

Solitary waves of the rotation-generalized Benjamin-Ono equation

Amin Esfahani

School of Mathematics and Computer Science
Damghan University
Damghan, Postal Code 36716-41167, Iran
E-mail: amin@impa.br, saesfahani@du.ac.ir

Steven Levandosky

Mathematics and Computer Science Department
College of the Holy Cross, Worcester, MA 01610
E-mail: spl@mathcs.holycross.edu

Abstract

This work studies the rotation-generalized Benjamin-Ono equation which is derived from the theory of weakly nonlinear long surface and internal waves in deep water under the presence of rotation. It is shown that the solitary-wave solutions are orbitally stable for certain wave speeds.

1 Introduction

In the present paper we are concerned with studying the rotation-generalized Benjamin-Ono (RGO) equation which can be written as

$$(u_t + \beta \mathcal{H}u_{xx} + (f(u))_x)_x = \gamma u, \quad x \in \mathbb{R}, t > 0, \quad (1.1)$$

where $\gamma > 0$ and $\beta \neq 0$ are real constants, f is a C^2 function which is homogeneous of degree $p > 1$ such that $sf(s) = pf'(s)$, and \mathcal{H} denotes the Hilbert transform defined by

$$\mathcal{H}u(x, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, t)}{x - z} dz,$$

where p.v. denotes the Cauchy principal value. When $f(u) = \frac{1}{2}u^2$, equation (1.1), which is so-called the rotation-modified Benjamin-Ono (RMBO) equation, models the propagation of long internal waves in a deep rotating fluid [15, 18, 22, 30]. In the context of shallow water the propagation of long waves in rotating fluid is described by the Ostrovsky equation [10, 16, 27, 28]

$$(u_t + \beta u_{xxx} + (u^2)_x)_x = \gamma u, \quad x \in \mathbb{R}, t > 0, \quad (1.2)$$

which is also called the rotation-modified Korteweg-de Vries (RMKdV) equation. See also [13, 14] for the two-dimensional long internal waves in a rotating fluid. The parameter γ is a measure of the effect of rotation. Setting $\gamma = 0$ in (1.1), integrating the result with respect to x and setting the constant of integration to zero, one obtains the generalized Benjamin-Ono (GBO) equation

$$u_t + \beta \mathcal{H}u_{xx} + (f(u))_x = 0. \quad (1.3)$$

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Most attention in this work is paid to the existence, the stability and the properties of localized traveling waves (commonly referred to as solitary waves) of (1.1). Using variational methods and the Pohozaev-type identities, we prove the existence and nonexistence of solitary waves for a range of the parameters of (1.1). We also show that our solitary waves (of (1.1)) are the ground states, i.e. they have minimal action. We also consider the effect of letting the rotation parameter γ approach zero. Actually we show that the ground state solitary waves converge to solitary waves of the GBO equation.

It was shown by Linares and Milanes [22] that the RMBO equation (1.1) is well-posed in the space

$$X_s = \{f \in H^s(\mathbb{R}); \partial_x^{-1} f \in H^s(\mathbb{R})\}$$

with norm

$$\|g\|_{X_s} = \|g\|_{H^s(\mathbb{R})} + \|\partial_x^{-1} g\|_{H^s(\mathbb{R})},$$

for $s > 3/2$, where the operator ∂_x^{-1} is defined via the Fourier transform as $\widehat{\partial_x^{-1} g}(\xi) = (i\xi)^{-1} \widehat{g}(\xi)$. The methods therein also imply the same result for the RGO equation (1.1).

It is also standard to show that the solution $u(t)$ obtained that way satisfies $E(u(t)) = E(u(0))$, $Q(u(t)) = Q(u(0))$ and $M(u(t)) = 0$, for $t \in [0, T)$ with the maximum existence time T , where

$$E(u) = \int_{\mathbb{R}} \frac{\beta}{2} (D_x^{1/2} u)^2 + \frac{\gamma}{2} (\partial_x^{-1} u)^2 + F(u) dx, \quad (1.4)$$

$$Q(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 dx \quad (1.5)$$

and

$$M(u) = \int_{\mathbb{R}} u dx \quad (1.6)$$

express, respectively, the energy, momentum and total mass, where $\widehat{D_x^{1/2} f}(\xi) = |\xi|^{1/2} \widehat{f}(\xi)$, $F' = f$ and $F(0) = 0$. It is also worth remarking that the sufficiently smooth solutions of (1.1) satisfy

$$\int_{\mathbb{R}} xu dx = 0.$$

These conserved quantities play an important role in our stability analysis.

