Lab on Taylor Polynomials

This Lab is accompanied by an Answer Sheet that you are to complete and turn in to your instructor.

In this Lab we will approximate complicated functions by simple functions. The simplest functions are polynomials. We will need to use commands defined in the student package. Enter

\texttt{with(Student[Calculus1]):}

at the beginning of your worksheet. (Recall that using a colon : instead of a semi-colon ; suppresses the blue-colored output.)

Maple has a built-in function called \( D \) that can be used to compute derivatives. For a previously defined function \( f \), the command \( D(f) \); will give you the derivative of \( f \) with respect to its input variable, whatever that may be. You can also use the command \( \text{diff}(f(x),x) \); for the derivative of \( f \).

\textbf{Practice:} Try defining some easy functions, whose derivatives you already know, to make sure \( D(f) \); works the way you’d expect. Recall that a function is defined, for example, as follows:

\[ f := x \rightarrow x \cdot \exp(x); \]

In many cases, it will be useful to name the derivative function. For example, you can do this by typing \( w := D(f) \);. Then, if you want to plot this derivative of \( f \), you simply type \( \text{plot}(w, -5 \ldots 5); \).

To get a second derivative, you can use either \( D(w) \); or \( D(D(f)) \).

\textbf{Practice:} Try defining a function, compute the derivative, re-name the derivative function (as described above), and then plot both the function and the derivative on the same set of axes. You can keep track of what curve represents which function by assigning colors to the curves. To specify the order, use square brackets \( [ ] \) instead of the usual \{ \}. For example, \( \text{plot}([f,w], -3 \ldots 3, \text{color} = \{\text{red, blue}\}); \) will plot the two functions on the interval \(-3 \leq x \leq 3\) where the function \( f \) is in red and the function \( w \) is in blue. Try evaluating the function and the derivative at various values of \( x \). Recall that when Maple can, it will try to supply you with an exact form of the result of a calculation. For example, if \( f(x) = \sin(x) \), then

\[ f\left(\frac{\pi}{4}\right) \text{ will return } \frac{\sqrt{2}}{2}. \]

Try it. If you want to get a decimal approximation instead, you can use the command \( \text{evalf}(f(\pi/4)); \).

\textbf{Problem #1:} For the following two functions, first calculate the derivative by hand and then use Maple to check your answer. If they aren’t a direct match, determine if they are equivalent by using algebra and/or trigonometric identities. On the solution paper that you turn in, write down both answers (the one by hand and the one given by Maple). Recall that \( \pi = \pi \), \( \exp(x) = e^x \), \( \tan(x) = \tan x \), and \( \ln(x) = \ln x \).

(a) \( f(x) = xe^{\tan(\pi)} \)

(b) \( g(x) = \ln(\ln(2x^3)) \)

\textbf{Problem #2:} Continue to consider the function \( f(x) = xe^{\tan(\pi)} \). Using Maple to help you, find a first degree polynomial \( p_1(x) = Ax + B \) that has the same value and the same first derivative as \( f \) when \( x = 0 \). In other words, you’ll need to determine the correct values for \( A \) and \( B \) to make it so that \( p_1(0) = f(0) \) and \( p_1'(0) = f'(0) \).

If you want to find the value of \( f \) at \( x = 0 \), you can type \( \text{evalf}(f(0)); \) Next, find a second degree polynomial of the form \( p_2(x) = Cx^2 + Dx + E \) that has the same value, the same first derivative, and the same second derivative as \( f \) when \( x = 0 \). In other words, determine the correct values for \( C, D \) and \( E \) using Maple to assist you. On the solution paper that you turn in, write down both answers.
Problem #3 Now, plot the two polynomials you’ve found, along with the function \( f \) on the same set of axes. Using colors, as described above, will help you keep track of what curve represents which function.

Warning: Be careful in choosing your range of \( x \) values for plotting. You may need to adjust the \( x \)-interval to get a better view of the graph. If you’re seeing dramatic vertical lines on your Maple plot, you are choosing a bad interval. Recall that \( \tan(x) \) is not defined when ever \( \cos(x) = 0 \). This means, for example, that graphing \( f \left( \frac{1}{2} \right) \) will cause big problems (do you see it?).

On your solution, answer the following questions.

a) For what range of \( x \) values does \( p_1(x) \) seem to be a good approximation for \( f(x) \)?

b) For what range of \( x \) values does \( p_2(x) \) seem to be a good approximation for \( f(x) \)?

c) What problem occurs when you try to plot on the interval \([-1,1]\)?

d) What does \( p_1(x) \) represent geometrically?

e) Of the two polynomials \( p_1(x) \) and \( p_2(x) \), which graph appears to be closer to the graph of \( f \)?

f) Evaluate the functions at the points listed in the table below out to 4 decimal places. Complete this table.

| \( x \) | \(-0.4\) | \(-0.2\) | \(-0.1\) | \(-0.01\) | \(0\) | \(0.01\) | \(0.1\) | \(0.2\) | \(0.4\) |
|---|---|---|---|---|---|---|---|---|
| \( f(x) \) | | | | | | | | | |
| \( p_1(x) \) | | | | | | | | | |
| \( p_2(x) \) | | | | | | | | | |

g) How do the values of \( p_1(x) \) and \( p_2(x) \) compare with the values of \( f(x) \)? Would you refine your answers to part (a) and (b) after taking this closer look at the function values? Explain.

h) How would you go about finding a polynomial \( p_3(x) \) of degree 3 that would be a good approximation for \( f \) close to \( x = 0 \)? What would you require of such a polynomial?

i) Can you generalize your requirements for a polynomial \( p_n(x) \) of (arbitrary) degree \( n \)?

