1. (3 points each) True or False. No explanation required.

(a) If \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges. \( T \quad F \)

(b) If \( \lim_{n \to \infty} a_n = 0.000001 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges. \( T \quad F \)

(c) If \( P \) is a cumulative distribution function then \( P' \) is a probability density function. \( T \quad F \)

(d) If a rod lying on the \( x \)-axis from \( x = -L \) to \( x = L \) has mass density \( \delta(x) \) and \( \delta \) is an even function (\( \delta(-x) = \delta(x) \)), then the center of mass is \( \bar{x} = 0 \). \( T \quad F \)

(e) If the Taylor Polynomial of degree 2 for \( f(x) \) at \( a = 0 \) is \( P_2(x) = 1 + x - x^2 \), then \( f \) is concave up near \( x = 0 \). \( T \quad F \)

(f) The integral \( \int_{-2}^{2} \pi(4-x^2) \, dx \) represents the volume of a sphere of radius 2. \( T \quad F \)

(g) Suppose \( p(x) \) is a probability density function, and \( p(10) = 1/2 \). Then the probability that \( 9.98 \leq x \leq 10.04 \) is about 0.03. There is a typo here. It should have read \( 9.98 \leq x \leq 10.04 \), in which case the correct answer is True. \( T \quad F \)

(h) Given any function \( f \), its Taylor series centered at \( a = 0 \) converges to \( f(x) \) for every \( x \). \( T \quad F \)

(i) If \( \sum_{n=0}^{\infty} a_n x^n \) is the Taylor series centered at \( a = 0 \) for \( f \), then \( \sum_{n=1}^{\infty} n a_n x^{n-1} \) is the Taylor series centered at \( a = 0 \) for \( f' \). \( T \quad F \)

2. The reflector behind a car headlight is made in the shape of the parabola \( y = \frac{3}{2} \sqrt{x} \), rotated around the \( x \)-axis, for \( 0 \leq x \leq 4 \).

(a) [10 points] Write down a Riemann sum that approximates the volume contained by this headlight.

\[ \text{Solution.} \quad \sum_{i=1}^{n} \pi \left( \frac{3}{2} \sqrt{x_i} \right)^2 \Delta x \]

(b) [10 points] Find the volume exactly.

\[ \text{Solution.} \quad \int_{0}^{4} \pi \frac{9}{4} x \, dx = \frac{9}{8} x^2 \bigg|_{0}^{4} = 18\pi \]

3. Two rods of length 3 are positioned along the \( x \)-axis from \( x = 0 \) to \( x = 3 \). The first rod has mass density \( \delta(x) \), and the second rod has mass density \( 2\delta(x) \).
(a) [10 points] What is the relationship, if any, between the masses of the two rods? Explain.

**Solution.** The masses of the rods are

\[ m_1 = \int_0^3 \delta(x) \, dx \]

and

\[ m_2 = \int_0^3 2\delta(x) \, dx = 2 \int_0^3 \delta(x) \, dx = 2m_1 \]

so the second rod has twice the mass of the first rod.

(b) [10 points] What is the relationship, if any, between the centers of mass of the rods? Explain.

**Solution.** The centers of mass of the rods are

\[ \bar{x}_1 = \frac{\int_0^3 x\delta(x) \, dx}{\int_0^3 \delta(x) \, dx} \]

and

\[ \bar{x}_2 = \frac{\int_0^3 2x\delta(x) \, dx}{\int_0^3 2\delta(x) \, dx} = \frac{2 \int_0^3 x\delta(x) \, dx}{2 \int_0^3 \delta(x) \, dx} = \frac{\int_0^3 x\delta(x) \, dx}{\int_0^3 \delta(x) \, dx} = \bar{x}_1 \]

so the rods have the same center of mass.

4. Let \( p(x) = \frac{2}{9}x \) for \( 0 \leq x \leq 3 \).

(a) [7 points] Verify that \( p \) is a probability density function.

**Solution.** The function \( p \) is nonnegative for \( 0 \leq x \leq 3 \), and

\[ \int_0^3 p(x) \, dx = \int_0^3 \frac{2}{9} x \, dx = \frac{1}{9} x^2 \bigg|_0^3 = 1 \]

so \( p \) is a probability density function.

(b) [7 points] Which of the following is the mean value of \( x \)? Show your work.

(i) \( \frac{2}{3} \)  
(ii) \( \frac{9}{4} \)  
(iii) 1.5  
(iv) \( \frac{3\sqrt{2}}{2} \)  
(v) 1

**Solution.**

\[ \mu = \int_0^3 xp(x) \, dx = \int_0^3 \frac{2}{9} x^2 \, dx = \frac{2}{27} x^3 \bigg|_0^3 = 2 \]

(c) [7 points] Which of the following is the cumulative distribution function? Show your work.
Solution.

\[ P(x) = \int_0^x p(t) \, dt = \int_0^x \frac{2}{9} t \, dt = \frac{1}{9} t^2 \bigg|_0^x = \frac{1}{9} x^2 \]

(d) [7 points] Which of the following is the median value of \( x \)? Show your work.

(i) \( 2 \)

(ii) \( \frac{9}{4} \)

(iii) \( 1.5 \)

(iv) \( \frac{3\sqrt{2}}{2} \)

(v) \( 1 \)

Solution. The median is where \( P(x) = 0.5 \). Solving \( \frac{1}{9} x^2 = 0.5 \) gives \( x = \frac{3\sqrt{2}}{2} \).

5. The distribution of IQ scores is often modelled by the normal distribution

\[ p(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{450}} \]

(a) [10 points] What are the mean and standard deviation of this distribution?

