Here are two versions of Taylor’s Theorem for multivariable functions.

**Theorem 1.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $k$-times differentiable. Then

$$f(x) = f(p) + Df_p(x-p) + \frac{1}{2} D^2 f_p(x-p, x-p) + \cdots + \frac{1}{k!} D^k f_p(x-p, x-p, \ldots, x-p) + R(x),$$

where

$$\lim_{x \to p} \frac{R(x)}{|x-p|^k} = 0.$$

**Proof.** We’ll just prove the case $k = 2$. So write

$$f(x) = f(p) + Df_p(x-p) + \frac{1}{2} D^2 f_p(x-p, x-p) + R(x)$$

Then we have $R(p) = 0$, $DR(p) = 0$ and $D^2 R(p) = 0$. By the Mean Value Theorem,

$$|R(x)| = |R(x) - R(p)| \leq M|x-p|$$

where $M = \sup \{\|DR(y)\| : y = p + t(x-p), 0 \leq t \leq 1\}$. Now

$$DR(y) = DR(p) + D^2 R(p)(y-p) + R_2(y)$$

where $\lim_{y \to p} \frac{R_2(y)}{|y-p|} = 0$.

Given $\epsilon > 0$, there is some $\delta > 0$ such that

$$|y-p| < \delta \implies \frac{R_2(y)}{|y-p|} < \epsilon.$$

Since $DR(p) = 0$ and $D^2 R(p) = 0$, we have $DR(y) = R_2(y)$. Thus $|x-p| < \delta$ implies $|y-p| < \delta$, which implies

$$\frac{\|DR(y)\|}{|y-p|} < \epsilon$$

so $\|DR(y)\| \leq \epsilon|y-p| \leq \epsilon|x-p|$ and thus $M \leq \epsilon|x-p|$. It then follows that $|R(x)| \leq \epsilon|x-p|^2$ whenever $|x-p| < \delta$. Hence

$$\lim_{x \to p} \frac{R(x)}{|x-p|^2} = 0$$

Another version of Taylor’s Theorem which gives a precise form of the remainder follows from the $C^1$ Mean Value Theorem.
Theorem 2. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be twice differentiable. Then

\[
f(x) = f(p) + Df(p)(x - p) + \beta(x - p, x - p)
\]

where

\[
\beta = \int_0^1 \int_0^1 D^2 f(p + st(x - p)) 
  \, dst \, dt.
\]

Proof. By the \( C^1 \) Mean Value Theorem

\[
f(x) = f(p) + \int_0^1 Df(p + t(x - p)) \, dt \cdot (x - p).
\]

Now apply the \( C^1 \) Mean Value Theorem to \( Df \) to get

\[
Df(y) = Df(p) + \int_0^1 D^2 f(p + s(y - p)) \, ds(y - p)
\]

Applying this with \( y = p + t(x - p) \) gives

\[
Df(p + t(x - p)) = Df(p) + \int_0^1 D^2 f(p + st(x - p)) \, ds \cdot t(y - p)
\]

so substituting for the integrand in the expression for \( f(x) \) above proves the theorem. \( \square \)