. Then $\hat{\check{f}} = f$.

\textit{Proof.} For $n \in \mathbb{N}$, define

$$I_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\xi^2/n^2 + i\xi} d\xi$$

Since the integrand converges pointwise to $\hat{f}(\xi) e^{i\xi}$ as $n \to \infty$ and since

$$|\hat{f}(\xi) e^{-\xi^2/n^2 + i\xi}| \leq |\hat{f}(\xi)|$$

and $|\hat{f}(\xi)|$ is integrable since $\hat{f} \in S$, it follows from the Dominated Convergence Theorem that

$$\lim_{n \to \infty} I_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi} d\xi = \check{\hat{f}}(x).$$

On the other hand,

$$I_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-i(y-x)\xi} e^{-\xi^2/n^2} dy d\xi$$

Since $|f(y)| e^{-\xi^2/n^2}$ is integrable over $\mathbb{R} \times \mathbb{R}$, we can apply Fubini’s theorem to get

$$I_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \left( \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-\xi^2/n^2} d\xi \right) dy$$

The term in parentheses is the Fourier transform of $e^{-\xi^2/n^2}$ evaluated at $y - x$, so applying Example 1 with $a = \epsilon$, it equals

$$\sqrt{\frac{n^2}{4\pi}} e^{-n^2(y-x)^2/4},$$

and thus

$$I_n(x) = \int_{-\infty}^{\infty} f(y) \frac{n}{\sqrt{4\pi}} e^{-n^2(y-x)^2/4} dy.$$ Setting

$$g_n(x) = \frac{n}{\sqrt{4\pi}} e^{-n^2x^2/4} = \frac{n}{2} \phi(nx/2)$$

where $\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, it follows that $g_n$ is an approximate identity and therefore

$$I_n = g_n \ast f \to f$$

pointwise as $n \to \infty$. Thus

$$\hat{\check{f}}(x) = \lim_{n \to \infty} I_n(x) = f(x).$$

\hfill \square
Fourier Transforms and Convolution

**Theorem 3.** Let \( f, g \in S \). Then

\[
\mathcal{F}(f \ast g) = \mathcal{F}(f) \mathcal{F}(g)
\]

\[
\mathcal{F}(fg) = \frac{1}{2\pi} \mathcal{F}(f) \ast \mathcal{F}(g)
\]

**Proof.** The first formula was assigned for homework. For the second formula, we first note that \( \mathcal{F}^{-1} f(y) = \frac{1}{2\pi} \mathcal{F} f(-y) \). So for any \( u, v \in S \), we have

\[
\mathcal{F}^{-1}(u \ast v)(\xi) = \frac{1}{2\pi} \mathcal{F}(u \ast v)(-\xi)
\]

\[
= \frac{1}{2\pi} \mathcal{F}u(-\xi)\mathcal{F}v(-\xi)
\]

\[
= 2\pi \mathcal{F}^{-1} u(\xi) \mathcal{F}^{-1} v(\xi).
\]

Letting \( u = \mathcal{F} f \) and \( v = \mathcal{F} g \) and using the Fourier inversion theorem, this becomes

\[
\mathcal{F}^{-1}(\mathcal{F} f \ast \mathcal{F} g)(\xi) = 2\pi f(\xi)g(\xi)
\]

Applying the inversion theorem once more gives

\[
\mathcal{F}(fg) = \frac{1}{2\pi} \mathcal{F} f \ast \mathcal{F} g
\]

as desired.

Here is an application of some of these results.

