College of the Holy Cross, Fall 2016  
Math 242, Midterm 1 Practice Questions  
Solutions

1. Use Axioms 1 though 9 to prove that if $x \cdot y = x \cdot z$ and $x \neq 0$ then $y = z$.

Solution. Suppose $x \cdot y = x \cdot z$. Since $x \neq 0$, Axiom 5(b) implies that there exists a real number $x^{-1}$ such that $x \cdot x^{-1} = 1$. By Axiom 1, this implies $x^{-1} \cdot x = 1$, so we have

\[
\begin{align*}
y &= y \cdot 1 & \text{by Axiom 4(b)} \\
  &= 1 \cdot y & \text{by Axiom 1} \\
  &= (x^{-1} \cdot x) \cdot y & \text{since } x^{-1} \cdot x = 1 \\
  &= x^{-1} \cdot (x \cdot y) & \text{by Axiom 2} \\
  &= x^{-1} \cdot (x \cdot z) & \text{by hypothesis} \\
  &= (x^{-1} \cdot x) \cdot z & \text{by Axiom 2} \\
  &= 1 \cdot z & \text{since } x^{-1} \cdot x = 1 \\
  &= z \cdot 1 & \text{by Axiom 1} \\
  &= z & \text{by Axiom 4(b)}.
\end{align*}
\]

2. (a) Show that $\sqrt{3}$ is irrational.

Solution. Suppose $\sqrt{3}$ is rational. Then we could write $\sqrt{3} = \frac{m}{n}$, where $m$ and $n$ are integers that have no common factors. This implies $m^2 = 3n^2$, so $m^2$ is a multiple of 3. Now there are 3 possibilities for $m$: $m = 3k$, $m = 3k + 1$ or $m = 3k + 2$ for some integer $k$. If $m = 3k + 1$, then $m^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. But this contradicts the fact that $m^2$ is a multiple of 3. Likewise, if $m = 3k + 2$, then $m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, again contradicting the fact that $m^2$ is a multiple of 3. The only possibility therefore is that $m = 3k$ for some integer $k$. This then implies $9k^2 = 3n^2$, so $n^2 = 3k^2$. Hence $n^2$ is a multiple of 3, and by reasoning as before, it follows that $n$ is a multiple of 3. But this means that both $m$ and $n$ share the common factor of 3, contrary to the assumption. This contradiction implies that $\sqrt{3}$ is irrational.

(b) Suppose $t > 0$ is irrational. Prove that $\sqrt{t}$ is irrational.

Solution. Suppose instead that $\sqrt{t}$ is rational. Then $\sqrt{t} = p/q$ where $p$ and $q$ are integers and $q \neq 0$. Then $t = p^2/q^2$ would also be rational, a contradiction.

3. Fix $r \neq 1$. Use the principle of induction to prove that the summation formula

$$
\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}
$$

holds for all $n \in \mathbb{N}$.
Solution. Denote this summation formula by \( S_n \). When \( n = 1 \), the statement reads \( 1 + r = \frac{1-r^2}{1-r} \), which is true since \( 1 - r^2 = (1 - r)(1 + r) \). Next suppose \( S_n \) is true for some \( n \geq 1 \). Then
\[
\sum_{k=0}^{n+1} r^k = \left( \sum_{k=0}^{n} r^k \right) + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1}
\]
since \( S_n \) is true
\[
= \frac{1 - r^{n+1} + r^{n+1}(1 - r)}{1 - r} = \frac{1 - r^{n+2}}{1 - r},
\]
so \( S_{n+1} \) is true. Thus \( S_n \) implies \( S_{n+1} \) and therefore by the principle of induction \( S_n \) is true for all \( n \geq 1 \).

