\[ \int_0^1 x^2 \sqrt{4 - x^2} \, dx = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \approx 0.6141848494 \]

1. Verify the exact value of the integral above, using a trig substitution.

Let \( x = 2 \sin \theta, \, dx = 2 \cos \theta \, d\theta \). Then \( \sqrt{4 - x^2} = 2 \cos \theta \), and the limits of integration become \( \theta = 0 \) to \( \theta = \pi/6 \):

\[ \int_0^1 x^2 \sqrt{4 - x^2} \, dx = \int_0^{\pi/6} 4 \sin^2 \theta \cdot 2 \cos \theta \cdot 2 \cos \theta \, d\theta = 16 \int_0^{\pi/6} \sin^2 \theta \cos^2 \theta \, d\theta \]

Now use the identities \( \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \) and \( \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \) to write this as

\[ 4 \int_0^{\pi/6} (1 - \cos 2\theta)(1 + \cos 2\theta) \, d\theta = 4 \int_0^{\pi/6} 1 - \cos^2(2\theta) \, d\theta \]

Now use the identity \( \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \) again to get

\[ 4 \int_0^{\pi/6} 1 - \frac{1}{2}(1 + \cos(4\theta)) \, d\theta = \int_0^{\pi/6} 2 - 2 \cos(4\theta) \, d\theta = 2 \theta - \frac{1}{2} \sin(4\theta) \bigg|_0^{\pi/6} = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \]

2. Complete the table below for the function \( f(x) = x^2 \sqrt{4 - x^2} \) over \([0, 1]\).

<table>
<thead>
<tr>
<th>( n )</th>
<th>LEFT(( n ))</th>
<th>RIGHT(( n ))</th>
<th>MID(( n ))</th>
<th>TRAP(( n ))</th>
<th>SIMP(( n ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2420614592</td>
<td>1.1080868630</td>
<td>0.5834612525</td>
<td>0.6750741612</td>
<td>0.6139988887</td>
</tr>
<tr>
<td>4</td>
<td>0.4127613558</td>
<td>0.8457740578</td>
<td>0.6066256665</td>
<td>0.6292677068</td>
<td>0.6141729467</td>
</tr>
<tr>
<td>8</td>
<td>0.5096934611</td>
<td>0.7261998120</td>
<td>0.6123028325</td>
<td>0.6179466366</td>
<td>0.6141841005</td>
</tr>
<tr>
<td>16</td>
<td>0.5609981468</td>
<td>0.6692513225</td>
<td>0.6137148364</td>
<td>0.6151247346</td>
<td>0.6141848024</td>
</tr>
</tbody>
</table>

\[ E(\( n \)) \approx \frac{\text{error}}{2^p} \]

3. For each method, when you double \( n \), the error should be (approximately) divided by \( 2^p \) for some power \( p \). Find that power for each of the methods. Put your results in the table below.

<table>
<thead>
<tr>
<th>Method</th>
<th>LEFT</th>
<th>RIGHT</th>
<th>MID</th>
<th>TRAP</th>
<th>SIMP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
For each method the basic error estimate is therefore

\[ E(n) \approx \frac{k}{n^p} \]

where \( p \) is the power in the table above.

4. In each method, determine the effect on the error of multiplying \( n \) by 10.

\[
\begin{array}{cccccc}
\text{Method} & \text{LEFT} & \text{RIGHT} & \text{MID} & \text{TRAP} & \text{SIMP} \\
\hline
\text{Error is divided by:} & 10 & 10 & 100 & 100 & 10000
\end{array}
\]

Next consider the definite integral

\[ \int_0^1 e^{x^2} \, dx, \]

which cannot be evaluated analytically.

5. Plot the graph of \( g(x) = e^{x^2} \) over the interval \([0, 1]\). For each of the methods LEFT, RIGHT, MID and TRAP, determine whether it will yield an overestimate or an underestimate. Use the graph of \( g \) to explain why.

