Stability and Instability of Fourth Order Solitary Waves

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Abstract

We study ground state traveling wave solutions of a fourth order wave equation. We find conditions on the speed of the waves which imply stability and instability of the solitary waves. The analysis depends on the variational characterization of the ground states rather than information about the linearized operator.

1 Introduction

This paper is an analysis of the stability of traveling wave solutions of the equation

$$u_{tt} + \Delta^2 u + u = f(u) \tag{1.1}$$

where $f(u) = |u|^{p-1}u$ for some $p > 1$. If $n \geq 5$ then we also require $p < 2^* - 1$ where

$$2^* = \frac{2n}{n-4} \tag{1.2}$$

is the critical exponent in the Sobolev embedding $H^2(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. We show that there exist solitary wave solutions of (1.1) and prove criteria for their stability and instability. Our results parallel those for the analogous second order Klein-Gordon equations (see [8],[25],[27]). We show that the solitary wave of speed c is stable when the action function $d_1(c)$, defined by (3.14), is convex and is unstable when $d_1(c)$ is concave.

The interesting feature of the problem is that the solitary wave satisfies a fourth order elliptic equation. The stability of solitary waves of second order equations has been studied in many papers, including [25], [26],[27] for the Klein-Gordon equations and [4],[32],[33] for the Schrodinger equation. Also, higher order equations such as the KdV equation ([3], [28]) and generalized Boussinesq equations ([19],[23]) have been examined. In each case, however, the solitary wave satisfies a second order ODE. Using a nodal analysis of the ground state it is possible in some cases to obtain information about the spectrum of the linearized operator. However, the solitary waves of (1.1) satisfy a genuine fourth order PDE (2.1), for which there is no maximum principle available. Thus the ground states may not necessarily be positive, and in fact may be oscillatory. So we cannot easily obtain this spectral information, and therefore some of the standard techniques for analyzing stability are no longer applicable. Instead we rely entirely on the variational characterization of the solitary wave.

In one dimension an equation similar to (1.1), with a different nonlinear term, has been studied as a model for the suspension bridge [20]. Numerical evidence in the case of an exponential nonlinearity suggests that traveling waves are unstable for small c and exhibit soliton-like behavior for c near the critical value $\sqrt{2}$ [21].

In Section 2 we prove the existence of a solitary wave. Solutions are obtained by using the method of concentrated compactness developed by Lions [17] to solve a constrained minimization problem. We use the scaling property of the pure power nonlinearity to verify the subadditivity conditions (2.9) and to scale away the Lagrange multiplier. In second order and pure fourth order problems we can verify these conditions and eliminate any multipliers by dilating in the independent variable [1]. The presence of both fourth and second order terms in (2.1), however, prohibits such an approach, and therefore we restrict our attention to homogeneous nonlinearities. We also note that the scale invariance allows us to solve the minimization without any restrictions on the dimension n.

The results in this section apply to a more general class of homogeneous nonlinear terms of variational type. For instance, consider $F \in C^1(\mathbb{R}^{n+1})$ such that $\nabla F(\lambda y, \lambda z) = \lambda^{p+1} F(y, z)$ for all $(y, z) \in \mathbb{R}^{n+1}$, $\lambda > 0$. If $F(u, \nabla u) \in L^1(\mathbb{R}^n)$ for every $u \in H^2(\mathbb{R}^n)$ and there exists some $u \in H^2(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} F(u, \nabla u)dx > 0$ then nonlinearities of the form

$$f(u, \nabla u) = \nabla_y F(u, \nabla u) - div_x(\nabla_z F(u, \nabla u)). \tag{1.3}$$

may be treated as well. In particular, the nonlinearities $f(u) = \pm|u|^p$ and $-3u^2 + (u_x)^2 - 2(uu_x)_x$ are of this form. The latter arises in the study of fifth order KdV equations [5], and will be the subject of another paper [16].

In Section 3 we show that the evolution equation admits solutions in the space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ which exist locally in time for given initial data in X, provided $p < 2^*/2$.

We discuss in Section 4 the properties of $d_1(c)$ we shall need for the stability analysis. Once again we use the scaling property of our nonlinearity to write $d_1(c)$ in terms of the minimum values of the functionals used to obtain the solitary wave. We also establish bounds on d_1 which imply concavity for small c and convexity for c near $\sqrt{2}$.

In Section 5 we show (Theorem 5.4) that the set of ground states is stable whenever $d_1(c)$ is strictly convex. The proof consists of a compacness argument based on the ideas of Shatah [25] and Cazenave and Lions [4]. We use the variational properties of the ground states, along with a convexity lemma of Shatah (Lemma 5.1), to establish the key inequality (5.5).

In Section 6 we use a Lyapunov functional construction due to Grillakis, Shatah and Strauss [8] to show (Theorem 6.2) that a given ground state is orbitally unstable when $d_1''(c) < 0$. We need to make the additional assumption that the there is a C^1 map $c \mapsto \varphi_c$, where φ_c is a ground state with speed c, in order to apply the implicit function theorem.

Finally, in Section 7 we consider standing wave solutions of (1.1). The results of Sections 5 and 6 extend quite easily to this case, and the scaling properties of the solitary wave equation (7.2) make it possible to determine explicitly the intervals of stability and instability.

2 Existence of Minimizers

In this section we prove the existence of traveling wave solutions. In the process an essential result concerning the compactness of minimizing sequences is established. Let $u(x,t) = \varphi(x+ct)$ for $c \in \mathbb{R}^n$ be a solution of (1.1). Then φ must solve

$$\Delta^2\varphi + \sum_{i,j=1}^{n} c_i c_j \varphi_{x_i x_j} + \varphi = |\varphi|^{p-1}\varphi \tag{2.1}$$

For $|c|^2 < 2$ we can obtain solutions of (2.1) by considering the following constrained minimization problem. Let

$$\begin{aligned} I_c(\psi) &= \frac{1}{2}\int_{\mathbb{R}^n} |\Delta\psi|^2 - |c\cdot\nabla\psi|^2 + |\psi|^2 dx \\ K(\psi) &= \frac{1}{p+1}\int_{\mathbb{R}^n} |\psi|^{p+1} dx \end{aligned} \tag{2.2}$$

and define

$$m_\lambda(c) = \inf\{I_c(\psi) : \psi \in H^2(\mathbb{R}^n),\ K(\psi) = \lambda\} \tag{2.3}$$

for $0 < \lambda \le 1$. We say a sequence $\{\psi_k\}_{k=1}^\infty$ in $H^2(\mathbb{R}^n)$ is a *minimizing sequence* if

$$\lim_{k\to\infty} I_c(\psi_k) = m_1(c) \qquad \text{and} \qquad \lim_{k\to\infty} K(\psi_k) = 1$$

Our main result in this section is the following.

Theorem 2.1 *For any n suppose $1 < p < 2^* - 1$ and $|c|^2 < 2$. Let $\{\psi_k\}_{k=1}^\infty$ be a minimizing sequence. Then there exists a subsequence $\{\psi_{k_j}\}$, $y_j \in \mathbb{R}^n$ and $\psi \in H^2(\mathbb{R}^n)$ such that $\psi_{k_j}(\cdot - y_j) \to \psi$ in $H^2(\mathbb{R}^n)$. The function ψ is a minimizer of I_c subject to the constraint $K(\psi) = 1$ and is therefore a weak solution of the Euler-Lagrange equation*

$$\Delta^2\psi + \sum_{i,j=1}^{n} c_i c_j \psi_{x_i x_j} + \psi = \mu|\psi|^{p-1}\psi.$$

Hence $\varphi = \mu^{\frac{1}{p-1}}\psi$ is the desired solution of (2.1). Solutions obtained in this manner will be referred to as *ground states*. If we multiply (2.1) by φ and integrate we see that

$$2I_c(\varphi_c) = (p+1)K(\varphi_c) \tag{2.4}$$

for any ground state vp_c. Since φ_c is obtained by minimizing $I_c(\psi)$ subject to the constraint $K(\psi) = 1$ and rescaling, it follows from the relation

$$m_1(c) = \inf_{0\neq\psi\in H^2(\mathbb{R}^n)} \frac{I_c(\psi)}{K(\psi)^{\frac{2}{p+1}}} \tag{2.5}$$

that

$$I_c(\varphi_c) = c_p\big(m_1(c)\big)^{\frac{p+1}{p-1}} \qquad\qquad c_p = \left(\frac{2}{p+1}\right)^{\frac{2}{p-1}} \tag{2.6}$$

Thus we may define the set of all ground states with speed c by

$$G_c = \left\{\psi \in H^2(\mathbb{R}^n) : (p+1)K(\psi) = 2I_c(\psi) = 2c_p(m_1(c))^{\frac{p+1}{p-1}}\right\} \tag{2.7}$$

We establish Theorem 2.1 by applying the method of concentrated compactness. By scaling it is easily seen that

$$m_\lambda(c) = \lambda^{\frac{2}{p+1}} m_1(c) \tag{2.8}$$

and therefore the strict subadditivity condition

$$m_\lambda(c) + m_{1-\lambda}(c) > m_1(c) \qquad \lambda \in (0,1) \tag{2.9}$$

holds. Let $\{\psi_k\}_{k=1}^\infty$ be a minimizing sequence and define a sequence in $L^1(\mathbb{R}^n)$ by

$$\rho_k = |\Delta\psi_k|^2 + |\psi_k|^2.$$

Since

$$I_c(\psi) \geq (1 - |c|^2/2)\int_{\mathbb{R}^n} |\Delta\psi|^2 + |\psi|^2 dx = (1-|c|^2/2)\|\psi\|^2_{H^2(\mathbb{R}^n)}, \tag{2.10}$$

for any $\psi \in H^2(\mathbb{R}^n)$, I_c is coercive for $|c|^2 < 2$, and therefore $\{\psi_k\}_{k=1}^\infty$ is bounded in $H^2(\mathbb{R}^n)$. So, upon passing to a subsequence if necessary, we may assume that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} \rho_k(x)dx = L > m_1(c)$$

and after normalizing appropriately we may suppose

$$\int_{\mathbb{R}^n} \rho_k(x)dx = L$$

for all k. By the concentration-compactness lemma of Lions [17], there is a subsequence (renamed to ρ_k) satisfying one of the following three conditions:

1. Tightness. There exist $y_k \in \mathbb{R}^n$ such that for any $\epsilon > 0$ there exists $R(\epsilon)$ so that for all k

$$\int_{B(y_k,R(\epsilon))} \rho_k \, dx \geq \int_{\mathbb{R}^n} \rho_k \, dx - \epsilon \tag{2.11}$$

2. Vanishing. For every $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} \rho_k \, dx = 0 \tag{2.12}$$

3. Dichotomy. There exists $\alpha \in (0, L)$ such that for any $\epsilon > 0$ there exist $R, R_k \to \infty, y_k \in \mathbb{R}^n$ and k_0 so that

$$\left|\int_{B(y_k,R)} \rho_k dx - \alpha\right| < \epsilon \;\text{ and }\; \left|\int_{R<|x-y_k|<R_k} \rho_k dx\right| < \epsilon \tag{2.13}$$

for $k \geq k_0$.

Lemma 2.2 *The sequence $\{\rho_k\}_{k=1}^{\infty}$ is tight modulo the sequence of translations $\{y_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$.*

Proof. The proof follows from arguments given in [17],[18] which we present here.

First suppose vanishing occurs. By the Sobolev inequality we have

$$\int_{B(y,R_0)} |\psi_k|^{p+1} dx \le C(R_0)\left(\int_{B(y,R_0)} |\Delta\psi_k|^2 + |\psi_k|^2 dx\right)^{\frac{p+1}{2}} \tag{2.14}$$

for all k and any $y \in \mathbb{R}^n$, since $p+1 < 2^*$ and $\{\psi_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)$. By (2.12) we can choose $k(\epsilon)$ so that $k > k(\epsilon)$ implies

$$\sup_{y\in\mathbb{R}^n} \int_{B(y,R_0)} \rho_k dx < \epsilon$$

so that by (2.14) we then have

$$\int_{B(y,R_0)} |\psi_k|^{p+1} dx \le C\epsilon^{\frac{p-1}{2}}\left(\int_{B(y,R_0)} |\Delta\psi_k|^2 + |\psi_k|^2 dx\right) \tag{2.15}$$

for $k > k(\epsilon)$ and any $y \in \mathbb{R}^n$. Using an elementary construction we may cover $\mathbb{R}^n$ with balls of radius 1 in such a way that each point of $\mathbb{R}^n$ is contained in at most $2n+1$ balls. Then, we sum (2.15) over these balls to get

$$\|\psi_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \le (2n+1)C\epsilon^{\frac{p-1}{2}}\|\psi_k\|_{H^2(\mathbb{R}^n)}^2 \le C\epsilon^{\frac{p-1}{2}}$$

This implies that $\lim_{k\to\infty} K(\psi_k) = 0$ and therefore the constraint is lost. Hence vanishing cannot occur.