4. Let \( A = \{3 - \frac{1}{n} : n \in \mathbb{N}\} \). Find lub \( A \), and prove your assertion.

Solution. lub \( A = 3 \). First, since \( 3 - \frac{1}{n} < 3 \) for any \( n \), 3 is an upper bound for \( A \). Now consider any number \( M < 3 \). Since \( 3 - M > 0 \), the Archimedian Property implies that there is some \( n \in \mathbb{N} \) such that \( 1/n < 3 - M \). This implies \( 3 - \frac{1}{n} > M \), so \( M \) is not an upper bound of \( A \). Therefore every upper bound \( M \) of \( A \) must satisfy \( M \geq 3 \). Hence 3 is the least upper bound of \( A \).

5. Let \( A = \{x \in \mathbb{R} \mid x^5 - 2x < 1000\} \).

(a) Prove that \( A \) is bounded above.

Solution. 10 is an upper bound for \( A \). To see this, suppose 10 is not an upper bound for \( A \). Then for some \( x \in A \) we have \( x > 10 \). But this would imply \( x^4 > 10^4 = 10000 \), so \( x^4 - 2 > 9998 \) and thus \( x^5 - 2x = x(x^4 - 2) > 10(9998) > 1000 \). This contradicts the fact that \( x^5 - 2x < 1000 \) for each \( x \in A \). Hence 10 is an upper bound for \( A \).

(b) Prove that \( A \) has a least upper bound.

Solution. By the previous problem \( A \) is bounded above by 10. Since \( 0^5 - 2(0) = 0 < 1000 \), 0 \( \in A \), and thus \( A \) is nonempty. The Least Upper Bound Axiom then implies that \( A \) has a least upper bound.

6. Suppose \( A \) and \( B \) are nonempty subsets of \( \mathbb{R} \) that are bounded above and satisfy lub \( A \) < lub \( B \). Prove that there exists some \( y \in B \) such that \( x < y \) for every \( x \in A \).

Solution. Since lub \( (A) < \) lub \( (B) \), lub \( (A) \) is not an upper bound for \( B \). Hence there exists some \( y \in B \) such that lub \( (A) < y \). Since lub \( (A) \) is an upper bound for \( A \), \( x \leq \) lub \( (A) \) for all \( x \in A \). By transitivity it follows that \( x < y \) for all \( x \in A \).

7. (a) Complete the following definition. A sequence \( x_n \) to converges to a real number \( a \) if

Solution. For any \( \epsilon > 0 \) there exists some \( n_0 \in \mathbb{N} \) such that \( |x_n - a| < \epsilon \) for all \( n \geq n_0 \).
(b) Use the definition of convergence to prove that \( \lim_{n \to \infty} \frac{3n}{2n-1} = \frac{3}{2} \).

**Solution.** Let \( \epsilon > 0 \). Choose \( n_0 \in \mathbb{N} \) such that \( n_0 > \frac{3}{2\epsilon} \) (this is possible by the Archimedian property since \( \frac{3}{2\epsilon} > 0 \)). Then for any \( n \geq n_0 \), we have \( n > \frac{3}{2\epsilon} \), so \( \frac{3}{2n} < \epsilon \) and thus

\[
\left| \frac{3n}{2n-1} - \frac{3}{2} \right| = \frac{3}{2(2n-1)} \leq \frac{3}{2n} < \epsilon
\]

since \( 2n-1 \geq n \) when \( n \geq 1 \)

for all \( n \geq n_0 \).

8. Suppose \( \lim a_n = 7 \). Show that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) we have \( a_n > 6.99 \).

**Solution.** Let \( \epsilon = 0.01 \). Then since \( \lim a_n = 7 \), there exists some \( n_0 \) such that \( |a_n - 7| < 0.01 \) for all \( n \geq n_0 \). This implies \(-0.01 < a_n - 7 < 0.01\), or equivalently \( 6.99 < a_n < 7.01 \). Hence \( a_n > 6.99 \) for all \( n \geq n_0 \).

9. True or False. If True, give a short proof. If False, give a counterexample.

(a) If the sequences \( \{x_n\} \) and \( \{y_n\} \) both diverge, then the sequence \( \{x_ny_n\} \) diverges.