We show in Theorems 6.1 and 7.2 that the function $d(c)$ defined by (6.1) determines the stability of the solitary waves in the sense that if $d''(c) > 0$, then $\mathcal{G}(\beta, c, \gamma)$ is \mathcal{X} -stable, while if $d''(c) < 0$, then \mathcal{O}_φ is \mathcal{X} -unstable, where the space \mathcal{X} is defined in (1.7). In Theorem 8.1, we use the ideas of [17], and provides sufficient conditions for instability directly in terms of the parameters β , γ and p .

We also investigate the properties of the function $d(c)$ which determines the stability of the ground states. Using an important scaling identity, together with numerical approximations of the solitary waves, we are able to numerically approximate $d(c)$.

Remark 1.1 *It is noteworthy that despite our regularity assumption on f , one can observe that all our results are valid for the nonlinearity $f(u) = \pm|u|u$.*

Notations

For each $r \in \mathbb{R}$, we define the translation operator by $\tau_r u = u(\cdot + r)$.

Given a solitary wave φ of (2.1), the orbit of φ is defined by the set $\mathcal{O}_\varphi = \{\tau_r \varphi; r \in \mathbb{R}\}$.

We shall denote by $\widehat{\varphi}$ the Fourier transform of φ , defined as

$$\widehat{\varphi}(\zeta) = \int_{\mathbb{R}} \varphi(\omega) e^{-i\omega \cdot \zeta} d\omega.$$

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$, the nonhomogeneous Sobolev space defined by the closure

$$\{\varphi \in \mathcal{S}'(\mathbb{R}) : \|\varphi\|_{H^s(\mathbb{R})} < \infty\},$$

with respect to the norm

$$\|\varphi\|_{H^s(\mathbb{R})} = \left\| (1 + |\zeta|)^{\frac{s}{2}} \widehat{\varphi}(\zeta) \right\|_{L^2(\mathbb{R})},$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions.

Let \mathcal{X} be the space defined by

$$\mathcal{X} = \left\{ f \in H^{1/2}(\mathbb{R}); (\xi^{-1} \widehat{f}(\xi))^\vee \in L^2(\mathbb{R}) \right\} \quad (1.7)$$

with the norm

$$\|f\|_{\mathcal{X}} = \|f\|_{H^{1/2}(\mathbb{R})} + \|\partial_x^{-1} f\|_{L^2(\mathbb{R})}.$$

2 Solitary Waves

By a solitary wave solution of the RGBO equation, we mean a traveling-wave solution of equation (1.1) of the form $\varphi(x - ct)$, where $\varphi \in \mathcal{X}$ and $c \in \mathbb{R}$ is the speed of wave propagation. Alternatively, it is a solution $\varphi(x)$ in \mathcal{X} of the stationary equation

$$\beta \mathcal{H} \varphi_x - c\varphi + f(\varphi) = \gamma \partial_x^{-2} \varphi. \quad (2.1)$$

We will prove existence of solitary waves in the space \mathcal{X} by considering the following variational problem. Define the functionals

$$I(u) = I(u; \beta, c, \gamma) = \int_{\mathbb{R}} \beta (D_x^{1/2} u)^2 - cu^2 + \gamma (\partial_x^{-1} u)^2 dx \quad (2.2)$$

and

$$K(u) = -(p+1) \int_{\mathbb{R}} F(u) dx; \quad (2.3)$$

and consider the following minimization problem

$$M_\lambda = \inf \{ I(u); u \in \mathcal{X}, K(u) = \lambda \}, \quad (2.4)$$

for some $\lambda > 0$.

First we observe that $M_\lambda > 0$ for any $\lambda > 0$. In fact, for $c \leq 0$

$$\max\{\beta, c, \gamma\} \|u\|_{\mathcal{X}}^2 \geq I(u) \geq \beta \int_{\mathbb{R}} (D_x^{1/2} u)^2 dx + \gamma \int_{\mathbb{R}} (\partial_x^{-1} u)^2 dx; \quad (2.5)$$

while for $c \in (0, c_*)$

$$\max\{\beta, c, \gamma\} \|u\|_{\mathcal{X}}^2 \geq I(u) \geq c_1 \beta \int_{\mathbb{R}} (D_x^{1/2} u)^2 dx + c_2 \gamma \int_{\mathbb{R}} (\partial_x^{-1} u)^2 dx, \quad (2.6)$$

where $c_1 = 1 - \sqrt{\frac{1}{2} + \frac{2c^3}{27\gamma\beta^2}}$ and $c_2 = \frac{27\beta^2 - 4c^3}{27\gamma\beta^2 + 4c^3}$. On the other hand, for $u \in \mathcal{X}$, we have

$$\|u\|_{L^{p+1}(\mathbb{R})}^{p+1} \leq C \|u\|_{H^{1/2}(\mathbb{R})}^{(3p+1)/3} \|\partial_x^{-1} u\|_{L^2(\mathbb{R})}^{2/3}. \quad (2.7)$$

