Let $S$ be a measurable subset of $\mathbb{R}^n$. We define

$$L^2(S) = \left\{ f : \int_S |f(x)|^2 \, dx < \infty \right\}$$

The norm on $L^2$ is defined by

$$\|f\|_{L^2} = \left( \int_S |f(x)|^2 \, dx \right)^{1/2}$$

and comes from the inner product

$$\langle f, g \rangle = \int_S f(x)g(x) \, dx.$$

**Theorem 1.** $L^2$ is a complete, normed, inner product space. (a.k.a. a *Hilbert space*)

When $m(S)$ is finite, $L^2$ functions are $L^1$.

**Lemma 1.** If $m(S) < \infty$, then $L^2(S) \subset L^1(S)$.

**Proof.** Let $f \in L^2(S)$. By the Cauchy Schwartz inequality,

$$\|f\|_{L^1} = \int |f(x)| \, dx = \int |f(x)| \cdot 1 \, dx = \langle f, 1 \rangle \leq \|f\|_{L^2} \|1\|_{L^2} = m(S)\|f\|_{L^2} < \infty,$$

so $f \in L^1(S)$. \qed

The following example illustrates that $L^2([0,1])$ is a proper subset of $L^1([0,1])$.

**Example 1.** Let $f(x) = x^{-1/2}$ on $[0,1]$. Then $\|f\|_{L^1} = 2$, while $\|f\|_{L^2} = \infty$, so $f \in L^1 \setminus L^2$.

The next example illustrates the necessity of the finiteness of $m(S)$ in the lemma.

**Example 2.** Let $S = [1, \infty)$ and let $f(x) = \frac{1}{x}$. Then $\|f\|_{L^2} = 1$, while $\|f\|_{L^1} = \infty$, so $f \in L^2 \setminus L^1$.

**Orthogonality**

Then main difference between $L^2$ and $L^1$ is that $L^2$ has an inner product.

**Definition 1.** We say that two functions $f$ and $g$ in $L^2$ are *orthogonal* if $\langle f, g \rangle = 0$.

**Example 3.** Let $S = [0, \pi]$ and let $f(x) = \sin x$, $g(x) = \cos x$. Then

$$\langle f, g \rangle = \int_0^\pi \sin x \cos x \, dx = 0$$

so $f$ and $g$ are orthogonal.
Definition 2. We say a collection of $L^2$ functions $\{\phi_k\}$ is **orthogonal** if each pair of functions in the collection is orthogonal. We say the collection is **orthonormal** if $\|\phi_k\| = 1$ for each $k$.

The advantage of orthogonal and orthonormal sets is that it is very easy to determine coefficients of elements in their span. For suppose $\{\phi_k\}_{k=1}^N$ is an orthogonal collection of $L^2$ functions, and $f$ is a function in span$\{\phi_k\}$. Then

$$f = \sum_{k=1}^N c_k \phi_k$$

for some coefficients $c_k$. To determine the coefficients, take the inner product of both sides with $\phi_j$ and use the fact that $\langle \phi_k, \phi_j \rangle = 0$ if $k \neq j$:

$$\langle f, \phi_j \rangle = \sum_{k=1}^N c_k \langle \phi_k, \phi_j \rangle = c_j \langle \phi_j, \phi_j \rangle$$

Thus

$$c_j = \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}$$

(and in the case that the collection $\{\phi_k\}$ is orthonormal, this becomes simply $c_j = \langle f, \phi_j \rangle$).

**Gram-Schmidt Process**

It is often necessary to construct an orthogonal or orthonormal collection from a given collection of functions. So suppose $\{\phi_k\}$ is a linearly independent collection of $L^2$ functions. We then define

$$\psi_1 = \phi_1$$
$$\psi_2 = \phi_2 - \frac{\langle \phi_2, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1$$
$$\psi_3 = \phi_3 - \frac{\langle \phi_3, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1 - \frac{\langle \phi_3, \psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} \psi_2$$
$$\psi_4 = \phi_4 - \frac{\langle \phi_4, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1 - \frac{\langle \phi_4, \psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} \psi_2 - \frac{\langle \phi_4, \psi_3 \rangle}{\langle \psi_3, \psi_3 \rangle} \psi_3$$

The new collection $\{\psi_k\}$ is orthogonal and has the same span as $\{\phi_k\}$. Dividing each $\psi_k$ by its norm produces an orthonormal set.

**Example 4.** Consider the collection of polynomials $\{1, x, x^2\}$ in $L^2([0, 1])$. Then $\psi_1 = 1$,

$$\psi_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{1}{2}$$

2
and

\[ \psi_3 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 - \frac{\langle x^2, x - 1/2 \rangle}{\langle x - 1/2, x - 1/2 \rangle} \cdot (x - 1/2) = x^2 - x + \frac{1}{6}, \]

so \{1, x - 1/2, x^2 - x + 1/6\} is a orthogonal set of polynomials.