The convolution of two functions \( f \) and \( g \) is the function

\[
(f \ast g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy.
\]

We first prove that, when the convolution is defined, it is symmetric.

**Theorem 1.** \( f \ast g = g \ast f \), provided one or the other is defined.

**Proof.** For fixed \( x \), consider the change of variable \( z = T(y) = x - y \). Since \(|\det(JT)| = 1\), we have

\[
(f \ast g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy = \int_{\mathbb{R}^n} f(z)g(x - z) \, dz = \int_{\mathbb{R}^n} g(x - z)f(z) \, dz = (g \ast f)(x).
\]

Next we consider two cases when the convolution is defined.

**Theorem 2.** If \( f \) is bounded and continuous and \( g \in L^1(\mathbb{R}^n) \), then \( f \ast g \) is bounded, and \( \|f \ast g\|_\infty \leq \|f\|_\infty \|g\|_{L^1} \).

**Proof.** For each fixed \( x \),

\[
|(f \ast g)(x)| \leq \int_{\mathbb{R}^n} |f(x - y)||g(y)| \, dy \leq \int_{\mathbb{R}^n} \|f\|_\infty |g(y)| \, dy = \|f\|_\infty \|g\|_{L^1}.
\]

**Theorem 3.** If \( f \) and \( g \) are in \( L^1 \), then \( f \ast g \) is in \( L^1 \) and \( \|f \ast g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1} \).

**Proof.** First suppose \( f \) and \( g \) are nonnegative. Then \( f \ast g \) is nonnegative, and using Fubini’s Theorem gives

\[
\|f \ast g\|_{L^1} = \int_{\mathbb{R}^n} (f \ast g)(x) \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y)g(y) \, dy \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y)g(y) \, dx \, dy
= \int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} f(x - y) \, dx \, dy
= \int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} f(x) \, dx \, dy
= \int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} g(y) \, dy
= \|f\|_{L^1} \|g\|_{L^1}.
\]
Now for arbitrary $f$ and $g$, notice that

$$)|(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(x - y)||g(y)| \, dy = (||f|| |g|)(x),$$

so

$$||f * g||_{L^1} \leq ||f|| \cdot ||g||_{L^1} \leq ||f||_{L^1} \cdot ||g||_{L^1} = ||f||_{L^1} \cdot ||g||_{L^1}.$$

\[\square\]

**Theorem 4.** Let $D_{x_i}$ denote partial differentiation with respect to $x_i$. If $D_{x_i} f$ is continuous and $f * g$ is defined in a neighborhood of $x$, then $D_{x_i} (f * g)(x) = ((D_{x_i} f) * g)(x)$.

**Proof.**

$$D_{x_i} (f * g)(x) = D_{x_i} \int_{\mathbb{R}^n} f(x - y)g(y) \, dy = \int_{\mathbb{R}^n} D_{x_i} f(x - y)g(y) \, dy = (D_{x_i} f) * g.$$  

\[\square\]

**Definition 1.** A family of functions $f_k$ on $\mathbb{R}^n$ is called an **approximate identity** if

1. $f_n(x) \geq 0$ for all $x \in \mathbb{R}^n$
2. $\int_{\mathbb{R}^n} f_k(x) \, dx = 1$
3. $\lim_{k \to \infty} \int_{|x| > \delta} f_k(x) \, dx = 0$ for all $\delta > 0$.

Sometimes it is more convenient to consider families of the form $f_s$ for a continuous parameter $s > 0$. In this case, modify condition 3 to read $\lim_{s \to 0} \int_{|x| > \delta} f_s(x) \, dx = 0$.

**Theorem 5.** Let $f_k$ be an approximate identity and let $g$ be a bounded continuous function. Then

$$\lim_{k \to \infty} (f_k * g)(x) = g(x)$$

for all $x$ in $\mathbb{R}^n$.