**Example 2.** Consider the heat equation in one dimension,

\[
\begin{align*}
  u_t &= ku_{xx} \\
  u(x, 0) &= \phi(x)
\end{align*}
\]

Take the Fourier transform of both sides with respect to \( x \), and use Theorem 1:

\[
\hat{u}_t = k\hat{u}_{xx} = k(i\xi)(i\xi)\hat{u} = -k\xi^2 \hat{u}
\]

For each fixed \( \xi \), solve the differential equation \( \hat{u}_t = -k\xi^2 \hat{u} \) to get

\[
\hat{u}(\xi, t) = C(\xi)e^{-kt\xi^2}
\]

To determine \( C(\xi) \), take the Fourier transform of the initial condition:

\[
\hat{u}(\xi, 0) = \hat{\phi}(\xi)
\]

Thus \( \hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-kt\xi^2} \). Now using the formula in Example 1, with \( a = 1/4kt \) gives

\[
e^{-x^2/4kt} = \sqrt{4\pi kt}e^{-k\xi^2t} \quad \Rightarrow \quad e^{-k\xi^2t} = \frac{1}{4\pi kt}e^{-x^2/4kt}
\]

Thus if we define
\[ S(x,t) = \frac{1}{4\pi kt} e^{-x^2/4kt} \]
then
\[ \hat{u} = \hat{\phi} \hat{S} = \hat{\phi} \ast S \]
so the Fourier inversion theorem implies
\[ u = \phi \ast S \]
So the solution of the heat equation is simply the initial data \( \phi \) convolved with the heat kernel \( S \). (See the handout on convolution.)

**L^2 Properties**

The Fourier transform behaves very nicely with respect to \( L^2 \).

**Lemma 1.** For \( f, g \in \mathcal{S} \), \( \langle \hat{f}, g \rangle = 2\pi \langle f, \hat{g} \rangle \).

**Proof.** By direct calculation
\[
\langle \hat{f}, g \rangle = \int_R \hat{f}(\xi)\overline{g(\xi)} \, d\xi \\
= \int_R \int_R f(x)\overline{g(\xi)} e^{-ix\xi} \, dx \, d\xi \\
= \int_R \int_R f(x)\overline{g(\xi)} e^{-ix\xi} \, d\xi \, dx \\
= \int_R f(x) \overline{\int_R g(\xi) e^{ix\xi} \, d\xi} \, dx \\
= \int_R f(x) 2\pi \overline{\hat{g}(x)} \, dx \\
= 2\pi \langle f, \hat{g} \rangle
\]
where once again the integrability of \( f \) and \( g \) justifies changing the order of integration. \( \square \)

**Theorem 4.** (Plancherel’s Formula) Let \( f, g \in \mathcal{S} \). Then
\[
\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle \\
\| \hat{f} \|_{L^2} = 2\pi \| f \|_{L^2}
\]

**Proof.** The second formula follows from the first by setting \( g = f \), while the first follows from the previous lemma by replacing \( g \) with \( \hat{g} \) and using the inversion theorem. \( \square \)
Remark. Plancherel’s formula seems to indicate that the Fourier transform is bounded as an operator from $L^2$ to $L^2$. However, one can find functions in $f \in L^2$ for which the integral which defines $\hat{f}$ is not an $L^2$ function. One must therefore be careful when thinking of $F$ as an operator on $L^2$.

Extension to $L^2$

Plancherel’s formula allows us to extend the Fourier transform to an operator $F : L^2 \to L^2$. The proof relies on the following facts:

1. $L^2$ is complete.

2. $S$ is dense in $L^2$.

To extend $F$ to all of $L^2$, use the density of $S$ to approximate elements of $L^2$ by elements of $S$, and take a limit. More precisely, let $f \in L^2$. Since $S$ is dense, there exists a sequence $f_k$ in $S$ such that

$$\lim_{k \to \infty} \| f_k - f \|_{L^2} = 0$$

Since $f_k$ is convergent, it is a Cauchy sequence. By Plancherel’s formula it follows that the sequence $F(f_k)$ is also a Cauchy sequence. By completeness of $L^2$ there exists a limit. Call this limit $F(f)$. Then

$$\| F(f) \|_{L^2} = \lim_{k \to \infty} \| F(f_k) \|_{L^2} = \lim_{k \to \infty} 2\pi \| f_k \|_{L^2} = 2\pi \| f \|_{L^2}$$

so Plancherel’s formula still holds for the extension.