Tchebyshev’s Inequality. Let \( f \) be nonnegative and measurable on some measurable set \( A \). Then
\[
m(\{x \in A : f(x) > \alpha\}) \leq \frac{1}{\alpha} \int_A f.
\]

Proof. Let \( B = \{x \in A : f(x) > \alpha\} \). Then \( \alpha \chi_B \leq f \) on \( A \), so
\[
\alpha m(B) = \int_A \alpha \chi_B \leq \int_A f \implies m(B) \leq \frac{1}{\alpha} \int_A f.
\]

Here are a couple of important consequences of Tchebyshev’s Inequality.

**Corollary 1.** Let \( f \) be nonnegative and measurable, and suppose that
\[
\int_A f = 0.
\]
Then \( f = 0 \) almost everywhere on \( A \).

Proof. Define \( A_k = \{x \in A : f(x) > \frac{1}{k}\} \). Then by Tchebyshev’s inequality,
\[
m(A_k) \leq k \int_A f = 0
\]
so each \( A_k \) is a zero set. Thus
\[
\{x \in A : f(x) > 0\} = \bigcup_{k=1}^{\infty} A_k
\]
is also zero set, so \( f = 0 \) almost everywhere.

**Corollary 2.** Suppose \( A \) is measurable, \( f \) is a measurable function on \( A \) and
\[
\int_B f = 0
\]
for every measurable subset \( B \) of \( A \). Then \( f = 0 \) almost everywhere on \( A \).

Proof. Homework.