Next suppose dichotomy occurs and choose $\xi_1,\ \xi_2 \in C^\infty(\mathbb{R})$ so that $0 \le \xi_1,\ \xi_2 \le 1$ and

$$\begin{aligned} &\xi_1(x) = 1 \quad \text{for } |x| \le 1 \qquad && \xi_2(x) = 1 \quad \text{for } |x| \ge 1 \\ &\xi_1(x) = 0 \quad \text{for } |x| \ge 2 && \xi_2(x) = 0 \quad \text{for } |x| \le \frac{1}{2} \end{aligned}$$

Then define

$$\psi_{k,1}(x) = \xi_1\left(\frac{|x-y_k|}{R}\right)\psi_k(x)$$

and

$$\psi_{k,2}(x) = \xi_2\left(\frac{|x-y_k|}{R_k}\right)\psi_k(x). \tag{2.16}$$

It is then easy to verify that

$$\begin{aligned} I_c(\psi_k) &= I_c(\psi_{k,1}) + I_c(\psi_{k,2}) + O(\epsilon) \\ K(\psi_k) &= K(\psi_{k,1}) + K(\psi_{k,2}) + O(\epsilon) \end{aligned} \tag{2.17}$$

for $k \geq k_0$. By passing to a further subsequence we can define

$$\begin{aligned}\lambda_1(\epsilon) &= \lim_{k\to\infty} K(\psi_{k,1}) \\ \lambda_2(\epsilon) &= \lim_{k\to\infty} K(\psi_{k,2}).\end{aligned} \tag{2.18}$$

We clearly have $\lambda_1(\epsilon), \lambda_2(\epsilon) \in [0,1]$ and

$$|\lambda_1(\epsilon) + \lambda_2(\epsilon) - 1| = O(\epsilon)$$

and we can therefore choose a sequence $\epsilon_j \to 0$ such that

$$\lambda_1 = \lim_{j\to\infty} \lambda_1(\epsilon)$$

exists. Then

$$\lambda_2 = \lim_{j\to\infty} \lambda_2(\epsilon) = 1 - \lambda_1$$

and there are two possibilities.

If $\lambda_1 \in (0,1)$ it follows from (2.17) and (2.8) that

$$\begin{aligned}I_c(\psi_k) &\geq I_{K(\psi_{k,1})} + I_{K(\psi_{k,2})} + O(\epsilon_j) \\ &= [K(\psi_{k,1})^{\frac{2}{p+1}} + K(\psi_{k,2})^{\frac{2}{p+1}}]m_1(c) + O(\epsilon_j)\end{aligned}$$

Since $\{\psi_k\}_{k=1}^{\infty}$ is a minimizing sequence we may send k to infinity and use (2.18) to obtain

$$m_1(c) \geq [\lambda_1(\epsilon_j)^{\frac{2}{p+1}} + \lambda_2(\epsilon_j)^{\frac{2}{p+1}}]m_1(c) + O(\epsilon_j)$$

Letting $j \to \infty$ we arrive at the contradiction

$$m_1(c) \geq [(\lambda_1)^{\frac{2}{p+1}} + (\lambda_2)^{\frac{2}{p+1}}]m_1(c) > m_1(c).$$

If $\lambda_1 = 0$ (and similarly if $\lambda_1 = 1$) we use (2.13), the coercivity of I_c and the fact that $\psi_{k,1}$ is supported in $B(y_k, 2R)$ to conclude that

$$\begin{aligned}I(\psi_{k,1}) &\geq (1 - |c|^2/2)\|\psi_{k,1}\|^2_{H^2(\mathbb{R}^n)} = (1 - |c|^2/2)\|\psi_{k,1}\|_{H^2_0(B(y_k,2R))} && \text{by (2.10)} \\ &\geq (1 - |c|^2/2)\int_{B(y_k,2R)} |\Delta\psi_{k,1}|^2 + |\psi_{k,1}|^2 dx \\ &= (1 - |c|^2/2)(\alpha - +O(\epsilon_j)) && \text{by (2.13).}\end{aligned}$$

Thus, using (2.17) and (2.8) again,

$$\begin{aligned}I_c(\psi_k) &\geq (1 - |c|^2/2)(\alpha + O(\epsilon_j)) + I_{K(\psi_{k,2})} + O(\epsilon_j) \\ &= (1 - |c|^2/2)\alpha + K(\psi_{k,2})^{\frac{2}{p+1}} m_1(c) + O(\epsilon_j)\end{aligned}$$

and sending k to infinity gives

$$m_1(c) \geq (1 - |c|^2/2)\alpha + \lambda_2(\epsilon_j)^{\frac{2}{p+1}} m_1(c) + O(\epsilon_j).$$

We let $j \to \infty$ to get the contradiction

$$m_1(c) \geq \frac{2}{3}(1 - |c|^2/2)\alpha + m_1(c) > m_1(c).$$

Hence dichotomy does not occur and the lemma is proved. ■

Proof of Theorem 2.1. By Lemma 2.2 there exist $y_k \in \mathbb{R}^n$ such that $\rho_k(\cdot + y_k)$ is tight. Since $K(\psi_k) \to 1$ this implies that $|\psi_k(\cdot + y_k)|^{p+1}$ is also tight. Now, since $\{\psi_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)$, there is a subsequence $\{\psi_{k_j}\}_{j=1}^{\infty}$ and $\psi \in H^2(\mathbb{R}^n)$ such that

$$\begin{aligned} \psi_{k_j}(\cdot + y_{k_j}) &\rightharpoonup \psi \in H^2(\mathbb{R}^n) \\ \psi_{k_j}(\cdot + y_{k_j}) &\to \psi \in L^2_{loc}(\mathbb{R}^n). \end{aligned}$$

Since $\{\psi_k\}$ is bounded in $L^{2^*}(\mathbb{R}^n)$ and $p+1 < 2^*$ it follows by interpolation that $\psi_{k_j}(\cdot + y_j) \to \psi$ in $L^{p+1}_{loc}(\mathbb{R}^n)$. We now claim that $\psi_{k_j}(\cdot + y_j) \to \psi$ strongly in $L^{p+1}(\mathbb{R}^n)$. Indeed, let $\epsilon > 0$ and choose R_0 such that

$$\int_{|x| \geq R_0} |\psi(x)|^{p+1} dx < \epsilon.$$

By (2.11) there exists $R(\epsilon) > R_0$ and $j_1(\epsilon)$ so that $j \geq j(\epsilon)$ implies

$$\int_{|x| \geq R(\epsilon)} |\psi_{k_j}(x + y_j)|^{p+1} dx < \epsilon.$$

By the convergence in $L^{p+1}_{loc}(\mathbb{R}^n)$ we can find $j_2(\epsilon) > j_1(\epsilon)$ so that for $j > j_2(\epsilon)$ we have

$$\|\psi_{k_j}(\cdot + y_j) - \psi\|^{p+1}_{L^{p+1}(B(0,R(\epsilon)))} < \epsilon$$

Thus

$$\int_{\mathbb{R}^n} |\psi_{k_j}(x + y_j) - \psi(x)|^{p+1} dx \leq \epsilon + 2^{p+1}\epsilon$$

and the claim is proved. Hence $K(\psi) = 1$. Since the weak convergence in $H^2(\mathbb{R}^n)$ implies $I(\psi) \leq m_1(c)$ we therefore have $I(\psi) = m_1(c)$. The lemma then follows since $\|\Delta\psi_{k_j}\|_{L^2(\mathbb{R}^n)} \to \|\Delta\psi\|_{L^2(\mathbb{R}^n)}$ and $\Delta\psi_{k_j} \rightharpoonup \Delta\psi$ in $L^2(\mathbb{R}^n)$. ■

Lemma 2.3 *For any n suppose $1 < p < 2^* - 1$ and let $\varphi \in H^2(\mathbb{R}^n)$ be a weak solution of (5.1). Then $\varphi \in H^5(\mathbb{R}^n)$.*

Proof. For $1 \leq n \leq 4$ the Sobolev inequality implies that $\varphi \in L^\infty(\mathbb{R}^n)$. Hence $|\varphi|^{p-1}\varphi \in H^1(\mathbb{R}^n)$. Since

$$(\Delta^2 + (c \cdot \nabla)^2 + Id)\varphi = |\varphi|^{p-1}\varphi. \tag{2.19}$$

it follows that $\varphi \in H^5(\mathbb{R}^n)$.

For $n = 5$ fix $1 < p < 2^* - 1 = 9$ and proceed as follows. Suppose $\varphi \in H^{s_k}(\mathbb{R}^5)$. If $s_k \geq 5/2$ then $\varphi \in L^\infty(\mathbb{R}^5)$ and therefore $\varphi \in H^5(\mathbb{R}^5)$ as above. Otherwise the Sobolev inequality implies that

$$|\varphi|^{p-1}\varphi \in L^{q_k}(\mathbb{R}^5) \qquad q_k = \frac{10}{p(5 - 2s_k)}.$$

If $s_k \geq 5/2 - 5/(2p)$ then $q_k \geq 2$ and $|\varphi|^{p-1}\varphi \in L^2(\mathbb{R}^5)$. Thus (2.19) shows that $\varphi \in H^4(\mathbb{R}^5)$. Otherwise if $s_k > 5/2 - 5/p$ then $q_k > 1$ and

$$|\varphi|^{p-1}\varphi \in H^{r_k}(\mathbb{R}^5) \qquad r_k = ps_k + \frac{5-5p}{2}$$

and therefore (2.19) implies

$$\varphi \in H^{s_{k+1}}(\mathbb{R}^5) \qquad s_{k+1} = g(s_k) = ps_k + \frac{13-5p}{2}.$$

Since $1 < p < 9$ we have $s_{k+1} > s_k > 5/2 - 5/p$. Thus the iteration proceeds until $s_k \geq 5/2 - 5/(2p)$. The fixed point of $g(s)$ is

$$\tilde{s} = \frac{5p-13}{2(p-1)} < 2 \qquad \text{since} \qquad 1 < p < 9$$

Thus if we take $s_0 = 2$ (i.e. $\varphi \in H^2(\mathbb{R}^5)$ as assumed) then $s_k \to +\infty$ as $k \to \infty$. So there is some $k = k(p)$ so that

$$4 > s_k \geq 5/2 - 5/(2p) > s_{k-1}$$

Hence $|\varphi|^{p-1}\varphi \in L^2(\mathbb{R}^5)$ and (2.19) implies that $\varphi \in H^4(\mathbb{R}^5)$. This shows that $\varphi \in L^\infty(\mathbb{R}^5)$, so that $|\varphi|^{p-1}\varphi \in H^1(\mathbb{R}^5)$ and (2.19) implies $\varphi \in H^5(\mathbb{R}^5)$.

For $n \geq 6$ we use the following bootstrap procedure. Suppose that $\varphi \in H^{s_k}(\mathbb{R}^n)$. If $s_k \geq n/2$ then $\varphi \in L^\infty(\mathbb{R}^n)$ and we conclude as above that $\varphi \in H^5(\mathbb{R}^n)$. Using the Sobolev inequality again shows that

$$\varphi \in L^{p_k}(\mathbb{R}^n) \qquad\qquad p_k = \frac{2n}{n-2s_k}$$

$$\varphi_{x_i} \in L^{q_k}(\mathbb{R}^n) \qquad\qquad q_k = \frac{2n}{n-2(s_k-1)}.$$

It then follows that

$$|\varphi|^{p-1}\varphi_{x_i} \in L^r(\mathbb{R}^n) \qquad \frac{2}{p} \leq r \leq r_k = \frac{2n}{p(n-2s_k)+2},$$

where $r_k = +\infty$ if the denominator is negative. If $r_k \geq 2$ then $|\varphi|^{p-1}\varphi_{x_i} \in L^2(\mathbb{R}^n)$ since $2/p < 2$. We then conclude using (2.19) that $\varphi \in H^5(\mathbb{R}^n)$. If $1 < r_k < 2$ then

$$|\varphi|^{p-1}\varphi_{x_i} \in H^{t_k}(\mathbb{R}^n) \qquad t_k = \frac{n}{2} - \frac{n}{r_k} = ps_k - 1 - \frac{n}{2}(p-1)$$

and therefore

$$(\Delta^2 + (c\cdot\nabla)^2 + Id)\varphi_{x_i} = p|\varphi|^{p-1}\varphi_{x_i}$$

implies

$$\varphi \in H^{s_{k+1}}(\mathbb{R}^n), \qquad s_{k+1} = g(s_k) = ps_k + 4 - \frac{n}{2}(p-1). \tag{2.20}$$

Notice that if $s_k \geq 2$ then $s_{k+1} > s_k$ since $p < 2^* - 1$. This implies that $r_{k+1} > r_k > 0$ and therefore the process may be iterated. The process stops only when $r_k \geq 2$ or $s_k \geq 5$, either of which results in $\varphi \in H^5(\mathbb{R}^n)$. The function $g(s)$ in (2.20) has only one fixed point,

$$\tilde{s} = \frac{n}{2} - \frac{4}{p-1}$$

and $\tilde{s} < 2$ since $p < 2^* - 1$. Also $g'(\tilde{s}) = p > 1$. Thus if we choose $s_0 = 2$ (i.e. $\varphi \in H^2(\mathbb{R}^n)$) then the sequence s_k approaches infinity as $k \to \infty$. Hence the process stops after finitely many iterations, yielding $\varphi \in H^5(\mathbb{R}^n)$, as long as $r_0 > 1$. To verify this hypothesis we note that for $n \geq 6$

$$r_0 = \frac{2n}{p(n-4)+2} > 1 \qquad \text{for} \qquad p < 2^* - 1$$

This proves the lemma. ■

The restriction on p for $n \geq 5$ permits the variational characterization of solutions of (2.1). It also allows us to solve the solitary wave equation for small c. We next show that the restriction $p < 2^* - 1$ is a necessary condition, in the sense that for $p > 2^* - 1$ there is some interval of speeds near zero for which the solitary wave equation does not have solutions in $H^2(\mathbb{R}^n)$. This is a consequence of the following Pohozaev type identity.

