The following test for convergence is known as the **Cauchy Condensation Test**. It applies to series with nonnegative, decreasing terms.

**Theorem.** Suppose \( a_n \geq 0 \) for all \( n \) and \( a_n \) is a decreasing sequence. Then the series \( \sum_{n=1}^{\infty} a_n \) converges if and only if the series \( \sum_{n=0}^{\infty} 2^n a_{2^n} \) converges.

**Note.** This is an if an only if statement, so this means that either both series converge or both series diverge. Thus the convergence/divergence of the series \( \sum_{n=1}^{\infty} a_n \) is determined by what happens with the series \( \sum_{n=0}^{\infty} 2^n a_{2^n} \).

**Proof.** Let

\[
 s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=0}^{n} 2^k a_{2^k}
\]

be the partial sums of the two series. First suppose the series \( \sum_{n=0}^{\infty} 2^n a_{2^n} \) converges. Then the sequence \( t_n \) converges and is therefore bounded, so there is some \( M \) such that \( t_n \leq M \) for all \( n \). Now observe that, since the sequence \( a_n \) is decreasing,

\[
 s_1 = a_1 = t_0 \\
 s_3 = a_1 + (a_2 + a_3) \leq a_1 + 2a_2 = t_1 \\
 s_7 = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \leq a_1 + 2a_2 + 4a_4 = t_2 \\
 s_{15} \leq a_1 + 2a_2 + 4a_4 + 8a_8 = t_3
\]

and in general \( s_{2^n - 1} \leq t_{n-1} \leq M \). This proves that the subsequence \( s_{2^n - 1} \) is bounded. But since \( s_n \) is increasing, this implies that \( s_n \) is bounded. So by the Monotone Convergence Theorem, \( s_n \) converges.

Conversely, suppose that \( \sum_{n=1}^{\infty} a_n \) converges. Then \( s_n \) converges and is therefore bounded, so there exists \( M \) such that \( s_n \leq M \) for all \( n \). Now notice that

\[
 2s_2 = 2a_1 + 2a_2 = a_1 + (a_1 + 2a_2) = a_1 + t_1 \\
 2s_4 = a_1 + (a_1 + 2a_2 + 2a_3 + 2a_4) \geq a_1 + (a_1 + 2a_2 + 4a_4) = a_1 + t_2 \\
 2s_8 = a_1 + (a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8) \geq a_1 + (a_1 + 2a_2 + 4a_4 + 8a_8) = a_1 + t_3 \\
 2s_{16} \geq a_1 + (a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16}) = a_1 + t_4
\]

and in general \( 2s_{2^n} \geq a_1 + t_n \). Thus \( t_n \leq 2s_{2^n} \leq 2M \) for all \( n \), so the sequence \( t_n \) is bounded. Since \( t_n \) is increasing, the Monotone Convergence Theorem implies that \( t_n \) converges.
**Example.** Let’s apply the Cauchy Condensation Test to the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some $p > 0$.

Here $a_n = \frac{1}{n^p}$, so

$$2^n a_{2^n} = \frac{2^n}{(2^n)^p} = (2^n)^{1-p} = (2^{1-p})^n.$$ 

Thus

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} (2^{1-p})^n.$$ 

This is a geometric series with ratio $r = 2^{1-p}$. It therefore converges when $2^{1-p} < 1$ and diverges when $2^{1-p} \geq 1$. Thus the $p$-series converges when $p > 1$ and diverges when $p \leq 1$. 