**Solution.** False, \( x_n = y_n = (-1)^n \) both diverge, but \( x_ny_n = 1 \) converges.

(b) If \( r \neq 0 \) is rational, and \( t \) is irrational, then \( t/r \) is irrational.

**Solution.** True. Suppose \( t/r \) is rational. Then \( t/r = m/n \) for some integers \( m \) and \( n \) with \( n \neq 0 \). Solving for \( t \) then gives \( t = rm/n \). Since \( r \) is rational, \( r = p/q \) where \( p \) and \( q \) are integers and \( q \neq 0 \). Thus \( t = (mp)/(nq) \). Since \( mp \) and \( nq \) are integers and \( nq \neq 0 \) it follows that \( t \) is rational, a contradiction. Hence \( t/r \) is irrational.

10. Suppose \( x_n \) converges to 0. Prove that \( \sqrt[n]{x_n} \) converges to 0.

**Solution.** Let \( \epsilon > 0 \). Since \( x_n \to 0 \), there exists some \( n_0 \in \mathbb{N} \) such that \( |x_n| < \epsilon^3 \) for all \( n \geq n_0 \). Thus \( |\sqrt[n]{x_n}| < \epsilon \) for all \( n \geq n_0 \).

11. Suppose \( \lim x_n = 0 \) and \( y_n \) is bounded. Prove that \( \lim x_ny_n = 0 \).

**Solution.** Since \( y_n \) is bounded, there exists a real number \( M > 0 \) such that \( |y_n| \leq M \) for all \( n \). Let \( \epsilon > 0 \). Since \( x_n \to 0 \), there exists some \( n_0 \in \mathbb{N} \) such that \( |x_n - 0| < \frac{\epsilon}{M} \) for all \( n \geq n_0 \). Thus

\[
|x_ny_n - 0| = |x_n||y_n| \leq M|x_n| < M \cdot \epsilon M = \epsilon
\]

for all \( n \geq n_0 \). Hence \( \lim x_ny_n = 0 \).

12. Let \( x_n \) be a sequence with the property that \( x_n^2 - 5x_n \) converges to 14.
13. Consider the sequence defined recursively by \( x_1 = 1 \) and \( x_{n+1} = \frac{x_n^2 + 8}{6} \).

(a) Prove that the sequence \( x_n \) is increasing.

**Solution.** By induction. Let \( S_n \) denote the statement \( x_n \leq x_{n+1} \). Since \( x_1 = 1 < \frac{9}{6} = x_2 \), this proves \( S_1 \) is true. Now suppose \( x_k \leq x_{k+1} \) for some \( k \geq 1 \). Then since both \( x_k \) and \( x_{k+1} \) are positive, this implies \( x_k^2 \leq x_{k+1}^2 \). Adding 8 and dividing by 6 then gives \( \frac{x_k^2 + 8}{6} \leq \frac{x_{k+1}^2 + 8}{6} \), which implies \( x_{k+1} \leq x_{k+2} \). Thus \( S_k \implies S_{k+1} \) for all \( k \geq 1 \), so by the principle of induction, \( S_n \) is true for all \( n \in \mathbb{N} \). Hence \( x_n \) is increasing.

(b) Prove that \( x_n \leq 3 \) for all \( n \).

**Solution.** By induction again. Let \( S_n \) denote the statement \( x_n \leq 3 \). The base case \( S_1 \) is true since \( x_1 = 1 < 3 \). So suppose \( x_k \leq 3 \) for some \( k \geq 1 \). Then since \( x_k \) is positive, \( x_k^2 \leq 9 \). Adding 8 and dividing by 6 then gives \( \frac{x_k^2 + 8}{6} \leq \frac{17}{6} < 3 \). Thus \( x_{k+1} \leq 3 \). Hence \( S_k \implies S_{k+1} \) for all \( k \geq 1 \), so by the principle of induction \( S_n \) is true for all \( n \in \mathbb{N} \).