Lemma 2.4 *Let φ be a solution of (5.1) lying in $H^2(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Then*

$$\alpha_n \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx - \beta_n \int_{\mathbb{R}^n} |c \cdot \nabla\varphi|^2 dx + \gamma_n \int_{\mathbb{R}^n} |\varphi|^2 dx = 0$$

where

$$\begin{aligned} \alpha_n &= n(p-1) - 4(p+1) \\ \beta_n &= n(p-1) - 2(p+1) \\ \gamma_n &= n(p-1) \end{aligned} \tag{2.21}$$

Proof. Since φ solves (5.1) it satisfies

$$I_c'(\varphi) = K'(\varphi).$$

So if we define

$$\psi_a(x) = \varphi(ax)$$

we have

$$\begin{aligned} \frac{d}{da}\Big(I_c(\psi_a) - K(\psi_a)\Big)\Big|_{a=1} &= \left\langle I_c'(\varphi) - K'(\varphi), a\sum_{i-1}^{n} \varphi_{x_i} \right\rangle \\ &= 0 \end{aligned} \tag{2.22}$$

So since

$$I_c(\psi_a) - K(\psi_a) = \frac{1}{2}a^{4-n}A - \frac{1}{2}a^{2-n}B + \frac{1}{2}a^{-n}C - \frac{1}{p+1}a^{-n}D$$

where

$$A = \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx \qquad\qquad B = \int_{\mathbb{R}^n} |c\cdot\nabla\varphi|^2 dx \tag{2.23}$$

$$C = \int_{\mathbb{R}^n} |\varphi|^2 dx \qquad\qquad D = \int_{\mathbb{R}^n} |\varphi|^{p+1} dx \tag{2.24}$$

it follows that

$$\frac{4-n}{2}A - \frac{2-n}{2}B - \frac{n}{2}C + \frac{n}{p+1}D = 0.$$

Since $2I_c(\varphi) = (p+1)K(\varphi)$ we also have

$$A - B + C - D = 0$$

and the lemma follows by eliminating D.

Lemma 2.5 *Let $n \geq 5$ and suppose $p > 2^* - 1$. Then there is some $c_n(p) > 0$ so that for $c \in [0, c_n(p))$ there is no solution of (5.1) in $H^2(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Furthermore $c_n(p) \to \sqrt{2}$ as $p \to \infty$.*

Proof. For $p > 2^* - 1$ the numbers α_n, β_n and γ_n defined in Lemma 2.4 are all positive. Using the notation in the proof of Lemma 2.4 we have

$$B \leq \frac{|c|^2}{2}\left(\frac{1}{\delta_n}A + \delta_n C\right)$$

where we choose δ_n to satisfy

$$\delta_n^2 = \frac{n(p-1)}{n(p-1) - 4(p+1)}.$$

If we also choose ϵ_n such that

$$\epsilon_n^2 = n(p-1)[n(p-1) - 4(p+1)]$$

then

$$\frac{1}{\delta_n} = \frac{\alpha_n}{\epsilon_n} \qquad \delta_n = \frac{\gamma_n}{\epsilon_n}$$

and therefore

$$\beta_n B \leq \frac{|c|^2\beta_n}{2\epsilon_n}(\alpha_n A + \gamma_n C)$$

This contradicts Lemma 2.4 if

$$|c|^2 < \frac{2\epsilon_n}{\beta_n} = \frac{2\sqrt{n(p-1)[n(p-1) - 4(p+1)]}}{n(p-1) - 2(p+1)}$$

The lemma then follows by defining $c_n^2 = 2\epsilon_n/\beta_n$. ■

3 Local Existence

We can write the evolution equation (1.1) as a system of two equations

$$\begin{aligned} u_t &= v \\ v_t &= -\Delta^2 u - u + |u|^{p-1}u \end{aligned} \tag{3.1}$$

If we denote $w = (u, v)$ then the functionals

$$\begin{aligned} E(w) &= \int_{\mathbb{R}^n} \frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|u|^2 - \frac{1}{p+1}|u|^{p+1} dx \\ Q(w) &= \int_{\mathbb{R}^n} v \nabla u dx \end{aligned} \tag{3.2}$$

are formally invariants of (3.1). The evolution equation may be written in terms of E as

$$\frac{dw}{dt} = JE'(w) \tag{3.3}$$

where $J : X^* \to X$ has domain $D(J) = L^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ and is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

By a *solution* of (1.1) on the interval $[0, t_0)$ we mean a function $w \in C([0, t_0); X)$ such that

$$\frac{d}{dt}\big\langle \mathbf{v}, w(t) \big\rangle = \big\langle E'(w(t)), -J\mathbf{v} \big\rangle \tag{3.4}$$

holds in the sense of distributions on $[0, t_0)$ for all $\mathbf{v} \in D(J)$, where $\big\langle \cdot, \cdot \big\rangle$ denotes the pairing of X^* with X. We shall assume the following throughout.

Assumption 3.1 *Given initial data $\mathbf{v} \in X$, there exists $t_0 > 0$ which depends only on $\|v\|_X$ and a unique solution w of* (1.1) *such that $w(0) = \mathbf{v}$, $E(w(t)) = E(\mathbf{v})$ and $Q(w(t)) = Q(\mathbf{v})$ for all $t \in [0, t_0)$.*

The following result shows that the assumption holds in dimension $n < 5$ with no restrictions on p and in dimension $n \geq 5$ if $p < 2^*/2$.

Theorem 3.2 *For any n suppose $1 < p < 2^*/2$. Then for every $w_0 \in X$, there exists $t_0 > 0$ such that the equation* (3.5) *has a unique integral solution $w(t) \in C([0, t_0); X)$ with $w(0) = w_0$ and if $t_0 < \infty$,*

$$\lim_{t \to t_0^-} \|w(t)\|_X = \infty.$$

Proof. The system (3.1) may be written in the form

$$w_t = Bw + P(w) \tag{3.5}$$

where

$$B = \begin{pmatrix} 0 & I \\ -\Delta^2 - I & 0 \end{pmatrix} \qquad P(w) = (0, |u|^{p-1}u)$$

The theorem follows from a theorem of Segal [24] once we show that B is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X and that f is locally Lipschitz on X [22]. This is the content of the following lemmas. ∎

Lemma 3.3 *The operator B is the infinitesimal generator of a C_0-semigroup of unitary operators on X.*

Proof. Define an inner product on X by

$$\big((u_1, v_1), (u_2, v_2)\big) = \int_{\mathbb{R}^n} (\Delta u_1 \Delta u_2 + u_1 u_2 + v_1 v_2) dx.$$

Then for $w \in D(B) = H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$,

$$\begin{aligned} \big(Bw, w\big) &= \big((v, -\Delta^2 u - u), (u, v)\big) \\ &= \int_{\mathbb{R}^n} (\Delta v \Delta u + vu - v\Delta^2 u - vu) dx = 0 \end{aligned}$$

and therefore B is skew adjoint. The lemma now follows from Stone's theorem. ∎

Lemma 3.4 *For any n let $1 < p < 2^*/2$. Then the map $P : X \to X$ given by $P(w) = (0, |u|^{p-1}u)$ is locally Lipschitz.*

Proof. Let $w_1, w_2 \in X$ and compute

$$\begin{aligned} \|f(w_1) - f(w_2)\|_X^2 &= \int_{\mathbb{R}^n} \big||u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\big|^2 dx \\ &= \int_{\mathbb{R}^n} \big|p(\lambda(x)u_1 + (1 - \lambda(x))u_2)^{p-1}(u_1 - u_2)\big|^2 dx \\ &\le p^2 \int_{\mathbb{R}^n} \big||u_1| + |u_2|\big|^{2(p-1)} |u_1 - u_2|^2 dx \\ &\le p^2 \||u_1| + |u_2|\|_{L^{2p}(\mathbb{R}^n)}^{2(p-1)} \|u_1 - u_2\|_{L^{2p}(\mathbb{R}^n)}^2 \\ &\le C(\|w_1\|_X + \|w_2\|_X)^{2(p-1)} \|w_1 - w_2\|_X^2 \end{aligned}$$

and therefore f is locally Lipschitz. ∎

Next let φ_c be a ground state with velocity c and denote by Φ_c the pair $(\varphi_c, c \cdot \nabla\varphi_c)$. Then Φ_c satisfies

$$E'(\Phi_c) - c \cdot Q'(\Phi_c) = 0. \tag{3.6}$$

We define, for $|c|^2 < 2$,

$$d(c) = E(\Phi_c) - c \cdot Q(\Phi_c). \tag{3.7}$$

The stability or instability of the ground state Φ_c will be determined by whether or not d is convex in c. In the next section we examine in detail the properties of $d(c)$. We conclude this section with the following definition.

Definition 3.5 *A set $S \subset X$ is stable with respect to* (1.1) *if, given $\epsilon > 0$ there exists $\delta > 0$ so that for any $g \in X$ with*

$$\inf_{v \in S} \|v - g\|_X < \delta,$$

the solution $w(t)$ of (1.1) *with initial data $w(0) = g$ can be extended to a solution in $C([0,\infty); X)$ and*

$$\sup_{0 \le t < \infty} \inf_{v \in S} \|w(t) - v\|_X < \epsilon.$$

Otherwise we say S is unstable.

4 Properties of $d(c)$

We first show that $d(c)$ is well defined. The relationship between the invariants E and Q of (1.1) and the functionals I and K used to find traveling waves is given by

$$E(w) - c \cdot Q(w) = I_c(u) - K(u) + \frac{1}{2} \int_{\mathbb{R}^n} |v - c \cdot \nabla u|^2 dx \tag{4.1}$$

It is this identity, along with the variational characterization of the ground states in terms of I_c and K, which will allow us to obtain stability (and instability) of the ground states without any detailed knowledge of the spectrum of the linearized operator.

Together with (2.4) and (3.7), (4.1) implies that

$$d(c) = I_c(\varphi_c) - K(\varphi_c) = \frac{p-1}{2} K(\varphi_c) = \frac{p-1}{p+1} I_c(\varphi_c) \tag{4.2}$$

for any ground state φ_c. Therefore $d(c)$ is well defined and by (2.6) we have

$$d(c) = \frac{c_p(p-1)}{p+1} \big(m_1(c)\big)^{\frac{p+1}{p-1}}. \tag{4.3}$$

By the invariance of the Laplacian under orthogonal transformations, it is easily shown that if $|c_1| = |c_2|$ then

$$m_1(c_1) = m_1(c_2)$$

and therefore $d(c)$ is radial. So

$$m_1(c) = m_*(|c|) \qquad d(c) = d_*(|c|) \tag{4.4}$$

where $m_*(s)$ and $d_*(s)$ are defined for $s \in [0, \sqrt{2})$. By (4.3) we also have

$$d_*(s) = \frac{c_p(p-1)}{p+1}\Big(m_*(s)\Big)^{\frac{p+1}{p-1}}. \tag{4.5}$$

We next investigate the smoothness of $d_*(c)$. In what follows $\nu = c/|c|$ will be a unit vector in $\mathbb{R}^n$ which represents the direction of the solitary wave and $s = |c|$ the speed of propagation. We will denote

$$Q_\nu(w) = \nu \cdot Q(w)$$

Lemma 4.1 *The function $d_*(s)$ is continuous, non-negative and strictly decreasing on the interval $[0, \sqrt{2})$.*

Proof. Suppose $0 \le s_0 < s < \sqrt{2}$. It is clear that $m_*(s_0) > m_*(s) > 0$. Let $c_0 = s_0\nu$, $c = s\nu$ and let u be a minimizer with velocity c. Then

$$\begin{aligned} m_*(s_0) \le \frac{I_{c_0}(u)}{K(u)^{\frac{2}{p+1}}} &= \frac{I_c(u)}{K(u)^{\frac{2}{p+1}}} + \frac{s^2 - s_0^2}{K(u)^{\frac{2}{p+1}}} \int_{\mathbb{R}^n} |\nu \cdot \nabla u|^2 dx \\ &= m_*(s) + \frac{s^2 - s_0^2}{K(u)^{\frac{2}{p+1}}} \int_{\mathbb{R}^n} |\nu \cdot \nabla u|^2 dx. \end{aligned} \tag{4.6}$$

So

$$\begin{aligned} |m_*(s) - m_*(s_0)| &\le \frac{3(s-s_0)}{K(u)^{\frac{2}{p+1}}} \int_{\mathbb{R}^n} |\nu \cdot \nabla u|^2 dx \le \frac{3(s-s_0) I_c(u)}{(2-s^2) K(u)^{\frac{2}{p+1}}} \\ &= 3(s-s_0)\frac{m_*(s)}{(2-s^2)} \le 3(s-s_0)\frac{m_*(0)}{(2-s^2)} \end{aligned} \tag{4.7}$$

and thus $m_*(s)$ is locally Lipschitz on $[0, \sqrt{2})$. The Lemma thus follows from (4.5). ∎

Motivated by the bounds obtained in Lemma 4.1 we define for $c = s\nu$,

$$\begin{aligned} \alpha(s) &= \inf\left\{ \int_{\mathbb{R}^n} |\nu \cdot \nabla \psi|^2 dx : \psi \in \tilde{G}_c \right\} \\ \beta(s) &= \sup\left\{ \int_{\mathbb{R}^n} |\nu \cdot \nabla \psi|^2 dx : \psi \in \tilde{G}_c \right\}. \end{aligned} \tag{4.8}$$

Lemma 4.2 *The left and right hand derivatives of d_* exist everywhere on $[0, \sqrt{2})$ and*

$$\begin{aligned} d_*'(s^-) &= -s\alpha(s) \\ d_*'(s^+) &= -s\beta(s). \end{aligned}$$

In particular, d_* is differentiable at $|c|$ with derivative

$$d_*'(|c|) = -Q_\nu(\Phi_c)$$

if and only if $\alpha(|c|) = \beta(|c|)$.