Indeed, by using the Sobolev embedding and an interpolation, we find

$$\|u\|_{L^{p+1}(\mathbb{R})}^{p+1} \leq C \|u\|_{H^{\frac{p-1}{2(p+1)}(\mathbb{R})}}^{p+1} \leq C \|u\|_{H^{1/2}(\mathbb{R})}^{(3p-1)/3} \|u\|_{H^{-1/4}(\mathbb{R})}^{4/3}. \quad (2.8)$$

Then the Cauchy-Schwarz inequality implies that

$$\|u\|_{H^{-1/4}(\mathbb{R})} \leq C \|u\|_{H^{1/2}(\mathbb{R})}^{1/2} \|\partial_x^{-1}u\|_{L^2(\mathbb{R})}^{1/2}. \quad (2.9)$$

Now inequality (2.7) is obtained from (2.8) and (2.9).

We also note that $\|u\|_{\mathcal{X}}^2$ is equivalent to $I(u)$. Indeed, (2.5), (2.6) and the inequality

$$\|u\|_{L^2(\mathbb{R})}^2 \leq A \|D_x^{1/2}u\|_{L^2(\mathbb{R})}^2 + B \|\partial_x^{-1}u\|_{L^2(\mathbb{R})}^2, \quad \text{where } A > 0, B > \frac{4}{27A^2}, \quad (2.10)$$

lead to the coercivity condition $I(u) \sim \|u\|_{\mathcal{X}}^2$.

Thus, it can be deduced from (2.7) that

$$\begin{aligned} \lambda = K(u) &\leq C \int_{\mathbb{R}} |u|^{p+1} dx \leq C \|u\|_{H^{1/2}(\mathbb{R})}^{(3p+1)/3} \|\partial_x^{-1}u\|_{L^2(\mathbb{R})}^{2/3} \\ &\leq C \left(\|u\|_{L^2(\mathbb{R})}^2 + \|D_x^{1/2}u\|_{L^2(\mathbb{R})}^2 + \|\partial_x^{-1}u\|_{L^2(\mathbb{R})}^2 \right)^{(p+1)/2}. \end{aligned}$$

Therefore we have $\lambda^{2/(p+1)} \leq CI(u)$, where $C = C(\beta, c, \gamma) > 0$. This implies that

$$M_\lambda \geq (\lambda/C)^{(p+1)/2} > 0.$$

Then if $\psi \in \mathcal{X}$ achieves the minimum of problem (2.4), for some $\lambda > 0$, then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\beta \mathcal{H} \psi_x - c\psi - \gamma \partial_x^{-2} \psi = -\mu f(\psi).$$

Hence $\varphi = \mu^{1/(p-1)} \psi$ satisfies (2.1). We denote the set of such solutions by $G(\beta, c, \gamma)$. By the homogeneity of I and K , $u \in G(\beta, c, \gamma)$ also achieve the minimum

$$m = m(\beta, c, \gamma) = \inf \left\{ \frac{I(u)}{(K(u))^{\frac{2}{p+1}}}; u \in \mathcal{X}, K(u) > 0 \right\}$$

and it follows that $M_\lambda = m\lambda^{\frac{2}{p+1}}$. We note that if we multiply (2.1) by φ and integrate, we find that $I(\varphi; \beta, c, \gamma) = K(\varphi)$. Thus we may characterize the set $G(\beta, c, \gamma)$ as

$$G(\beta, c, \gamma) = \left\{ \varphi \in \mathcal{X}; K(\varphi) = I(\varphi; \beta, c, \gamma) = (m(\beta, c, \gamma))^{\frac{p+1}{p-1}} \right\}.$$

We now seek to prove that this set is nonempty.

We say that a sequence ψ_n is a minimizing sequence if for some $\lambda > 0$, $\lim_{n \rightarrow \infty} K(\psi_n) = \lambda$ and $\lim_{n \rightarrow \infty} I(\psi_n) = M_\lambda$.