Problem #4 The polynomial \( p_1(x) \) above is called the Taylor polynomial of degree 1 for \( f \) near \( x = 0 \) and the polynomial \( p_2(x) \) is called the Taylor polynomial of degree 2 for \( f \) near \( x = 0 \). By repeating your work from Problem #1 write down the general formula for \( p_1(x) \) and \( p_2(x) \) for an arbitrary function \( f \). Using your conclusions from Problem #3, give the formula for the Taylor polynomial of degree \( n \) for \( f \) near \( x = 0 \). It might be easier to work this problem out by hand rather than with Maple.

Problem #5 Find the Taylor polynomials of degree 5, 6, and 7 for \( f(x) = \sin x \) near \( x = 0 \). Why are some Taylor polynomials for \( f(x) = \sin x \) the same?

Problem #6 Take the general formula for a Taylor polynomial of degree \( n \) for \( f \) near \( x = a \) found in your text book. Show that \( p_n^{(i)}(a) = f^{(i)}(a) \) for all \( i = 0,1,...,n \). Recall that in this notation, the superscript \( i \) refers to the order of the derivative. The Taylor polynomials near \( x = a \) are good approximations for \( f \) at values of \( x \) that are close to \( x = a \). The higher the degree of the polynomial, the better the approximation and the larger the interval around \( x = a \) on which the polynomial is a good approximation.
Problem #7 Find the Taylor polynomials of degree 5, 6, and 7 for \( f(x) = \sin x \) near \( x = \frac{\pi}{2} \). Do this by hand first and list on your answer sheet.

Maple can find the Taylor polynomials for you! To check your answer, type

\[
\text{TaylorApproximation}(\sin(x), x=\pi/2, \text{order}=5..7);
\]

You can also plot the function together with the Taylor polynomials using the command:

\[
\text{TaylorApproximation}(\sin(x), x=\pi/2, \text{order}=5..7, \text{output}=\text{plot}, \text{view}=[-10..10,-2..2]);
\]

(The view option command allows you to tell Maple what ranges of \( x \)'s and \( y \)'s to display.)

You can also use the animation feature of Maple to see the polynomials displayed one after another.

\[
\text{TaylorApproximation}(\sin(x), x=\pi/2, \text{order}=1..10, \text{view}=[-1..7,-2..2], \text{output}=\text{animation});
\]

To see the animation, after you hit enter you’ll see a set of axes appear. Click on it. Then click on the “animation” button on the top of the tool bar. Click play to see the polynomials displayed one after another. You an also click “next” several times to display one polynomial after another at your own speed.

Practice: Find, plot, and animate the Taylor polynomials of degrees 1 up to 15 for the function \( y = e^x \) near \( x = 0 \). You will need to find a viewing window that allows you to understand how the polynomials approximate the function. Conjecture the form of the Taylor polynomial of degree \( n \) for \( y = e^x \) near \( x = 0 \). Warning! Maple multiplies out the factorials!

To complete this lab, use Maple to find the Taylor polynomials of degree 5, 6, and 7 for \( y = f(x) \) near \( x = 0 \) for some complicated function that you choose. Do not choose one that was done in your text book, but invent your own function. Take the time to view the plots. Include the Taylor polynomials and cut and paste a picture of the plots onto your lab solution. Be careful to select the best viewing window that shows the function and the Taylor polynomials most completely.
Answer Sheet for Lab on Taylor Polynomials

Name: ___________________________________________________

Problem #1: For the following two functions, write the first derivative formulas, plus an explanation of why Maple’s answer for the first function looked different.

(a) \( f(x) = xe^{\tan(x)} \)

(b) \( g(x) = \ln(\ln(2y^3)) \)

Problem #2: For the function \( f(x) = xe^{\tan(x)} \), the polynomials \( p_1(x) = Ax + B \) and \( p_2(x) = Cx^2 + Dx + E \) have the forms:

Problem #3

a) For what range of \( x \) values does \( p_1(x) \) seem to be a good approximation for \( f(x) \)?

b) For what range of \( x \) values does \( p_2(x) \) seem to be a good approximation for \( f(x) \)?

c) What problem occurs when you try to plot on the interval \([-1,1]\)?

d) What does \( p_1(x) \) represent geometrically?

e) Of the two polynomials \( p_1(x) \) and \( p_2(x) \), which graph appears to be closer to the graph of \( f \)?
f) Evaluate the functions at the points listed in the table below out to 4 decimal places. Complete this table.

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</table>

g) How do the values of $p_1(x)$ and $p_2(x)$ compare with the values of $f$? Would you refine your answers to part (a) and (b) after taking this closer look at the function values? Explain.

h) How would you go about finding a polynomial $p_3(x)$ of degree 3 that would be a good approximation for $f$ close to $x = 0$? What would you require of such a polynomial?

i) Can you generalize your requirements for a polynomial $p_n(x)$ of (arbitrary) degree $n$?

Problem #4 The general formula for $p_1(x)$ and $p_2(x)$ for an arbitrary function $f$ are given as

Problem #5 The Taylor polynomials of degree 5, 6, and 7 for $f(x) = \sin x$ near $x = 0$ are listed here. Why are some Taylor polynomials for $f(x) = \sin x$ the same?
Problem #6  Take the general formula for a Taylor polynomial of degree \( n \) for \( f \) near \( x = a \) found in your text book. Show that \( p_n^{(i)}(a) = f^{(i)}(a) \) for all \( i = 0,1,...n \).

Problem #7  The Taylor polynomials of degree 5, 6, and 7 for \( f(x) = \sin x \) near \( x = \frac{\pi}{2} \) are given by:

The Taylor polynomials of degree 5, 6, and 7 for \( y = f(x) \) near \( x = 0 \) for a function that I chose are given here. List the function itself, plus the three Taylor polynomial formulas. Attach a print out of the plots that show the function and the polynomials on the same set of axes.