Proof. From (4.6) we see that, for $s_0 < s$,

$$\frac{-(s_0+s)\beta(s_0)}{(m_*(s_0))^{\frac{2}{p-1}}} \geq \frac{m_*(s)-m_*(s_0)}{s-s_0} \geq \frac{-(s_0+s)\alpha(s)}{(m_*(s))^{\frac{2}{p-1}}} . \tag{4.9}$$

We now claim that

$$\limsup_{s\to s_0} \alpha(s) \leq \beta(s_0).$$

Let $\{s_k\}$ be any sequence so that $s_k \to s_0$ and let $\psi_{c_k} \in \tilde{S}_{c_k}$, where $c_k = s_k\nu$. Then if we set $c_0 = s_0\nu$, the continuity of m_* implies

$$I_{c_0}(\psi_k) \to (m_*(s_0))^{\frac{p+1}{p-1}}$$

and

$$K(\psi_{c_k}) \to (m_*(s_0))^{\frac{p+1}{p-1}}.$$

Thus $\{\psi_{c_k}\}_{k=1}^{\infty}$ is a minimizing sequence for I_{c_0} and, by Theorem 2.1, has a strongly convergent subsequence ψ_{c_j} (modulo translations) to some $\psi_{c_0} \in \tilde{S}_{c_0}$. Hence

$$\int_{\mathbb{R}^n} |\nu\cdot\nabla\psi_{c_j}|^2 dx \to \int_{\mathbb{R}^n} |\nu\cdot\nabla\psi_{c_0}|^2 dx$$

and thus

$$\limsup_{j\to\infty} \alpha(s_j) \leq \beta(s_0)$$

which proves the claim. Applying the claim to (4.8) shows that

$$\lim_{s\to s_0^+} \frac{m_*(s)-m_*(s_0)}{s-s_0} = \frac{-2s_0\beta(s_0)}{(m_*(s_0))^{\frac{2}{p-1}}}.$$

Similary we see that

$$\lim_{s\to s_0^-} \frac{m_*(s)-m_*(s_0)}{s-s_0} = \frac{-2s_0\alpha(s_0)}{(m_*(s_0))^{\frac{2}{p-1}}}$$

Thus the left and right derivatives of $m(s)$ exist everywhere and are equal whenever $\alpha(s) = \beta(s)$. The lemma therefore follows from (4.3). ■

Lemma 4.3 $\alpha(s_1) = \beta(s_1) \iff$ *α is right continuous at s_1* $\iff$ *β is left continuous at s_1.*

Proof. By the inequalities in (4.9),

$$\frac{\beta(s_2)}{(m_*(s_2))^{\frac{2}{p-1}}} \geq \frac{\alpha(s_2)}{(m_*(s_2))^{\frac{2}{p-1}}} \geq \frac{\beta(s_1)}{(m_*(s_1))^{\frac{2}{p-1}}} \geq \frac{\alpha(s_1)}{(m_*(s_1))^{\frac{2}{p-1}}} \tag{4.10}$$

for $s_1 < s_2$. Thus $\alpha(s)/(m_*(s))^{\frac{2}{p-1}}$ and $\beta(s)/(m_*(s))^{\frac{2}{p-1}}$ are increasing functions of s. Applying Theorem 2.1 once shows that $\alpha(s)$ is lower semicontinuous and $\beta(s)$ is upper semicontinuous. Hence $\alpha(s)$ is left continuous and $\beta(s)$ is right continuous. By (4.10) it follows that

$$\lim_{s\to s_1^+} \alpha(s) = \beta(s_1)$$

and

$$\lim_{s\to s_1^-}\beta(s)=\alpha(s_1)$$

and the lemma is proved. ■

Corollary 4.4 *The function $d_*(s)$ is differentiable at all but countably many points of the interval $[0,\sqrt{2})$.*

Proof. The monotonicity of $\alpha/m_*^{2/(p-1)}$ and $\beta/m_*^{2/(p-1)}$, together with the continuity of m_* show that $\alpha(s)$ and $\beta(s)$ are continuous at all but countably many points of $[0,\sqrt{2})$. Hence by Lemma 4.2 and Lemma 4.3, it follows that $d_*(s)$ is differentiable at all but countably many points of $[0,\sqrt{2})$. ■

We now wish to obtain bounds on $d_*(s)$ in order to determine regions of convexity and concavity. We first find an upper bound on $d_*(s)$ for s near $\sqrt{2}$.

Lemma 4.5 *Suppose $1<p<2^*-1$. Then*

$$d_*(s)\le C(2-s^2)^\gamma$$

where $\gamma=\frac{2n-(n-2)(p+1)}{2(p-1)}$.

Proof. We consider the case $n=1$ only, as the result for $n>1$ follows similarly. First let $\zeta_1=\frac{\sqrt{2-s^2}}{2}$, $\zeta_2=\frac{\sqrt{2+s^2}}{2}$. Then $\zeta=\zeta_1+\zeta_2 i$ solves $\zeta^4+s^2\zeta^2+1=0$ and therefore $e^{\pm\zeta x}$, $e^{\pm\bar{\zeta}x}$ are solutions of the linear equation $\varphi_{xxxx}+s^2\varphi_{xx}+\varphi=0$. Define $g_s\in H^2(\mathbb{R})$ by

$$g_s(x)=e^{-\zeta_1|x|}\left(\cos\zeta_2|x|+\frac{\zeta_1}{\zeta_2}\sin\zeta_2|x|\right)$$

Integrating by parts we see that

$$I_c(g_s)=2\sqrt{2-s^2}$$

Also, for s near $\sqrt{2}$ we have for some constant C that

$$K(g_s)\ge\frac{C}{\sqrt{2-s^2}}$$

and therefore

$$m_*(s)\le\frac{I_c(g_s)}{K(g_s)^{\frac{2}{p+1}}}\le C(2-s^2)^{\frac{p+3}{2(p+1)}}$$

Finally (4.3) implies

$$d_*(s)\le C(2-s^2)^{\frac{p+3}{2(p-1)}}$$

which proves the lemma. ■

Remark 4.6 *If $p<1+\frac{4}{n}$ then $\gamma>1$ and hence $(2-c^2)^\gamma$ vanishes to first order at $c=\sqrt{2}$. The positivity and monotonicity of d_* then imply the existence of intervals of convexity arbitrarily close to $\sqrt{2}$.*

We next establish a lower bound on d_* under the assumption that d_* is differentiable.

Lemma 4.7 *Suppose that $d_*(s)$ is differentiable on $[0, \sqrt{2})$. Then*

$$d_*(c) \geq d_*(0)\left(1 - \frac{s^2}{2}\right)^{\frac{p+1}{p-1}}$$

Furthermore, suppose that Assumption 6.1 holds (see Section 6). Then there is an interval $\mathcal{I}$ containing zero such that $d_*''(s) < 0$ for all s in $\mathcal{I}$.

Proof. Set $s = |c|$. Then, if φ_c is any ground state, we have by Lemma 4.2

$$d_*'(s) = -s \int_{\mathbb{R}^n} |\nu \cdot \nabla \varphi_c|^2 dx$$

By (4.2) and (2.10),

$$\begin{aligned} d_*(s) = \frac{p-1}{p+1} I_c(\varphi_c) &\geq \frac{p-1}{p+1}\left(1 - \frac{s^2}{2}\right)\|\varphi(c)\|^2_{H^2(\mathbb{R}^n)} \\ &= \frac{p-1}{p+1}\left(1 - \frac{s^2}{2}\right)\left(I_c(\varphi_c) + s^2 \int_{\mathbb{R}^n} |\nu \cdot \nabla \varphi_c|^2 dx\right) \\ &= \left(1 - \frac{s^2}{2}\right)\left(d_*(s) - \frac{p-1}{2(p+1)} s d_*'(s)\right) \end{aligned}$$

Thus

$$\frac{d_*'(s)}{d_*(s)} \geq \left(\frac{p+1}{p-1}\right)\frac{-2s}{2-s^2}$$

and the first statement follows. To prove the second statement we differentiate (6.3) to obtain

$$\begin{aligned} d_*''(s) &= -\int_{\mathbb{R}^n} |\nu \cdot \nabla \varphi(c)|^2 dx - 2s \int_{\mathbb{R}^n} (\nu \cdot \nabla \varphi(c))(\nu \cdot \nabla \varphi'(c)) dx \\ &\leq -(1-s) \int_{\mathbb{R}^n} |\nu \cdot \nabla \varphi(c)|^2 dx + s \int_{\mathbb{R}^n} |\nu \cdot \nabla \varphi'(c)|^2 dx \end{aligned}$$

Since $\varphi(c)$ is C^2 from $[0, \sqrt{2})$ to X, it follows that $d_*''(s) < 0$ if s is chosen small enough. ■

Remark 4.8 *Lemma 4.7 shows that there are intervals of concavity of $d_*(s)$ arbitrarily close to $c = 0$.*

We shall need the following lemmas in order to relate the properties of $d(c)$ and $d_*(s)$.

Lemma 4.9 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2 and radial, with $f(x) = g(|x|)$. Then, for $x \neq 0$, $D^2 f(x)$ is singular if and only if $g''(|x|) = 0$ or $g'(|x|) = 0$. ($D^2 f(0) = g''(0) I$)*

Proof. Since

$$f_{x_i x_j} = \left(\frac{g''(|x|)}{|x|^2} - \frac{g'(|x|)}{|x|^3}\right) x_i x_j + \frac{g'(|x|)}{|x|}\delta_{ij},$$

we have

$$D^2 f(x) = M(|x|)(x \otimes x) + N(|x|)I$$

where

$$M(r) = \left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3}\right) \qquad N(r) = \frac{g'(r)}{r}.$$

Now $D^2 f(x)$ is singular if and only if $-N(|x|)/M(|x|)$ is an eigenvalue of $x \otimes x$. Since the eigenvalues of $x \otimes x$ are zero and $|x|^2$, we have either $N(|x|) = 0$, which implies $g'(|x|) = 0$, or $-N(|x|) = |x|^2 M(|x|)$, which implies $g''(|x|) = 0$. ■

Lemma 4.10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2 with $f(x) = g(|x|)$ and suppose $g''(|x|) \leq 0$ and $g'(|x|) \leq 0$. Then, for $x \neq 0$, $D^2 f(x)$ is negative-semidefinite. Further, for $x \neq 0$, $D^2 f(x)$ is negative-definite if and only if $g''(|x|) < 0$ and $g'(|x|) < 0$.*

Proof. Since $D^2 f(x)$ is symmetric it suffices to show that it has only nonpositive(negative) eigenvalues. If λ is an eigenvalue of $D^2 f$, then

$$\lambda = N(|x|) \qquad \text{or} \qquad \lambda = N(|x|) + |x|^2 M(|x|)$$

in which case $\lambda = g'(|x|)/|x|$ or $\lambda = g''(|x|)$. ■

So if $d_*(s)$ is differentiable and $d_*''(s) < 0$ we have by (4.4), Lemma 4.2 and Lemma 4.10 that $D^2 d(c)$ is negative definite for every c with $|c| = s$.

5 Stability

Recall that the set of ground states was defined by

$$G_c = \left\{\psi \in H^2(\mathbb{R}^n) : (p+1)K(\psi) = 2I_c(\psi) = 2c_p(m_1(c))^{\frac{p+1}{p-1}}\right\} \tag{5.1}$$

We extend this to a subset of X in the natural way and write

$$\mathcal{G}_c = \left\{\Psi = (\psi, c \cdot \nabla\psi) : \psi \in G_c\right\}$$

The tubular neighborhood of radius ϵ about $\mathcal{G}_c$ is defined by

$$U_{c,\epsilon} = \{w \in X : \inf\{\|w - \Psi\|_X : \Psi \in \mathcal{G}_c\} < \epsilon\}$$

We show that the set of ground states $\mathcal{G}_c$ is stable whenever $d_*(s)$ is strictly convex in a neighborhood of $|c|$. The variational nature of the ground states is used to show that sequences of later time data are minimizing sequences, provided the initial data is chosen close enough to $\mathcal{G}_c$. First we state without proof a lemma due to Shatah [25] concerning strictly convex functions.