Theorem 2.1 *Let $p \geq 2$, $\beta > 0$, $\gamma > 0$ and $c < c_* = 3(\beta^2\gamma/4)^{1/3}$. Let $\{\psi_n\}_n$ be a minimizing sequence for some $\lambda > 0$. Then there exist a subsequence (renamed ψ_n) and scalars $y_n \in \mathbb{R}$ and $\psi \in \mathcal{X}$ such that $\psi_n(\cdot + y_n) \rightarrow \psi$ in \mathcal{X} . The function ψ achieves the minimum $I(\psi) = M_\lambda$ subject to the constraint $K(\psi) = \lambda$.*

Proof. The result is an application of the concentration compactness lemma of Lions [23], similar to [1, 6, 25]. We give the sketch of the proof here.

Let $\{\psi_n\}$ be a minimizing sequence, then we deduce from the coercivity of I that the sequence $\{\psi_n\}$ is bounded in \mathcal{X} , so if we define

$$\rho_n = |D_x^{1/2}\psi_n|^2 + |\partial_x^{-1}\psi_n|^2,$$

then after extracting a subsequence, we may assume $\lim_{n \rightarrow \infty} \|\rho_n\|_{L^1(\mathbb{R})} = L > 0$. We may assume further after normalizing that $\|\rho_n\|_{L^1(\mathbb{R})} = L$ for all n . By the concentration compactness lemma, a

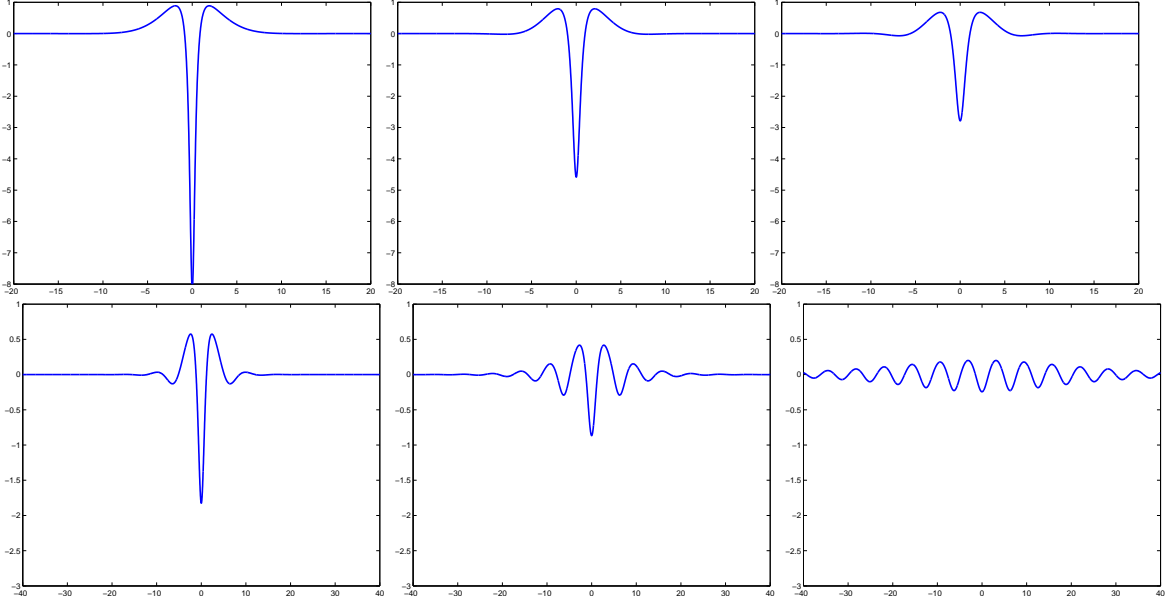


Figure 1: Solitary waves of the rotation generalized Benjamin-Ono equation with $f(u) = u^2$, $\beta = 2$, $\gamma = 1$ and $c = -1, 1, 2, 2.5, 2.9$ and 2.99 . Note that $c_* = 3$ when $\beta = 2$ and $\gamma = 1$.

further subsequence ρ_n satisfies one of vanishing, dichotomy or compactness conditions. We can easily see that $M_\lambda = \lambda^{2/(p+1)} M_1$, so that the strict subadditivity condition

