1. Suppose $U$ is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is twice differentiable, with $(D^2 f)_p = 0$ for all $p \in U$. What can you say about $f$?

2. (a) For invertible matrices $A$ and $B$, verify the identity

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}.$$  

Use this to prove that $\|B^{-1} - A^{-1}\| \leq \|B^{-1}\|\|B - A\|\|A^{-1}\|$.

(b) Let $A$ be an invertible matrix. Show that $\|A^{-1}\| > 0$.

(c) Let $A$ be invertible. Show that if $\|B - A\| < \frac{1}{\|A^{-1}\|}$, then $B$ is invertible. (Hint: Suppose $B$ is not invertible. Then there is some nonzero vector $v$ such that $Bv = 0$. Let $w = Av$ and show that $|A^{-1}w| > \|A^{-1}\||w|$, a contradiction.) This shows that the set of invertible matrices is an open subset of the set of square matrices.

(d) Show that if $\|B - A\| \leq \frac{1}{2\|A^{-1}\|}$, then $\|B^{-1}\| \leq 2\|A^{-1}\|$. (Hint: Write $\|B^{-1}\| = \|B^{-1} - A^{-1} + A^{-1}\|$ and use the inequality in part (a).)

(e) Conclude that the inverse operator $\text{Inv}(X) = X^{-1}$ is a continuous function on the space of invertible matrices.

3. Consider the system of equations

$$x^2 + y^2 = 4$$
$$e^y + xy = 3$$

Solutions are roots of $F(x, y) = (x^2 + y^2 - 4, e^y + xy - 3)$. Apply one step of Newton’s Method with initial guess $(1, 0)$. 