Lemma 5.1 *Let h be any function which is strictly convex in an interval $\mathcal{I}$ about s. Then given $\epsilon > 0$ there exists $N(\epsilon) > 0$ so that for $s_1 \in \mathcal{I}$, $|s_1 - s| \geq \epsilon$ we have*

1. $s_1 < s < s_0$, $|s_0 - s| < \epsilon/2, s_0 \in I \Rightarrow$
$$\frac{h(s_1) - h(s_0)}{s_1 - s_0} \leq \frac{h(s) - h(s_0)}{s - s_0} - \frac{1}{N(\epsilon)} \tag{5.2}$$

2. $s_0 < s < s_1$, $|s_0 - s| < \epsilon/2, s_0 \in \mathcal{I} \Rightarrow$
$$\frac{h(s_1) - h(s_0)}{s_1 - s_0} \geq \frac{h(s) - h(s_0)}{s - s_0} + \frac{1}{N(\epsilon)} \tag{5.3}$$

Lemma 5.2 *Suppose that $d_*(s)$ is strictly convex in an interval $\mathcal{I}$ around s. Then for every $\epsilon > 0$, there exists $N(\epsilon) > 0$ so that for $s_1 \in \mathcal{I}$ with $|s_1 - s| \geq \epsilon$ we have*

$$d_*(s_1) \geq d_*(s) - s\beta(s)(s_1 - s) + \frac{1}{N(\epsilon)}(s - s_1)$$

for $s_1 < s$ and

$$d_*(s_1) \geq d_*(s) - s\alpha(s)(s_1 - s) + \frac{1}{N(\epsilon)}(s_1 - s)$$

for $s_1 > s$.

Proof. This follows by taking limits in (1) and (2) of Lemma 4.1 as $s_0 \to s$ and using the inequalities in (4.9). ■

The next lemma uses the variational characterization of ground states to establish the key inequality in the proof of stability. First we use the fact that $d_*(s)$ is continuous and strictly monotone on $[0, \sqrt{2})$ to define, for $w = (u, v)$ near Φ_c

$$\begin{aligned} s(w) &= d_*^{-1}\left(\frac{p-1}{2}K(u)\right) \\ c(w) &= s(w)\nu \end{aligned} \tag{5.4}$$

Lemma 5.3 *Suppose that $d_*(s)$ is strictly convex in an interval $\mathcal{I}$ containing $|c|$. Then there exists $\epsilon > 0$ so that for all $w \in U_{c,\epsilon}$ and any $\Psi_c = (\psi_c, c \cdot \psi_c) \in \mathcal{G}_c$,*

$$E(w) - E(\Psi_c) - s(w)(Q_\nu(w) - Q_\nu(\Psi_c)) \geq \frac{1}{N(\epsilon)}\left|s(w) - |c|\right| \tag{5.5}$$

Proof. Since $s(w)$ is a continuous function of w, we may choose ϵ small enough that $s(U_{c,\epsilon})$ is a subset of the interval $\mathcal{I}$. Then equations (4.1) and (4.2) imply

$$\begin{aligned} E(w) - s(w)Q_\nu(w) &= I_{c(w)}(u) - K(u) + \frac{1}{2}\int_{\mathbb{R}^n} |v - c(w)\cdot\nabla u|^2 dx \\ &\geq I_{c(w)}(u) - K(u) \\ &\geq I_{c(w)}(\psi_{c(w)}) - K(\psi_{c(w)}) = d_*(s(w)) \end{aligned}$$

since $\dfrac{2}{p-1}d_*(s(w)) = K(u) = K(\psi_{c(w)})$ and $\psi_{c(w)}$ minimizes $I_{c(w)}$ subject to this constraint. On the other hand, by Lemma 5.2 and (4.1) we have

$$\begin{aligned} d_*(s(w)) &\geq d_*(|c|) - Q_\nu(\Psi_c)(s(w) - |c|) + \frac{1}{N(\epsilon)}\,|s(w) - |c|| \\ &= E(\Psi_c) - s(w)Q_\nu(\Psi_c) + \frac{1}{N(\epsilon)}\,|s(w) - |c||\,. \end{aligned}$$

This proves the lemma. ■

Theorem 5.4 *Suppose that* Assumption 3.1 *holds and that* $1 < p < 2^* - 1$*. If* $d_*(s)$ *is strictly convex in an interval around* $|c|$*, then the set of ground states* $\mathcal{G}_c$ *is stable.*

Proof. Assume that $\mathcal{G}_c$ is unstable and choose initial data $w_k(0) \in U_{c,1/k}$. The by Assumption 3.1 the solution $w_k(t) = (u_k(t), v_k(t))$ with this initial data is continuous in t we can find t_k such that

$$\inf_{\Psi\in\mathcal{G}_c} \|w_k(t_k) - \Psi\|_X = \delta. \tag{5.6}$$

Since $U_{c,1/k}$ is bounded for each k and since E and Q_1 are invariants of the equation, we can find $\Psi_k \in \mathcal{G}_c$ and a constant C such that

$$|E(w_k(t_k)) - E(\Psi_k)| < \frac{C}{k}$$

$$|Q(w_k(t_k)) - Q(\Psi_k)| < \frac{C}{k}.$$

Now choose δ small enough so that Lemma 5.3 applies. Then

$$E(w_k(t_k)) - E(\Psi_k) - s(w_k(t_k))(Q_\nu(w_k(t_k)) - Q_\nu(\Psi_k)) \geq \frac{1}{N(\epsilon)}|s(w_k(t_k)) - |c||.$$

So, letting $k \to \infty$, it follows that $s(w_k(t_k)) \to |c|$. By (5.4) and the continuty of d_* we then have

$$\lim_{k\to\infty} K(u_k(t_k)) = \frac{2}{p-1}d_*(s). \tag{5.7}$$

Since (4.1) implies

$$\begin{aligned} I_c(u_k(t_k)) &= E(w_k(t_k)) - c\cdot Q(w_k(t_k)) + K(u_k(t_k)) \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^n} |v_k(t_k) - c\cdot\nabla u_k(t_k)|^2 dx \end{aligned} \tag{5.8}$$

we also have

$$\begin{aligned}\limsup_{k\to\infty} I_c(u_k(t_k)) &\le d_*(s) + \frac{2}{p-1}d_*(s)\\ &= \frac{p+1}{p-1}d_*(s).\end{aligned} \tag{5.9}$$

By (5.7) we therefore have

$$\lim_{k\to\infty} I_c(u_k(t_k)) = \frac{p+1}{p-1}d_*(s).$$

So $(2d_*(s)/(p-1))^{-\frac{1}{p+1}}u_k(t_k)$ is a minimizing sequence, and by Theorem 2.1 there exist $\psi_k \in G_c$ such that

$$\lim_{k\to\infty} \|u_k(t_k) - \psi_k\|_{H^2(\mathbb{R}^n)} = 0 \tag{5.10}$$

Finally, by (5.8)

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |v_k(t_k) - c\cdot\nabla u_k(t_k)|^2 dx = 0$$

and by (5.10)

$$\lim_{k\to\infty} \|c\cdot\nabla u_k(t_k) - c\cdot\nabla\psi_k\|_{L^2(\mathbb{R}^n)} = 0.$$

Thus $\|v_k(t_k) - c\cdot\nabla\psi_k\|_{L^2(\mathbb{R}^n)} \to 0$ and, together with (5.10), this implies

$$\lim_{k\to\infty} \|w_k(t_k) - \Psi_k\|_X = 0,$$

which contradicts (5.6). ■

Remark 5.5 *Together with the bound in Lemma 4.5, Theorem 5.4 implies the existence of stable traveling waves for some speeds near $\sqrt{2}$ when $p < 1 + \frac{4}{n}$. This differs from the second order wave equation (in one dimension) for which all traveling waves are unstable [8].*

6 Instability

We once again consider ground state traveling wave solutions of (1.1) and obtain conditions on the wave speed which implies their instability. As in Section 5 we write the velocity as $c = s\nu$, where it is assumed that the direction ν is fixed. Our main assumption in this section is the following.

Assumption 6.1 *For fixed ν there exists a C^2 map*

$$\phi : [0, \sqrt{2}) \to H^2(\mathbb{R}^n) \tag{6.1}$$

such that $\phi(s)$ is a ground state with velocity $s\nu$.

For fixed velocity $c_0 = s_0\nu$, the map in (6.1) can be extended to a map from a neighborhood U of c_0 in $\mathbb{R}^n$ to $H^2(\mathbb{R}^n)$ in the following manner. Let $c \in \mathbb{R}^n$ and choose an orthogonal transformation A_c such that $A_c\nu = c/|c|$. Then define $\varphi(c)(x) = \phi(|c|)(A^t x)$. By the invariance of the Laplacian under orthogonal transformations it follows that $\varphi(c)$ is a ground state with velocity c. For c near c_0 it is possible to choose A_c smoothly in c. We may then define the C^2 map from U to X by

$$\Phi(c) = (\varphi(c), c \cdot \nabla\varphi(c)) \tag{6.2}$$

and denote by $\varphi'(c)$ and $\Phi'(c)$ their gradients with respect to c. Together with (4.2) this implies that $d(c)$ is differentiable and we have by Lemma 4.2,

$$\begin{aligned} \nabla_c d(c) &= -Q(\Phi(c)) = -c \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi(c)|^2 dx \\ d_*'(|c|) &= -\nu \cdot Q(\Phi(|c|)) = -|c| \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi(c)|^2 dx \le 0. \end{aligned} \tag{6.3}$$

Solutions of (1.1) are invariant under the group action $T : \mathbb{R}^n \times X \to X$ defined by

$$T(r)w(x) = w(x + r) \tag{6.4}$$

The set of translates of $\Phi(c)$ is given by

$$\mathcal{T}_c = \{T(r)\Phi(c) : r \in \mathbb{R}^n\} \tag{6.5}$$

The main result of this section is the following.

Theorem 6.2 *For any n suppose that $1 < p < 2^* - 1$ and that Assumptions 3.1 and 6.1 hold. If $d_*''(|c|) < 0$, then $\mathcal{T}_c$ is unstable.*

Remark 6.3 *It follows from Theorem 6.2 and Lemma 4.7 that there is an interval $\mathcal{I}_0 = [0, s_0)$ of speeds such that $\mathcal{T}_c$ is unstable for $|c| \in \mathcal{I}_0$.*

In the remainder of this section we shall use $\langle \cdot, \cdot \rangle$ to denote the pairing of X^* with X and $\langle \cdot, \cdot \rangle_X$ to denote the inner product in X. We will denoted by D and D^2 the gradient and Hessian, respectively and, for $w = (u, v) \in X$ we will denote $Dw = (Du, Dv)$. When necessary, subscripts will be used to distinguish derivatives with respect to different variables (e.g. $D_c\Phi(c)$ denotes the derivative of the map Φ with respect to c, while $D_x\Phi(c)$ denotes the spatial gradient of $\Phi(c)$). Derivatives of functions of a single real variable will be denoted by $'$ and $''$. We first reformulate the local existence assumption of Chapter 3 as follows.

Lemma 6.4 *Suppose that Assumption 3.1 holds. For any n suppose $1 < p < 2^* - 1$ and define $W = L^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. Given initial data $g \in X$ there is some $T > 0$ depending only on $\|g\|_X$, and a unique solution $w = (u, v)$ of (3.1) in $C([0, T), X) \cap C^1([0, T), W^*)$ such that $w(0) = g$ and $E(w(t)) = E(g)$ for $t \in [0, T)$.*

Proof. It suffices to show that $w_t = (u_t, v_t) \in C([0,T), W^*)$, where $W^* = L^2(\mathbb{R}^n) \oplus H^{-2}(\mathbb{R}^n)$. This follows immediately from (3.5) since both B and P map X continuously into W^*. ■

It follows from Lemma 6.4 that the solution w satisfies

$$\frac{d}{dt}\langle v, w(t)\rangle = \langle E'(w(t)), -Jv\rangle \tag{6.6}$$

for all $v \in W$, where $J : X^* \to X$ has domain $D(J) = W$ and is defined by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.7}$$

The proof of the next lemma is trivial.