**Proof.** Fix $x$ and let $\epsilon > 0$. By the continuity of $g$, there exists $\delta > 0$ such that $|y - x| < \delta$ implies $|g(y) - g(x)| < \epsilon/2$. Since

$$g(x) = g(x) \int_{\mathbb{R}^n} f_k(y) \, dy = \int_{\mathbb{R}^n} g(x)f_k(y) \, dy,$$

we have

$$(f_k * g)(x) - g(x) = (g * f_k)(x) - g(x)$$

$$= \int_{\mathbb{R}^n} g(x - y)f_k(y) \, dy - \int_{\mathbb{R}^n} g(x)f_k(y) \, dy$$

$$= \int_{\mathbb{R}^n} (g(x - y) - g(x))f_k(y) \, dy.$$  

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Now split this integral into $I_1 = \int_{|y|<\delta} f_k(y) |g(x-y) - g(x)| dy$ and $I_2 = \int_{|y|\geq\delta} f_k(y) |g(x-y) - g(x)| dy$. By the choice of $\delta$, and the first two properties of an approximate identity, 

$$|I_1| \leq \int_{|y|<\delta} f_k(y) |g(x-y) - g(x)| dy < \frac{\epsilon}{2} \int_{|y|<\delta} f_k(y) dy \leq \frac{\epsilon}{2} \int_{\mathbb{R}^n} f_k(y) dy = \frac{\epsilon}{2}.$$ 

Since $g$ is bounded, we have 

$$|I_2| \leq \int_{|y|\geq\delta} f_k(y) |g(x-y) - g(x)| dy \leq 2\|g\|_{\infty} \int_{|y|\geq\delta} f_k(y) dy.$$ 

By the third property of an approximate identity there is some $K$ such that the last term is less than $\epsilon/2$ for $k \geq K$. Thus $|(f_k * g)(x) - g(x)| < \epsilon$ for $k \geq K$. 

Notice that the choice of $K$ depends only on $\delta$, which in turn depends on $x$ and $\epsilon$. If $g$ were assumed to be uniformly continuous, then $\delta$ would depend only on $\epsilon$, so $K$ would depend only on $\epsilon$, which means that $f_k * g$ would converge uniformly to $g$.

The Weierstrass Approximation Theorem is a direct consequence of the previous result. For by defining 

$$f_n(x) = \begin{cases} c_n(1-x^2)^n & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$ 

where the coefficients $c_n$ are chosen so that $\int_{-\infty}^{\infty} f_n(x) dx = 1$, it follows that the family $f_n$ is an approximate identity. Given any continuous function $g$ on $[-1, 1]$, since $g$ is uniformly continuous the sequence $f_n * g$ converges uniformly to $g$ on $[-1, 1]$. Since each $f_n$ is a polynomial of degree it follows that $f_n * g$ is also a polynomial. Thus polynomials are dense in $C([-1, 1])$.

Here is another application.

**Example 1.** (Solutions of the Heat Equation) Let $u(x, t)$ denote the temperature at a point $x$ on a metal rod at time $t$. It can be shown that the function $u$ must satisfy the partial differential equation 

$$u_t = ku_{xx},$$ 

which is known as the **heat equation**. Given the temperature distribution of the rod at time zero, $\phi(x)$, we wish to determine the solution of this equation which satisfies this initial condition $u(x, 0) = \phi(x)$. One way to find the solution uses the function 

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt},$$ 

called the **heat kernel**. It is not difficult to check that it has the following properties:

- $S_t = kS_{xx}$ for $t > 0$
- $S(x, t) \geq 0$ for $t > 0$
- $\int_{-\infty}^{\infty} S(x, t) dx = 1$ for $t > 0$
- $\lim_{t \to 0} \int_{|x|>\delta} S(x, t) dx = 0$ for all $\delta > 0$
The first property just states that $S$ is a solution of the heat equation, while the remaining properties say that $S$ is an approximate identity, when viewed as a family of functions parametrized by $t$.

Now define $u = S * \phi$. That is, let

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} \phi(y) \, dy$$

for $t > 0$. We claim that $u$ is the solution we desire. First, since the derivative of a convolution can be passed onto one of the factors,

$$u_t = (S * \phi)_t = S_t * \phi = (kS_{xx}) * \phi = k(S * \phi)_{xx} = ku_{xx},$$

and therefore $u$ is a solution of the heat equation. Now the expression for $u$ above is actually undefined at $t = 0$, so we have to explain the manner in which $u$ satisfies the initial condition. Since $S$ is an approximate identity, we have

$$\lim_{t \to 0} u(x, t) = \lim_{t \to 0} S(x, t) * \phi(x) = \phi(x)$$

Therefore, extending the definition of $u$ to $t = 0$ by $u(x, 0) = \phi(x)$ defines a continuous function $u(x, t)$ which satisfies the heat equation and the initial condition.