The function is increasing so

\[ \text{LEFT}(n) \leq \int_0^1 e^{x^2} \, dx \leq \text{RIGHT}(n) \]

The function is concave down, so

\[ \text{MID}(n) \leq \int_0^1 e^{x^2} \, dx \leq \text{TRAP}(n) \]

6. Compute LEFT(10) and RIGHT(10). What is the most \( E(10) \) could be for each method? Why?

\[ \text{LEFT}(10) = 1.381260601 \text{ and } \text{RIGHT}(10) = 1.553088784. \] Since the actual value is between these, the error is at most \( \text{RIGHT}(10) - \text{LEFT}(10) = 0.171828183 \).

7. How large does \( n \) need to be in order for LEFT(\( n \)) and RIGHT(\( n \)) to approximate the integral to 9 decimal places? (For 9 decimal places accuracy, \( E(n) \) must be at most \( 0.5 \times 10^{-9} \).) By the result of Question 4, each time \( n \) is multiplied by 10, the error is divided by 10. Since we need to divide \( E(10) \approx 0.171828183 \) by \( 10^9 \) to obtain a number less than \( 0.5 \times 10^{-9} \), we need to multiply \( n = 10 \) by \( 10^9 \). So we need \( n \) to be around \( 10^{10} \).

8. Compute MID(10) and TRAP(10). What is the most \( E(10) \) could be for each method? Why?

\[ \text{MID}(10) = 1.460393091 \text{ and } \text{TRAP}(10) = 1.467174692. \] Since the actual value is between these, the error is at most \( \text{TRAP}(10) - \text{MID}(10) = 0.006781601 \).
9. How large does $n$ need to be in order for $\text{MID}(n)$ and $\text{TRAP}(n)$ to approximate the integral to 9 decimal places?

By the result of Question 4, each time $n$ is multiplied by 10, the error is divided by 100. Since we need to divide $E(10) \approx 0.006781601$ by $10^8$ to obtain a number less than $0.5 \times 10^{-9}$, we need to multiply $n = 10$ by $10^4$. So we need $n$ to be around 100000.

10. (Do before or after lab.) Explain why $\text{SIMP}(n)$ is always between $\text{MID}(n)$ and $\text{TRAP}(n)$.

Hint: Either $\text{MID}(n) \leq \text{TRAP}(n)$ or $\text{TRAP}(n) \leq \text{MID}(n)$. Consider each case separately.

If $\text{MID}(n) \leq \text{TRAP}(n)$, then
\[
\frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3} \leq \frac{2 \cdot \text{TRAP}(n) + \text{TRAP}(n)}{3} = \text{TRAP}(n)
\]
and
\[
\frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3} \geq \frac{2 \cdot \text{MID}(n) + \text{MID}(n)}{3} = \text{MID}(n)
\]
so
\[
\text{MID}(n) \leq \text{SIMP}(n) \leq \text{TRAP}(n)
\]
On the other hand, if $\text{TRAP}(n) \leq \text{MID}(n)$, then
\[
\frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3} \leq \frac{2 \cdot \text{MID}(n) + \text{MID}(n)}{3} = \text{MID}(n)
\]
and
\[
\frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3} \geq \frac{2 \cdot \text{TRAP}(n) + \text{TRAP}(n)}{3} = \text{TRAP}(n)
\]
so
\[
\text{TRAP}(n) \leq \text{SIMP}(n) \leq \text{MID}(n)
\]

11. Compute $\text{SIMP}(10)$. What is the most $E(10)$ could be for this method? Use the results of questions 8 and 10.

$\text{SIMP}(10) = 1.462653625$. Since $\text{SIMP}(10)$ and the exact value are both between $\text{MID}(10)$ and $\text{TRAP}(10)$, the error is at most $\text{TRAP}(10) - \text{MID}(10) = 0.006781601$.

12. How large does $n$ need to be in order for $\text{SIMP}(n)$ to approximate the integral to 9 decimal places? Write out the value of the integral to 9 decimal places.

By the result of Question 4, each time $n$ is multiplied by 10, the error is divided by 10000. Since we need to divide $E(10) \approx 0.006781601$ by $10^8$ to obtain a number less than $0.5 \times 10^{-9}$, we need to multiply $n = 10$ by $10^2$. So we need $n$ to be around 1000. Computing $\text{SIMP}(1000)$ gives the value $1.462651746$. 