Lemma 6.5 *For any non-zero $w \in X$, if $T(r_n)w \to w$ in X then $r_n \to 0$.*

We next define the ϵ-neighborhood of $\mathcal{T}_c$ by

$$V_{c,\epsilon} = \{w \in X : \inf_{\Phi \in \mathcal{T}_c} \|w - \Phi\|_X < \epsilon\}$$

Lemma 6.6 *There exists $\epsilon > 0$ and a C^1 map $\rho : V_{c,\epsilon} \to \mathbb{R}^n$ such that for any $w \in V_{c,\epsilon}$ and $r \in \mathbb{R}^n$ we have*

$$\begin{aligned} &(1) \quad \|T(\rho(w))w - \Phi(c)\|_X \le \|T(r)w - \Phi(c)\|_X \\ &(2) \quad \big\langle T(\rho(w))w, D_x\Phi(c)\big\rangle_X = 0 \\ &(3) \quad \rho(T(r)w) = \rho(w) - r \\ &(4) \quad J\rho' : V_{c,\epsilon} \to X. \end{aligned} \tag{6.8}$$

Proof. Define $G : X \times \mathbb{R}^n \to \mathbb{R}$ by

$$G(w,\rho) = \|T(\rho)w - \varphi_c\|_X^2 = \|T(\rho)w\|_X^2 + \|\Phi(c)\|_X^2 - 2\big\langle w, T(-\rho)\Phi(c)\big\rangle_X. \tag{6.9}$$

Since $\varphi \in H^5(\mathbb{R}^n)$ we may compute

$$\begin{aligned} D_\rho G(w,\rho) &= 2\big\langle T(\rho)w, D_x\Phi(c)\big\rangle_X \\ D_\rho^2 G(w,\rho) &= -2\big\langle T(\rho)w, D_x^2\Phi(c)\big\rangle_X \end{aligned}$$

For any non-zero $\xi \in \mathbb{R}^n$ we have

$$D_\rho^2 G(\Phi(c), 0)\xi\cdot\xi = 2\|\xi \cdot D_x\Phi(c)\|_X^2 > 0 \tag{6.10}$$

since otherwise $\varphi(c)$ would be constant in the direction ξ and therefore not in $H^2(\mathbb{R}^n)$. Hence $D_\rho^2 G(\Phi(c), 0)$ is positive definite. Since $D_\rho G(\Phi(c), 0) = 0$, the Implicit Function Theorem implies the existence of a neighborhood U of $\Phi(c)$ in X, a ball $B(0,\tilde{\epsilon})$ and a C^2

map $\rho : U \to B(0, \tilde{\epsilon})$ such that $D_\rho G(w, \rho(w)) = 0$ and $D_\rho^2(w, \rho(w))$ is positive definite for all $w \in U$. Thus $\rho(w)$ is the unique minimizer of $G(w, \cdot)$ in $B(0, \tilde{\epsilon})$ for each $w \in U$. By Lemma 6.5, there exists $\delta > 0$ such that $\|T(r)\Phi(c) - \Phi(c)\|_X < \delta$ implies $r \in B(0, \tilde{\epsilon})$. Choose $\epsilon < \delta/2$ so that $V_\epsilon = \{w : \|w - \Phi(c)\|_X < \epsilon\} \subset U$. Then (1) and (2) hold for $w \in V_\epsilon$.

To show that (3) holds compute

$$\begin{aligned} \|T(\rho(w) - r)T(r)w - \Phi(c)\|_X &\leq \|T(\rho(T(r)w) + r)w - \Phi(c)\|_X \\ &= \|T(\rho(T(r)w))T(r)w - \Phi(c)\|_X \\ &\leq \|T(\rho(w) - r)T(r)w - \Phi(c)\|_X. \end{aligned} \tag{6.11}$$

Thus (3) follows if we can show that $\rho(w) - r \in B(0, \tilde{\epsilon})$ when $w, T(r)w \in V_\epsilon$. Since

$$\begin{aligned} \|T(\rho(w) - r)\varphi - \Phi(c)\|_X &\leq \|\Phi(c) - T(r)w\|_X + \|T(\rho(w))w - \Phi(c)\|_X \\ &\leq 2\epsilon < \delta, \end{aligned} \tag{6.12}$$

it follows from our choice of δ that $\rho(w) - r \in (-\tilde{\epsilon}, \tilde{\epsilon})$. Now we extend ρ to $w \in V_{c,\epsilon}$ by first choosing r such that $T(r)w \in V_\epsilon$. Then let $\rho(w) = \rho(T(r)w) - r$. Since (3) holds in V_ϵ, $\rho(w)$ is independent of the choice of r and properties (1)-(3) follow for $w \in V_{c,\epsilon}$.

To prove (4) we differentiate (2) to obtain

$$\big\langle T(\rho(w))y, D_x\Phi(c)\big\rangle_X - \big\langle T(\rho(w))w, D_x^2\Phi(c)\big\rangle_X \big\langle \rho'(w), y\big\rangle \tag{6.13}$$

Thus

$$\big\langle T(\rho(w))w, D_x^2\Phi(c)\big\rangle_X \big\langle \rho'(w), y\big\rangle = \big\langle T(-\rho(w))ID_x\Phi(c), y\big\rangle \tag{6.14}$$

where $I = (\Delta^2 + Id, Id)$ is the natural isomorphism from X to X^*. Since the matrix $\big\langle T(\rho(w))w, D_x^2\Phi(c)\big\rangle$ is negative definite at $w = \Phi(c)$ it follows by continuity that its inverse exists for all $w \in V_{c,\epsilon}$ for ϵ small enough. Thus

$$\rho'(w) = \big\langle T(\rho(w))w, D_x^2\Phi(c)\big\rangle^{-1} IT(-\rho(w))D_x\Phi(c). \tag{6.15}$$

Since $\varphi(c) \in H^5(\mathbb{R}^n)$ it follows that $J\rho'(w) \in X$. This proves the lemma. ∎

The next lemma proves the existence of an "unstable direction" and depends on the fact that there is an element of X for which the linearized operator

$$H_c = E''(\Phi(c)) - c \cdot Q''(\Phi(c)) \tag{6.16}$$

has negative spectrum. In fact, evaluating H_c on $\Phi(c)$ yields

$$\big\langle H_c\Phi(c), \Phi(c)\big\rangle = -(p^2 - 1)K(\varphi(c)) < 0. \tag{6.17}$$

We need to modify $\Phi(c)$ in order to get a vector orthogonal to $Q'(\Phi(c))$.

Lemma 6.7 *If $d_*''(|c|) < 0$ then there exists $y^u \in Y \equiv H^4(\mathbb{R}^n) \times H^4(\mathbb{R}^n)$ such that*

$$\begin{aligned} &(1) \quad \big\langle H_c y^u, y^u\big\rangle < 0 \\ &(2) \quad \big\langle Q'(\Phi(c), y^u\big\rangle = 0. \end{aligned} \tag{6.18}$$

Proof. Denote by $B(0,\sqrt{2})$ the ball in $\mathbb{R}^n$ of radius $\sqrt{2}$ centered at the origin, and define $q : B(0,\sqrt{2}) \times \mathbb{R} \to \mathbb{R}^n$ by

$$q(h,\sigma) = Q(\Phi(h) + \sigma\Phi(c)), \tag{6.19}$$

then

$$\begin{aligned} D_h q(c,0) &= \langle Q'(\Phi(c)), \Phi'(c)\rangle = D_c\big(Q'(\Phi(c))\big) \\ &= -D^2 d(c) \qquad \text{by (6.3).} \end{aligned}$$

Since $d''_*(|c|) < 0$ and $d'_*(|c|) < 0$ it follows from Lemma 4.10 that $D^2d(c)$ is negative definite. Therefore the Implicit Function Theorem implies that there exist $\epsilon > 0$ and a C^2 function $h : (-\epsilon,\epsilon) \to B(0,\sqrt{2})$ such that $h(0) = c$ and

$$Q(\Phi(h(\sigma)) + \sigma\Phi(c)) = Q(\Phi(c)) \tag{6.20}$$

for $\sigma \in (-\epsilon,\epsilon)$. Next define

$$\tilde{\Psi}(\sigma) = \Phi(h(\sigma)) + \sigma\Phi(c) \tag{6.21}$$

Since $h(0) = c$, $\Psi(0) = \Phi(c)$ and we define

$$\tilde{y} = \tilde{\Psi}'(0) = h'(0)\cdot D_c\Phi(c) + \Phi(c) \tag{6.22}$$

By (6.20),

$$\big\langle Q'(\Phi(c)), \tilde{y}\big\rangle = \frac{d}{d\sigma} Q(\Phi(h(\sigma)) + \sigma\Phi(c))\Big|_{\sigma=0} = 0 \tag{6.23}$$

and thus (2) holds for $\tilde{y}$. To show that (1) holds define $\mathcal{E}(\sigma) = E(\tilde{\Psi}(\sigma))$. We claim that

$$\begin{aligned} \mathcal{E}(0) &= E(\Phi(c)) \\ \mathcal{E}'(0) &= 0 \\ \mathcal{E}''(0) &= \big\langle H_c\tilde{y}, \tilde{y}\big\rangle \end{aligned} \tag{6.24}$$

The first statement is obvious, while the second follows by adding the equations

$$\begin{aligned} \mathcal{E}'(0) &= \big\langle E'(\Phi(c)), \tilde{y}\big\rangle \\ 0 &= -c\cdot\big\langle Q'(\Phi(c)), \tilde{y}\big\rangle \end{aligned}$$

and using the fact that $\Phi(c)$ satisfies (3.6). We next compute

$$\begin{aligned} \mathcal{E}''(\sigma) &= \big\langle E''(\tilde{\Psi}(\sigma)), \tilde{\Psi}'(\sigma)\big\rangle + \big\langle E'(\tilde{\Psi}(\sigma)), \tilde{\Psi}''(\sigma)\big\rangle \\ 0 &= -c\cdot\big\langle Q''(\tilde{\Psi}(\sigma)), \tilde{\Psi}'(\sigma)\big\rangle + \big\langle Q'(\tilde{\Psi}(\sigma)), \tilde{\Psi}''(\sigma)\big\rangle \end{aligned}$$

When we evaluate these at $\sigma = 0$ and add, (3.6) implies that we are left with

$$\mathcal{E}''(0) = \big\langle\, (E''(\Phi(c)) - c\cdot Q''(\Phi(c)))\,\tilde{\Psi}'(0), \tilde{\Psi}'(0)\big\rangle = \big\langle H_c\tilde{y}, \tilde{y}\big\rangle$$

This proves (6.24). We now consider the Taylor expansions of E and Q at $\Phi(h(\sigma))$. First write

$$\begin{aligned} E(u+v) &= E(u) + \big\langle E'(u), v\big\rangle + \frac{1}{2}\big\langle E''(u)v, v\big\rangle + o\,\|v\|^2) \\ Q(u+v) &= Q(u) + \big\langle Q'(u), v\big\rangle + \frac{1}{2}\big\langle Q''(u)v, v\big\rangle + o\,\|v\|^2) \end{aligned}$$

With $u = \Phi(h(\sigma))$ and $v = \sigma\Phi(c)$ we have $\tilde{\Psi}(\sigma) = u + v$. If we multiply the latter equation by $-h(\sigma)$ and use (3.7), (3.6) and (6.16) (all with $c = h(\sigma)$) we obtain

$$E(\tilde{\Psi}(\sigma)) - h(\sigma) \cdot Q(\tilde{\Psi}(\sigma)) = d(h(\sigma)) + \frac{1}{2}\sigma^2 \big\langle H_{h(\sigma)}\Phi(c), \Phi(c)\big\rangle + o\left(\sigma^2\right)$$

Hence

$$\mathcal{E}(\sigma) = d(h(\sigma)) + h(\sigma) \cdot Q(\Phi(c)) + \frac{1}{2}\sigma^2 \big\langle H_{h(\sigma)}\Phi(c), \Phi(c)\big\rangle + o\left(\sigma^2\right)$$

By the concavity of d and (6.3)

$$\begin{aligned} d(h(\sigma)) &\leq d(c) + (h(\sigma) - c) \cdot Dd(c) \\ &= E(\Phi(c)) - h(\sigma) \cdot Q(\Phi(c)) \end{aligned} \tag{6.25}$$

so that

$$\begin{aligned} \mathcal{E}(\sigma) &\leq E(\Phi(c)) + \frac{1}{2}\sigma^2 \big\langle H_{h(\sigma)}\Phi(c), \Phi(c)\big\rangle + o\left(\sigma^2\right) \\ &< E(\Phi(c)) + \frac{1}{4}\sigma^2 \big\langle H_c\Phi(c), \Phi(c)\big\rangle + o\left(\sigma^2\right) \end{aligned}$$

for $\sigma \neq 0$ small enough. Together with the first two parts of (6.24) this implies that $\mathcal{E}''(0) < 0$ and therefore by the last part of (6.24)

$$\big\langle H_c\tilde{y}, \tilde{y}\big\rangle < 0. \tag{6.26}$$

Thus both (1) and (2) hold for $\tilde{y}$.

Since $H^4(\mathbb{R}^n) \times H^4(\mathbb{R}^n)$ is dense in X we can perturb $\tilde{y}$ slightly to obtain a vector in Y satisfying (1) and (2). First let $Q_i(w)$ denote the i^{th} component of $Q(w)$. Without loss of generality we may suppose that the $Q_i'(\Phi(c))$ are linearly independent. We will define inductively $y_1, \ldots, y_n$ in Y with the property

$$\big\langle Q_i'(\Phi(c)), y_j\big\rangle = \delta_{ij} \tag{6.27}$$

Let y be any element of Y for which $\big\langle Q_1'(\Phi(c)), y\big\rangle \neq 0$, and let

$$y_1 = \frac{y}{\big\langle Q_1'(\Phi(c)), y\big\rangle}.$$

Next, assume that we have constructed $y_1, \ldots, y_k$ such that (6.27) is satisfied for $1 \leq i, j \leq k$, and choose any $y \in Y$ such that

$$\big\langle Q_{k+1}'(\Phi(c)), y\big\rangle - \sum_{i=1}^{k} \big\langle Q_i'(\Phi(c)), y\big\rangle\big\langle Q_{k+1}'(\Phi(c)), y_i\big\rangle \neq 0. \tag{6.28}$$

This can be done by the assumption of independence. We now define

$$y_{k+1} = y - \sum_{i=1}^{k} \big\langle Q_i'(\Phi(c)), y\big\rangle y_i$$

Then by the induction hypothesis, $\langle Q_i'(\Phi(c)), y_{k+1}\rangle = 0$ for $i = 1, \ldots, k$ and by (6.28) $\langle Q_{k+1}'(\Phi(c)), y_{k+1}\rangle \neq 0$. If we now subtract

$$\frac{\langle Q_{k+1}'(\Phi(c)), y_i\rangle}{\langle Q_{k+1}'(\Phi(c)), y_{k+1}\rangle} y_{k+1}$$

from y_i for each $i = 1, \ldots, k$ and normalize y_{k+1}, we obtain a collection $y_1, \ldots, y_{k+1}$ satisfying (6.27).