Solution. \( \mu = 100 \) and \( \sigma = 15 \).

(b) [10 points] Describe in words what the integral

\[ \int_{115}^{120} p(x) \, dx \]

represents.

Solution. The integral represents the proportion of people who scored between 115 and 120 on the IQ test.

6. (a) [10 points] Find the sum of the finite geometric series

\[ 2 + 2(1.1) + 2(1.1)^2 + 2(1.1)^3 + \cdots + 2(1.1)^{100} \]

Solution.

\[ \frac{2(-(1.1)^{101})}{1 - 1.1} = 303153.47 \]
(b) Consider the infinite geometric series

\[ 3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \cdots \]

(i) [8 points] Write the series in the form \( \sum_{n=0}^{\infty} ax^n \). That is, find \( a \) and \( x \).

**Solution.** \( a = 3 \) and \( x = -\frac{2}{3} \), so the series is \( \sum_{n=0}^{\infty} 3 \left( -\frac{2}{3} \right)^n \).

(ii) [8 points] Does the series converge? If so, what is the sum? If not, why?

**Solution.** Since \( \left| -\frac{2}{3} \right| < 1 \), the series converges. Its sum is

\[ \frac{3}{1 - \left( -\frac{2}{3} \right)} = \frac{9}{5} \]

7. (a) [10 points] Use the Integral Test to decide whether the series

\[ \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3} \]

converges or diverges. Show all your work.

**Solution.** The function \( f(x) = \frac{x}{(x^2+1)^3} \) is nonnegative and decreasing for \( x \geq 1 \) so the Integral Test applies. Letting \( w = x^2 + 1 \), \( dw = 2x \, dx \) gives

\[
\int_{1}^{\infty} \frac{x}{(x^2+1)^3} \, dx = \int_{2}^{\infty} \frac{1}{2} w^{-3} \, dw \\
= \lim_{b \to \infty} \int_{2}^{b} \frac{1}{2} w^{-3} \, dw \\
= \lim_{b \to \infty} -\frac{1}{4w^2} \bigg|_{2}^{b} \\
= \lim_{b \to \infty} -\frac{1}{4b^2} + \frac{1}{16} = \frac{1}{16}
\]

Since this improper integral converges, the series converges.

(b) [10 points] Decide whether the series

\[ \sum_{n=1}^{\infty} \cos \left( \frac{1}{n^2} \right) \]

converges or diverges. Show all your work.

**Solution.** Since

\[ \lim_{n \to \infty} \cos \left( \frac{1}{n^2} \right) = \cos(0) = 1 \neq 0, \]

the series diverges. (Theorem 9.2, part 3.)
8. (a) [10 points] Suppose \( f(2) = 5, \ f'(2) = -1, \ f''(2) = 6 \) and \( f'''(2) = 1 \). Write down the third degree Taylor polynomial of \( f \) centered at \( a = 2 \).

Solution.

\[
P_3(x) = 5 - (x - 2) + \frac{6}{2!} (x - 2)^2 + \frac{1}{3!} (x - 2)^3
\]

(b) [12 points] Use the third-degree Taylor polynomial for \( \sin x \), centered at \( a = 0 \), to approximate the integral

\[
\int_0^1 \frac{\sin x}{x} \, dx.
\]

Solution. \( P(x) = x - \frac{1}{6} x^3 \), so

\[
\int_0^1 \frac{\sin x}{x} \, dx \approx \int_0^1 1 - \frac{1}{6} x^2 \, dx = x - \frac{1}{18} x^3 \bigg|_0^1 = \frac{17}{18}
\]

9. (a) [10 points] The series

\[
1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots
\]

is obtained by plugging in \( x = \pi \) to the Taylor series centered at \( a = 0 \) for some function. Find the function, and use it to find the exact sum of this series.

Solution. Since

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots
\]

the series above is the Taylor series for \( \cos x \) evaluated at \( x = \pi \) and therefore converges to \( \cos \pi = -1 \).

(b) [12 points] Let \( f(x) = e^{x^2} \sin x \). Use the Taylor series for \( e^x \) and \( \sin x \), centered at \( a = 0 \), to find the fifth degree Taylor Polynomial of \( f \) centered at \( a = 0 \).

Solution.

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

and

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots
\]

so

\[
e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots
\]

and thus

\[
e^{x^2} \sin x = \left( 1 + x^2 + \frac{x^4}{2!} + \cdots \right) \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right)
\]

\[
= x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^3 - \frac{x^5}{3!} + x^5 + \text{terms of degree greater than 5}
\]
So, after combining like terms and simplifying,

\[ P_5(x) = x + \frac{5}{6}x^3 + \frac{41}{120}x^5 \]

10. (a) [10 points] Give an estimate for the error when \( \cos x \) is approximated by its \( n \)th degree Taylor polynomial \( P_n(x) \) centered at \( a = 0 \) on the interval \([-2, 2]\).

**Solution.** The basic error estimate is

\[ |E_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!} \]

Since \( f^{(n+1)}(x) \) is either \( \pm \sin x \) or \( \pm \cos x \), the largest its absolute value could be on the interval \([-2, 2]\) is 1, so \( M = 1 \). Also, \( |x| \leq 2 \) on \([-2, 2]\), so

\[ |E_n(x)| \leq \frac{2^{n+1}}{(n+1)!} \]

(b) [10 points] How large must \( n \) be in order for the error to be less than 0.001?

**Solution.** Since \( 2^7/7! = 0.0254 \) and \( 2^8/8! = 0.0063 < 0.01 \), the smallest \( n \) that will give the desired estimate is \( n + 1 = 8 \), i.e. \( n = 7 \).