(c) Prove that the sequence \( x_n \) converges, and find its limit.

**Solution.** By parts (a) and (b), the sequence \( x_n \) is monotone and bounded (above by 3 and below by \( x_1 = 1 \) since \( x_n \) is increasing), so \( x_n \) converges by the Monotone Convergence Theorem. Let \( a = \lim x_n \). Then, using the algebraic limit theorem,

\[
a = \lim x_{n+1} = \lim \frac{x_n^2 + 8}{6} = \frac{a^2 + 8}{6},
\]

so \( a^2 + 8 = 6a \) which implies \( a = 2 \) or \( a = 4 \). But since \( x_n \leq 3 \) for all \( n \), we have \( a \leq 3 \). Thus \( a = 2 \).

14. Consider the sequence defined recursively by \( y_1 = 5 \) and \( y_{n+1} = \frac{y_n^2 + 8}{6} \).

(a) Use induction to prove that \( y_n \geq 5 \) for all \( n \).

**Solution.** When \( n = 1 \), we have \( y_1 = 5 \geq 5 \). Now suppose \( y_n \geq 5 \) for some \( n \geq 1 \). Then \( y_n^2 \geq 25 \), so \( y_n^2 + 8 \geq 33 \) and therefore \( y_{n+1} = \frac{y_n^2 + 8}{6} \geq \frac{33}{6} > 5 \), so \( y_{n+1} \geq 5 \).

(b) Prove by contradiction that \( y_n \) diverges.

**Solution.** Suppose \( y_n \) converges. Let \( a = \lim y_n \). Then by the Algebraic Limit Theorem, \( \lim y_{n+1} = \lim \frac{y_n^2 + 8}{6} = \frac{a^2 + 8}{6} \). But \( \lim y_{n+1} = \lim y_n = a \), so we have \( a = \frac{a^2 + 8}{6} \). The solutions of this equation are \( a = 2 \) and \( a = 4 \). But since \( y_n \geq 5 \) for all \( n \), \( a \) must be at least 5, a contradiction.
15. Determine whether or not each sequence converges, and find the limit of those that converge.

(a) $x_n = \frac{3}{5n^2 + 4}$ if $n$ is even and $x_n = \frac{n}{1 - 5n}$ if $n$ is odd.

**Solution.** Using the fact that $\lim \frac{1}{k} = \lim \frac{1}{k^2} = 0$ together with the algebraic limit theorem, we have

$$\lim x_{2k} = \lim \frac{3}{5(2k)^2 + 4} = \lim \frac{3}{20 + \frac{4}{k^2}} = \frac{0}{20} = 0$$

and

$$\lim x_{2k+1} = \lim \frac{2k + 1}{1 - 5(2k + 1)} = \lim \frac{2 + \frac{1}{k}}{-\frac{4}{k} - 10} = \frac{2}{-10} = -\frac{1}{5}.$$ 

Thus $x_n$ has subsequences that converge to different limits, so $x_n$ does not converge.

(b) $x_n = \frac{3n}{5n + 4}$ if $n$ is even and $x_n = \frac{1 - 3n}{1 - 5n}$ if $n$ is odd.

**Solution.** Using the fact that $\lim \frac{1}{k} = \lim \frac{1}{k^2} = 0$ together with the algebraic limit theorem, we have

$$\lim x_{2k} = \lim \frac{6k}{10k + 4} = \lim \frac{6}{10 + \frac{4}{k}} = \frac{6}{10} = \frac{3}{5}$$

and

$$\lim x_{2k+1} = \lim \frac{1 - 3(2k + 1)}{1 - 5(2k + 1)} = \lim \frac{-6 - \frac{2}{k}}{-\frac{5}{k} - 10} = \frac{-6}{-10} = \frac{3}{5}.$$ 

Therefore, by the theorem we proved in class, since both $x_{2k}$ and $x_{2k+1}$ converge to $\frac{3}{5}$, $x_n$ converges to $\frac{3}{5}$.