Having chosen $y_1, \ldots, y_n$ satisfying (6.27) we let $\epsilon > 0$ be given and choose $y_\epsilon \in Y$ such that $\|y_\epsilon - \tilde{y}\|_X < \epsilon$. Define

$$y^u = y_\epsilon - \sum_{i=1}^{n} \langle Q_i'(\Phi(c)), y_\epsilon\rangle y_i$$

Then $y^u \in Y$ and, by (6.27), $\langle Q_i'(\Phi(c)), y^u\rangle = 0$ for $i = 1, \ldots, n$. If ϵ is chosen small enough it follows from (6.26) that $\langle H_c y^u, y^u\rangle < 0$. Thus (1) and (2) hold for y^u. Later we will need the first components of y^u and $\tilde{y}$ to be close in the sense that

$$\|y_1^u - \tilde{y}_1\|_{L^{p+1}(\mathbb{R}^n)} \leq \frac{1}{2}\|\varphi(c)\|_{L^{p+1}(\mathbb{R}^n)}. \tag{6.29}$$

This again follows by choosing ϵ small enough. ■

We now define the Lyapunov functional $A : V_{c,\epsilon} \to \mathbb{R}$ by

$$\boxed{A(w) = -\langle J^{-1}y^u, T(\rho(w))w\rangle.}$$

Lemma 6.8 *The functional A is C^1 on $V_{c,\epsilon}$ and*

$$\begin{aligned} &(1) \quad A(T(r)w) = A(w) \qquad \textit{for any } r \in \mathbb{R}^n, \\ &(2) \quad JA'(\Phi(c)) = -y^u, \\ &(3) \quad \langle Q'(w), JA'(w)\rangle = 0. \end{aligned} \tag{6.30}$$

Proof. Part (1) follows from Lemma 6.6(3). For $y \in V_{c,\epsilon} \cap Y$ and $w \in X$, we have

$$\langle A'(y), w\rangle = -\langle J^{-1}y^u, T(\rho(y))w\rangle - \langle J^{-1}y^u, T(\rho(y))Dy\rangle \cdot \langle \rho'(y), w\rangle. \tag{6.31}$$

By Lemma 6.7, $J^{-1}\nabla y^u \in X^n$ and therefore A' extends to all of $V_{c,\epsilon}$. So A is C^1 and

$$\begin{aligned} \langle A'(\Phi(c)), w\rangle &= -\langle J^{-1}y^u, w\rangle - \langle J^{-1}y^u, D_x\Phi(c)\rangle \cdot \langle \rho'(\Phi(c)), w\rangle \\ &= -\langle J^{-1}y^u, w\rangle + \langle Q'(\Phi(c)), y^u\rangle \cdot \langle \rho'(\Phi(c)), w\rangle \end{aligned} \tag{6.32}$$

By Lemma 6.7(2) the last term on the right hand side vanishes, and therefore (2) holds. Differentiating (1) with respect to r at $r = 0$ proves (3). ■

We now wish to construct a curve in X through $\Phi(c)$ in the unstable direction y^u, on which the functional Q is constant and such that E is maximized at $\Phi(c)$. First let $W(\lambda, w_0)$ denote the solution of

$$\frac{dW}{d\lambda} = -JA'(W)$$

with initial data $W(0, w_0) = w_0 \in V_{c,\epsilon}$ and let the components of W be given by $W_1(\lambda, w_0)$, $W_2(\lambda, w_0)$. By Lemma 6.8 (3),

$$\frac{\partial Q}{\partial \lambda}\Big(W(\lambda, w_0)\Big) = 0$$

and thus Q is constant in λ on W. Also,

$$\frac{\partial W}{\partial \lambda}\Big(\lambda, \Phi(c)\Big)\Big|_{\lambda=0} = y^u \tag{6.33}$$

by Lemma 6.8 (2) and by Lemma 6.8 (1)

$$T(r)W(\lambda, w_0) = W(\lambda, T(r)w_0). \tag{6.34}$$

The next lemma shows that there is a point along the curve $W(\lambda, w_0)$ at which the functional K attains the value $K(\varphi(c))$. This allows us to once again exploit the variational characterization of $\varphi(c)$.

Lemma 6.9 *If $d_*''(|c|) < 0$ then there exists $\epsilon > 0$ and a C^1 functional $\lambda : V_{c,\epsilon} \to \mathbb{R}$ so that for $w \in V_{c,\epsilon}$*

$$K(W_1(\lambda(w), w)) = K(\varphi(c)) = \frac{2}{p-1} d(c). \tag{6.35}$$

Proof. As in Lemma 6.6 we first show that there is a C^1 functional λ defined on the neighborhood V_ϵ of $\Phi(c)$ with values in $(-\tilde{\epsilon}, \tilde{\epsilon})$ such that for $w \in V_\epsilon$, $\lambda(w)$ is the unique solution of (6.35) in $(-\tilde{\epsilon}, \tilde{\epsilon})$. Since $W_1(0, \Phi(c)) = \Phi(c)$, this follows from the implicit function theorem and (6.34) once it is shown that

$$\frac{\partial K}{\partial \lambda}(W_1(\lambda, \Phi(c)))\Big|_{\lambda=0} \neq 0.$$

By (6.33) we have

$$\frac{\partial K}{\partial \lambda}\Big(W_1(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} = \big\langle K'(\varphi(c)), y_1^u \big\rangle \tag{6.36}$$

which may be broken up as

$$\big\langle K'(\varphi(c)), y_1^u \big\rangle = \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)(y_1^u - \tilde{y}_1)dx + \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)\tilde{y}_1 dx. \tag{6.37}$$

By (6.29) we can bound the first integral in (6.37) by

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)(y_1^u - \tilde{y}_1)dx \right| &\leq \|\varphi(c)\|^p_{L^{p+1}(\mathbb{R}^n)} \|y_1^u - \tilde{y}_1\|_{L^{p+1}(\mathbb{R}^n)} \\ &\leq \frac{1}{2}\|\varphi(c)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}. \end{aligned} \tag{6.38}$$

Using (6.22) we can rewrite the second term as

$$\int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)\tilde{y}_1 dx = h'(0) \cdot \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c) D_c\Phi(c) dx \\ + \|\varphi(c)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}. \tag{6.39}$$

But since $d(c) = \dfrac{p-1}{2} K(\varphi(c))$, (6.3) implies

$$-Q(\Phi(c)) = Dd(c) = \frac{p-1}{2} \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)\Phi'(c) dx$$

and therefore

$$h'(0) \cdot \int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c) D_c\Phi(c) dx = -\frac{2}{p-1} h'(0) \cdot Q(\Phi(c)). \tag{6.40}$$

Using Lemma 6.7 and equations (6.22), (3.6) we see that

$$\begin{aligned} 0 &= \big\langle Q'(\Phi(c)), \tilde{y} \big\rangle = \big\langle Q'(\Phi(c)), h'(0) \cdot D\Phi(c) + \Phi(c) \big\rangle \\ &= h'(0) \cdot D_c(Q(\Phi(c))) + \big\langle Q'(\Phi(c)), \Phi(c) \big\rangle \\ &= -h'(0) \cdot D^2 d(c) + 2Q(\Phi(c)). \end{aligned} \tag{6.41}$$

Thus since $D^2 d$ is negative definite, (6.41) implies

$$-\frac{2}{p-1} h'(0) \cdot Q(\Phi(c)) = -\frac{1}{p-1} h'(0) \cdot D^2 d(c) \cdot h'(0) \geq 0 \tag{6.42}$$

In view of (6.39), (6.40) and (6.42) we therefore have

$$\int_{\mathbb{R}^n} |\varphi(c)|^{p-1}\varphi(c)\tilde{y}_1 dx \geq \|\varphi(c)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}$$

which, together with (6.38) implies

$$\frac{\partial K}{\partial \lambda}\Big(W_1(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} \geq \frac{1}{2}\|\varphi(c)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} > 0.$$

Next let $w \in V_\epsilon$ and suppose that r is small enough that $T(r)w \in V_\epsilon$ as well. Then by (6.34),

$$K(W_1(\lambda(w), T(r)w)) = K((T(r)W(\lambda(w), w))_1) = K(W_1(\lambda(w), w)) = K(\varphi(c))$$

Since $\lambda(T(r)w)$ is the unique solution of $K(W_1(\lambda, T(r)w)) = K(\varphi(c))$ in $(-\tilde{\epsilon}, \tilde{\epsilon})$, it follows that

$$\lambda(T(r)w) = \lambda(w) \tag{6.43}$$

Thus λ may be extended to $V_{c,\epsilon}$ as follows. Let $w \in V_{c,\epsilon}$ and choose r such that $T(r)w \in V_\epsilon$. Then set $\lambda(w) = \lambda(T(r)w)$. To see that this definition is independent of the choice of r, suppose $T(r_1)w$ and $T(r_2)w$ are both in V_ϵ. Then $T(r_1)w = T(r_1 - r_2)T(r_2)w$ and (6.43) implies that $\lambda(T(r_1)w) = \lambda(T(r_2)w)$. ■

Lemma 6.10 *Suppose $d_*''(s) < 0$. Then there exists $\epsilon > 0$ and a C^1 functional $\lambda : V_{c,\epsilon} \cap \{Q(w) = Q(\Phi(c))\} \to \mathbb{R}$ such that*

$$E(W(\lambda(w), w)) \geq E(\Phi(c)). \tag{6.44}$$

Proof. Let $w \in V_{c,\epsilon}$ with $Q(w) = Q(\Phi(c))$ and let $\lambda(w)$ be given by Lemma 6.9. Then since $\varphi(c)$ minimizes I_c subject to the constraint $K(u) = K(\varphi(c))$ we have, using (4.1) and (6.35),

$$\begin{aligned}
E(W(\lambda(w), w)) &= I_c(W_1(\lambda(w), w)) + c \cdot Q(W(\lambda(w), w)) \\
&\quad - K(W_1(\lambda(w), w)) + \frac{1}{2}\int_{\mathbb{R}^n} |(W_2 - c \cdot \nabla W_1)(\lambda(w), w)|^2 dx \\
&\geq I_c(W_1(\lambda(w), w)) + c \cdot Q(W(\lambda(w), w)) - K(W_1(\lambda(w), w)) \\
&= I_c(W_1(\lambda(w), w)) + c \cdot Q(\Phi(c)) - K(\varphi(c)) \\
&\geq I_c(\varphi(c)) + c \cdot Q(\Phi(c)) - K(\varphi(c)) = E(\Phi(c))
\end{aligned}$$

which proves the lemma. ■

Lemma 6.11 *Let $w \in V_{c,\epsilon}$ with $Q(w) = Q(\Phi(c))$ and $\lambda(w) \neq 0$. If $d''(s) < 0$ then*

$$E(\Phi(c)) < E(w) + \lambda(w)R(w)$$

where $R(w) \equiv \big\langle E'(w), -JA'(w)\big\rangle$.

Proof. The lemma follows by computing the second order Taylor expansion of $E(W(\lambda, w))$ at $\lambda = 0$.

$$\frac{\partial E}{\partial \lambda}\Big(W(\lambda, w)\Big)\Big|_{\lambda=0} = \left\langle E'(w), \frac{\partial W}{\partial \lambda}(\lambda, w)\Big|_{\lambda=0}\right\rangle = R(w) \tag{6.45}$$

and

$$\frac{\partial^2 E}{\partial \lambda^2}\Big(W(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} = \big\langle E''(\Phi(c))y^u, y^u\big\rangle + \left\langle E'(\Phi(c)), \frac{\partial^2 W}{\partial \lambda^2}(0, \Phi(c))\right\rangle. \tag{6.46}$$

Since $Q(W(\lambda, w)) = Q(\Phi(c))$,

$$\frac{\partial Q}{\partial \lambda}\Big(W(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} = \frac{\partial^2 Q}{\partial \lambda^2}\Big(W(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} = 0$$

so that

$$0 = c \cdot \big\langle Q''(\Phi(c))y^u, y^u\big\rangle + c \cdot \left\langle Q'(\Phi(c)), \frac{\partial^2 W}{\partial \lambda^2}(0, \Phi(c))\right\rangle. \tag{6.47}$$

Subtracting (6.47) from (6.46) and using (3.6) yields

$$\frac{\partial^2 E}{\partial \lambda^2}\Big(W(\lambda, \Phi(c))\Big)\Big|_{\lambda=0} = \big\langle (E'' - c \cdot Q'')(\Phi(c))y^u, y^u\big\rangle < 0.$$

Thus, for $\lambda \neq 0$ small and ϵ small enough it follows that if $w \in V_{c,\epsilon}$ then

$$E(W(\lambda, w)) < E(w) + \lambda R(w).$$

So if w also satisfies $Q(w) = Q(\Phi(c))$we have by that Lemma 6.10

$$E(\Phi(c)) \leq E(W(\lambda(w), w)) < E(w) + \lambda(w)R(w)$$

as long as $\lambda(w) \neq 0$. ■

Lemma 6.12 *There exists $\delta > 0$ and a C^2 curve $\Psi : (-\delta, \delta) \to V_{c,\epsilon}$ such that $\Psi(0) = \Phi(c)$, $\Psi'(0) = y$, $Q(\Psi(\tau)) = Q(\Phi(c))$, $R(\Psi(\tau))$ changes sign at $\tau = 0$ and $E(\Psi(\tau))$ has a strict local maximum at $\tau = 0$.*

