1. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$. Compute $\|T\|$, and find a vector in $v \in \mathbb{R}^2$ such that $\|T(v)\| = \|T\| \|v\|$. 

2. Find linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\|S \circ T\| < \|S\| \|T\|$. Hint: Try projections. 

3. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ and let $B$ be the box $[1, 3] \times [2, 5] \times [-1, 1]$ in $\mathbb{R}^3$. What is the volume of $T(B)$? 

4. Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space $V$ are called equivalent if there exist positive constants $c$ and $C$ such that $c \|v\|_1 \leq \|v\|_2 \leq C \|v\|_1$ for all $v \in V$. 

(a) Let $d_1$ and $d_2$ be the metrics defined by two equivalent norms: $d_1(v, w) = |v - w|_1$ and $d_2(v, w) = |v - w|_2$. Show that a subset $U$ of $V$ is open with respect to $d_1$ if and only if it is open with respect to $d_2$. (Therefore equivalent norms on $V$ define the same topologies on $V$.) 

(b) Prove that any two norms on a finite-dimensional vector space are equivalent. (Hint: Use Theorem 2 in the book.)

(c) On the vector space of continuous functions $C([a, b])$ consider the norms 

$$ |f|_{L^1} = \int_a^b |f(x)| \, dx \quad \text{and} \quad |f|_{\text{max}} = \max\{ |f(x)| : x \in [a, b] \} $$

i. Show that $|f|_{L^1} \leq (b - a) |f|_{\text{max}}$ for all $f \in C([a, b])$ 

ii. Show that there is no constant $C$ such that $|f|_{\text{max}} \leq C |f|_{L^1}$ for all $f \in C([a, b])$. (Therefore the $L^1$ and max norms are not comparable.) Hint: Find a sequence of functions $f_n$ such that $|f_n|_{L^1} \rightarrow 0$ but $|f_n|_{\text{max}} \not\rightarrow 0$. 

5. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f(s, t) = (st, s^2 - t^2)$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $g(x, y) = (x + y^2, y - x^2)$. Let $p = (2, 1)$, $q = f(p)$. 

(a) Compute the matrices for $(Df)_p$ and $(Dg)_q$. 

(b) Use the Chain Rule to calculate the matrix for $(D(g \circ f))_p$. 

(c) Compute $h = g \circ f$ and use it to compute $(D(g \circ f))_p$ directly. 

6. Suppose that $U$ is a connected open subset of $\mathbb{R}^n$ and that $f : U \rightarrow \mathbb{R}^m$ is differentiable and $(Df)_p = 0$ for all $p \in U$. Prove that $f$ is constant.