So far we know only how to integrate bounded functions over bounded domains. If the domain of integration is unbounded, or the function is unbounded on the domain of integration, the integral is called improper. To define an integral over a domain $D \subset \mathbb{R}^2$, we first make the following definition.

**Definition 1.** A sequence $S_k$ of compact Riemann measurable subsets of $\mathbb{R}^2$ is said to exhaust $D$ if

1. $S_k \subset S_{k+1}$ for all $k$
2. $D = \bigcup S_k$

There are many ways to exhaust any given region.

**Example 1.** The sets

$$S_k = [-k, k] \times [-k, k] \quad \text{and} \quad T_k = B_k(0,0)$$

both exhaust $\mathbb{R}^2$.

We now define the improper integral.

**Definition 2.** Let $D$ be a subset of $\mathbb{R}^2$ and suppose that $f$ is bounded on every compact subset of $D$. If

$$\lim_{k \to \infty} \int_{S_k} f(x, y) \, dA = L$$

for every sequence $S_k$ that exhausts $D$, then we define

$$\int_D f(x, y) \, dA = L.$$ 

Otherwise the integral does not exist.

**Example 2.** Consider the improper integral

$$\int_{\mathbb{R}^2} \sin x \sin y \, dA.$$ 

The sequence $S_k = [-2\pi k, 2\pi k] \times [-2\pi k, 2\pi k]$ gives

$$\lim_{k \to \infty} \int_{S_k} \sin x \sin y \, dA = \lim_{k \to \infty} \int_{-2\pi k}^{2\pi k} \int_{-2\pi k}^{2\pi k} \sin x \sin y \, dy \, dx = \lim_{k \to \infty} 0 = 0$$

while the sequence $T_k = [-2\pi k, 2\pi (k+1)] \times [-2\pi k, 2\pi (k+1)]$ gives

$$\lim_{k \to \infty} \int_{T_k} \sin x \sin y \, dA = \lim_{k \to \infty} \int_{-2\pi k}^{(2k+1)\pi} \int_{-2\pi k}^{(2k+1)\pi} \sin x \sin y \, dy \, dx = \lim_{k \to \infty} 4 = 4,$$

so the integral does not exist.
For nonnegative functions however it suffices to consider any one sequence that exhausts $D$.

**Theorem 1.** Suppose $f(x, y) \geq 0$ for all $(x, y) \in D$, and that $f$ is bounded on every compact subset of $D$. Let $S_k$ and $T_k$ be two sequences that exhaust $D$. Then

$$\lim_{k \to \infty} \int_{S_k} f(x, y) \, dA = \lim_{k \to \infty} \int_{T_k} f(x, y) \, dA.$$  

**Proof.** Fix a set $S_k$. Let $M = \sup\{|f(x)| : x \in S_k\}$, and let $\epsilon > 0$ be given. Since $T_n$ is Riemann measurable, its boundary $\partial T_n$ is a zero set. Therefore there exist open rectangles $R_{nj}$ such that

$$\partial T_n \subset \bigcup_j R_{nj} \quad \sum_j |R_{nj}| < \frac{\epsilon}{2^n M}.$$  

Let $B_n = \bigcup_j R_{nj}$. The $|B_n| < \frac{\epsilon}{2^n M}$ and $T_n \cup B_n$ is open. The sets $T_n \cup B_n$ form an open covering of the compact set $S_k$. Therefore some finite collection $T_n \cup B_n$, $i = 1, \ldots, N$ covers $S_k$. Since the $T_k$ are nested, this implies

$$S_k \subset T_{n_N} \cup \left( \bigcup_{i=1}^N B_{n_i} \cap S_k \right)$$

Since $f$ is nonnegative, this implies

$$\int_{S_k} f \, dA \leq \int_{T_{n_N}} f \, dA + \sum_{i=1}^N \int_{B_{n_i} \cap S_k} f \, dA$$

$$\leq \int_{T_{n_N}} f \, dA + \sum_{i=1}^N \int_{B_{n_i} \cap S_k} M \, dA$$

$$\leq \int_{T_{n_N}} f \, dA + M \sum_{i=1}^N |B_{n_i}|$$

$$\leq \int_{T_{n_N}} f \, dA + M \sum_{n=1}^{\infty} \frac{\epsilon}{2^n M}$$

$$= \int_{T_{n_N}} f \, dA + \epsilon.$$  

Now since $f$ is nonnegative, the sequence $\int_{T_k} f \, dA$ is nondecreasing, so

$$\int_{S_k} f \, dA \leq \lim_{k \to \infty} \int_{T_k} f \, dA + \epsilon.$$  

Since $\epsilon$ was arbitrary, this shows that

$$\int_{S_k} f \, dA \leq \lim_{k \to \infty} \int_{T_k} f \, dA.$$  

2
and since this is true for every \( S_k \), we have
\[
\lim_{k \to \infty} \int_{S_k} f \, dA \leq \lim_{k \to \infty} \int_{T_k} f \, dA.
\]
Similarly the reverse inequality holds.

**Example 3.** Let \( D \) be the first quadrant. To compute
\[
\int_D e^{-x^2-y^2} \, dA
\]
let’s use the sequence \( S_k = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq k^2\} \) and switch to polar coordinates. The change of variables \( \varphi(r, \theta) = (r \cos \theta, r \sin \theta) \) satisfies \( \det(J \varphi(r, \theta)) = r \), and \( S_k = \varphi(R_k) \), where \( R_k = [0, k] \times [0, \pi/2] \). By the Change of Variables formula,
\[
\int_{S_k} e^{-x^2-y^2} \, dA = \int_{R_k} e^{-r^2} r \, dA
\]
By Fubini’s Theorem,
\[
\int_{R_k} e^{-r^2} r \, dA = \int_0^{\pi/2} \int_0^k e^{-r^2} r \, dr \, d\theta = \frac{1}{4} \pi (1 - e^{-k^2}).
\]
Therefore
\[
\int_D e^{-x^2-y^2} \, dA = \lim_{k \to \infty} \pi (1 - e^{-k^2}) = \frac{1}{4} \pi.
\]
A nice consequence of this calculation is revealed if we consider a different exhaustion of \( D \). Let \( T_k = [0, k] \times [0, k] \). Then by Theorem 1 and Fubini’s Theorem,
\[
\frac{1}{4} \pi = \lim_{k \to \infty} \int_{T_k} e^{-x^2-y^2} \, dA = \lim_{k \to \infty} \int_0^k \int_0^k e^{-x^2} e^{-y^2} \, dy \, dx
\]
\[
= \lim_{k \to \infty} \int_0^k e^{-x^2} \int_0^k e^{-y^2} \, dy \, dx = \lim_{k \to \infty} \int_0^k e^{-y^2} \, dy \int_0^k e^{-x^2} \, dx
\]
\[
= \lim_{k \to \infty} \left[ \int_0^k e^{-x^2} \, dx \right]^2 = \left[ \lim_{k \to \infty} \int_0^k e^{-x^2} \, dx \right]^2,
\]
so
\[
\int_0^{\infty} e^{-x^2} \, dx = \lim_{k \to \infty} \int_0^k e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi}.
\]