Proof. Since $\langle Q'(\Phi(c)), y^u\rangle = 0$, y^u is tangent to the manifold $\mathcal{M} = \{w \in X : Q(w) = Q(\varphi)\}$, and thus there is a curve $\Psi(\tau)$ in $\mathcal{M}$ with $\Psi(0) = \Phi(c)$ and $\Psi'(0) = y^u$. To show that $E(\Psi(\tau))$ is maximized at $s = 0$ we differentiate in τ to obtain

$$\begin{aligned} \frac{dE}{d\tau}\Big(\Psi(\tau)\Big)\Big|_{\tau=0} &= \frac{d}{d\tau}\Big(E(\Psi(\tau)) - c \cdot Q(\Psi(\tau))\Big)\Big|_{\tau=0} \\ &= \big\langle E'(\Phi(c)) - c \cdot Q'(\Phi(c)), y^u\big\rangle = 0 \end{aligned}$$

by (3.6). Also

$$\begin{aligned} \frac{d^2E}{d\tau^2}\Big(\Psi(\tau)\Big) &= \big\langle [E''(\Psi(\tau)) - c \cdot Q''(\Psi(\tau))]\Psi'(\tau), \Psi'(\tau)\big\rangle \\ &\quad + \big\langle E'(\Psi(s)) - c \cdot Q'(\Psi(\tau)), \Psi''(\tau)\big\rangle \end{aligned}$$

and therefore

$$\frac{d^2E}{d\tau^2}\Big(\Psi(\tau)\Big)\Big|_{\tau=0} = \big\langle H_c y^u, y^u\big\rangle_X < 0$$

by Lemma 6.6 (1). Thus $E(\Psi(\tau))$ is locally maximized at $\tau = 0$. Next recall that λ is defined by (6.35) and satisfies

$$K(\Phi(c)) = K(W_1(\lambda(\Psi(\tau)), \Psi(\tau))) = \int_{\mathbb{R}^n} |W_1(\lambda(\Psi(\tau)), \Psi(\tau))|^{p+1} dx.$$

Differentiating at $\tau = 0$ gives

$$0 = \left\langle K'(\varphi(c)), \left(\frac{\partial W_1}{\partial \lambda}\frac{\partial \lambda(\Psi(\tau))}{\partial \tau} + \frac{\partial W_1}{\partial u}\frac{\partial \Psi_1}{\partial \tau} + \frac{\partial W_1}{\partial v}\frac{\partial \Psi_2}{\partial \tau}\right)\Big|_{\tau=0}\right\rangle \tag{6.48}$$

Since $W(0, w) = w = (u, v) = (W_1(0, w), W_2(0, w))$,

$$\frac{\partial W_1}{\partial u}\Big(0, w\Big) = Id \qquad \text{and} \qquad \frac{\partial W_1}{\partial v}\Big(0, w\Big) = 0.$$

Thus, since $\frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = \frac{\partial W_1}{\partial \lambda}(\lambda, \Phi(c))\Big|_{\lambda=0} = y_1^u$, (6.48) becomes

$$0 = \left(\frac{\partial \lambda(\Psi(\tau))}{\partial \tau}\Big|_{\tau=0} + 1\right)\big\langle K'(\varphi(c)), y_1^u\big\rangle$$

The latter pairing was shown to be positive in the proof of Lemma 6.9. Thus

$$\frac{\partial \lambda(\Psi(\tau))}{\partial \tau}\Big|_{\tau=0} = -1$$

and, since $\lambda(\Phi(c)) = 0$, we have shown that $\lambda(\Psi(\tau))$ changes sign at $\tau = 0$. Furthermore, $\lambda(\Psi(\tau)) \neq 0$ for all small $\tau \neq 0$ and therefore we may apply Lemma 6.11 to obtain

$$0 < E(\Phi(c)) - E(\Psi(\tau)) < \lambda(\Psi(\tau))R(\Psi(\tau)).$$

Hence $R(\Psi(\tau))$ changes sign at $\tau = 0$. ■

Proof of Theorem 6.2 Fix $\epsilon > 0$ small enough so that Lemma 6.12 applies. Choose τ near zero so that $\lambda(\Psi(\tau)) > 0$ and choose initial data $w_0 = \Psi(\tau)$. Then by Lemma 6.12, $Q(w_0) = Q(\Phi(c))$, $E(w_0) < E(\Phi(c))$ and we may assume that $R(w_0) > 0$. By Assumption3.1, there is an interval $[0, t_0)$ on which a solution $w(t)$ exists and satisfies $w(0) = w_0$, $Q(w(t)) = Q(w_0)$, and $E(w(t)) = E(w_0)$. We may suppose that $t_0 = \infty$ because otherwise $\mathcal{T}_c$ is unstable by definition. Now, by Lemma 6.11,

$$\begin{aligned} 0 < E(\Phi(c)) - E(w_0) &= E(\Phi(c)) - E(w(t)) \\ &< \lambda(w(t))R(w(t)) \end{aligned}$$

for all $t > 0$. Thus by the continuity of P, $R(w(t)) > 0$ for all $t > 0$. We may assume that $\lambda(w(t)) < 1$ so that

$$R(w(t)) > E(\Phi(c)) - E(w_0) \equiv \epsilon_0 > 0.$$

Now let $W = D(J)$ with the graph norm $\|w_0\|_W^2 = \|w_0\|_{X^*}^2 + \|Jw_0\|_X^2$. Then $J : W \to X$ and $J^* : X^* \to W^*$ are continuous and by Lemma 6.4 we have

$$\frac{d}{dt}\big\langle w_0, w(t)\big\rangle = \big\langle E'(w(t)), -Jw_0\big\rangle = \big\langle -J^*E'(w(t)), w_0\big\rangle_{W^*,W}$$

Hence

$$\frac{dw}{dt} = -J^*E'(w)$$

So (by [8], Lemma 4.6) we may compute

$$\begin{aligned} \frac{dA}{dt}\Big(w(t)\Big) &= \left\langle \frac{dw(t)}{dt}, A'(w(t))\right\rangle_{W^*,W} \\ &= \big\langle -J^*E'(w(t)), A'(w(t))\big\rangle_{W^*,W} \\ &= \big\langle E'(w(t)), -JA'(w(t))\big\rangle = R(w(t)) > \epsilon_0 \end{aligned}$$

But A is bounded on $V_{c,\epsilon}$ and hence $w(t)$ must leave $V_{c,\epsilon}$ in finite time, and therefore $\mathcal{T}_c$ is unstable. ■

Remark 6.13 *By* (4.7) *we see that* $d''(s) < 0$ *in some interval around zero. Thus at small speeds travelling waves are unstable.*

7 Standing Waves

In this section we extend our results to include the (easier) case of standing wave solutions of (1.1).

By a standing wave we mean a solution of (1.1) of the form

$$u(x,t) = e^{i\omega t}\varphi(x) \tag{7.1}$$

where the space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ now consists of complex valued functions and has inner product given by

$$\big\langle (u_1, v_1), (u_2, v_2) \big\rangle = Re \int_{\mathbb{R}^n} \Delta u_1 \overline{\Delta u_2} + u_1 \overline{u_2} + v_1 \overline{v_2} dx$$

Substituting (7.1) into (1.1) shows that φ must satisfy

$$\Delta^2 \varphi + (1-\omega^2)\varphi = |\varphi|^{p-1}\varphi \tag{7.2}$$

We solve (7.2) for $\omega^2 < 1$ using the method of Section 1 to show that minimizing sequences for the pair

$$\begin{aligned} I_\omega(u) &= \int_{\mathbb{R}^n} |\Delta u|^2 + (1-\omega^2)|u|^2 dx \\ K(u) &= \int_{\mathbb{R}^n} |u|^{p+1} dx \end{aligned}$$

are relatively compact in $H^2(\mathbb{R}^n)$ up to translation. The absence of second order terms in (7.2) allows us to use the scaling property of the nonlinearity to make a choice of the ground state which is smooth in ω. If φ_0 is a ground state solution of (7.2) with $\omega = 0$ (i.e. φ_0 is a stationary state of (1.1)), then

$$\varphi_w(x) = (1-\omega^2)^{\frac{1}{p-1}} \varphi_0((1-\omega^2)^{\frac{1}{4}} x) \tag{7.3}$$

is a ground state with frequency ω. Next we consider the invariants of (1.1) relevant to standing waves

$$\begin{aligned} E(w) &= \int_{\mathbb{R}^n} \frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|u|^2 - \frac{1}{p+1}|u|^{p+1} dx \\ Q(w) &= Im \int_{\mathbb{R}^n} \bar{u} v dx \end{aligned}$$

If φ is any ground state with frequency ω, we define $\Phi = (\varphi, i\omega\varphi)$ and it follows that

$$E'(\boldsymbol{\varphi}) - \omega Q'(\boldsymbol{\varphi}) = 0$$

We define the action function $d(\omega)$ as before by

$$d(\omega) = E(\boldsymbol{\varphi}) - \omega Q(\boldsymbol{\varphi})$$

By the relation

$$E(w) - \omega Q(w) = \frac{1}{2}I_\omega(u) - \frac{1}{p+1}K(u) + \frac{1}{2}\int_{\mathbb{R}^n} |v - i\omega u|^2 dx$$

we see that $d(\omega)$ is well defined and

$$d(\omega) = \frac{p-1}{2(p+1)}I_\omega(\varphi) = \frac{p-1}{2(p+1)}K(\varphi)$$

which, by (7.3), yields the explicit formula

$$d(\omega) = \frac{p-1}{2(p+1)}K(\varphi_0)(1-\omega^2)^{\tilde{\gamma}} \qquad \tilde{\gamma} = \frac{p+1}{p-1} - \frac{n}{4}. \tag{7.4}$$

If we define the set of ground states with frequency ω to be

$$S_\omega = \{\psi \in H^2(\mathbb{R}^n) | I_\omega(\psi) = K(\psi) = \frac{2(p+1)}{p-1}d(\omega)\}$$

then we have the following stability result.

Theorem 7.1 *Suppose that* Assumption 3.1 *holds and that* $1 < p < 2^* - 1$. *If* $d''(\omega) > 0$, *then* S_ω *is stable.*

Proof. We define

$$\omega(w) = d^{-1}\left(\frac{p-1}{2(p+1)}K(u)\right) = \left(1 - \left(\frac{K(u)}{K(\varphi_0)}\right)^{\frac{1}{\tilde{\gamma}}}\right)^{\frac{1}{2}}$$

for w near S_ω. Under the assumption $d''(\omega) > 0$ we can improve the inequality (5.5) to

$$E(w) - E(\boldsymbol{\psi}) - \omega(w)(Q(w) - Q(\boldsymbol{\psi})) \geq \frac{1}{4}d''(\omega)|\omega(w) - \omega|^2$$

for any $\boldsymbol{\psi} \in S_\omega$ and w near S_ω. The rest of the proof is identical to the proof of Theorem 5.4. ■

Solutions of (1.1) are invariant under the group action $T : \mathbb{R} \times X \to X$ given by

$$T(s)w = e^{i\omega s}w = (e^{i\omega s}u, e^{i\omega s}v)$$

Given a ground state $\boldsymbol{\varphi}$ with frequency ω we define its orbit under T by

$$\mathcal{T}_\omega = \{T(s)\boldsymbol{\varphi} | s \in \mathbb{R}\}$$

With these definitions we have

Theorem 7.2 *Suppose that* Assumption 3.1 *holds and* $1 < p < 2^* - 1$. *If* $d''(\omega) < 0$, *then* $\mathcal{T}_\omega$ *is unstable.*

Proof. First, Lemma 6.5 and Lemma 6.6 are true for T as given above, σ, $\tau \in \mathbb{R}$, modulo 2π. Also, Theorem 6.7 follows more easily in this case since we no longer insist that the unstable direction y^u have any regularity properties. Thus we may define the Lyapunov functional by

$$A(w) = -\left\langle J^{-1}y^u, T(\sigma(w))w\right\rangle$$

The rest of the proof follows exactly as in Section 6. ■

Using the expression (7.4) for $d(\omega)$ we may now explicitly determine the intervals in which ground states are stable and unstable. We compute

$$d''(\omega) = 2\tilde{\gamma}(1-\omega^2)^{\tilde{\gamma}-2}(w^2(2\tilde{\gamma}-1)-1)$$

Thus if $\tilde{\gamma} \leq \frac{1}{2}$ then $d''(\omega) < 0$ for all $\omega^2 < 1$. That is, when

$$p \geq 1 + \frac{8}{n-2}$$

all ground states are unstable. On the other hand, if

$$p < 1 + \frac{8}{n-2}$$

then ground states are stable in the interval $\omega^2 > \frac{1}{2\tilde{\gamma}-1}$ and unstable in the interval $\omega^2 < \frac{1}{2\tilde{\gamma}-1}$. Ground states at the critical value $\omega^2 = \frac{1}{2\tilde{\gamma}-1}$ are also unstable since, by the smooth choice of ground states, there are unstable states arbitrarily nearby.

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