$$M_\alpha + M_{\lambda-\alpha} > M_\lambda$$

holds for any $\alpha \in (0, \lambda)$. In the same manner as in [19, 20, 25], it follows from the coercivity of I , inequality (2.7), and the subadditivity condition that both vanishing and dichotomy may be ruled out, and therefore the sequence ρ_n is compact. Now set $\varphi_n(x) = \psi(x + y_n)$. Since φ_n is bounded in \mathcal{X} , a subsequence φ_n converges weakly to some $\psi \in \mathcal{X}$, and by the weak lower semicontinuity of I over \mathcal{X} , we have $I(\psi) \leq \liminf_{n \rightarrow \infty} I(\varphi_n) = M_\lambda$. Moreover, weak convergence in \mathcal{X} , compactness of ρ_n , and inequality (2.7) imply strong convergence of φ_n to ψ in $L^{p+1}(\mathbb{R})$. Therefore $K(\psi) = \lim K(\varphi_n) = \lambda$, so $I(\psi) \geq M_\lambda$. Together with the inequality above, this implies $I(\psi) = M_\lambda$, so ψ is a minimizer of I subject to the constraint $K(\cdot) = \lambda$. Finally, since I is equivalent to the norm on \mathcal{X} , $\varphi_n \rightharpoonup \psi$, and $I(\varphi_n) \rightarrow I(\psi)$, it follows that φ_n converges strongly to ψ in \mathcal{X} . \square

Theorem 2.2 *Let β, γ and c be as in Theorem 2.1, and $f \in C^k$, for some nonnegative integer k . Then any weak solution φ of (2.1) is a $H^{k+1}(\mathbb{R})$ -function. Moreover $\partial_x^{-1}\varphi \in H^{k+2}(\mathbb{R})$ and $\partial_x^{-2}\varphi \in H^{k+3}(\mathbb{R})$.*

Proof. First we write (2.1) in the form of a convolution equation

$$\varphi = h * g(\varphi), \tag{2.11}$$

where $g(\varphi) = -f(\varphi)$ and

$$\widehat{h}(\xi) = \frac{\xi^2}{\beta\xi^2|\xi| - c\xi^2 + \gamma}. \tag{2.12}$$

Since $\frac{\xi^3}{\beta\xi^2|\xi| - c\xi^2 + \gamma}$ and $\frac{1}{\beta\xi^2|\xi| - c\xi^2 + \gamma}$ are bounded for any $\xi \in \mathbb{R}$, then (2.11) and the Sobolev embedding implies that $\partial_x^{-2}\varphi, \varphi_x \in L^2(\mathbb{R})$. Therefore we find that $\varphi \in H^1(\mathbb{R})$; so that $\varphi \in L^\infty(\mathbb{R})$. Hence $(g(\varphi))_x \in L^2(\mathbb{R})$. Since

$$(\beta\mathcal{H}\varphi_x - c\varphi - g(\varphi))_x = \gamma\partial_x^{-1}\varphi$$

it follows that $\mathcal{H}\varphi_{xx} \in L^2(\mathbb{R})$ and consequently $\varphi_{xx} \in L^2(\mathbb{R})$ and $\varphi \in H^2(\mathbb{R})$. Repeating this process yields the property $\varphi \in H^{k+1}(\mathbb{R})$.

A similar argument shows that $\partial_x^{-1}\varphi \in H^{k+2}(\mathbb{R})$ and $\partial_x^{-2}\varphi \in H^{k+3}(\mathbb{R})$. \square

Theorem 2.3 *Let $p > 1$ be an integer, β, γ and c be as in Theorem 2.1 and φ be a nontrivial solution of (2.1). There exist $\kappa > 0$ and an holomorphic function ψ of variable z , defined in the domain*

$$\mathcal{H}_\kappa = \{z \in \mathbb{C}; |\operatorname{Im}(z)| < \kappa\},$$

such that $\psi(x) = \varphi(x)$ for all $x \in \mathbb{R}$.

Proof. By the Cauchy-Schwarz inequality, we have that $\widehat{\varphi} \in L^1(\mathbb{R})$. The equation (2.1) implies that

$$|\widehat{\varphi}|(\xi) \leq \overbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}^p(\xi), \quad (2.13)$$

$$|\xi| |\widehat{\varphi}|(\xi) \leq \underbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}_p(\xi), \quad (2.14)$$

We denote $\mathcal{T}_1(|\widehat{\varphi}|) = |\widehat{\varphi}|$ and for $m \geq 1$, $\mathcal{T}_{m+1}(|\widehat{\varphi}|) = \mathcal{T}_m(|\widehat{\varphi}|) * |\widehat{\varphi}|$. It can be easily seen by induction that for all $m \in \mathbb{N}$,

$$|\xi|^m |\widehat{\varphi}|(\xi) \leq (m-1)! p^{m-1} \mathcal{T}_{(m+1)(p-1)+1}(|\widehat{\varphi}|)(\xi). \quad (2.15)$$

Therefore we have

$$\begin{aligned} |\xi|^m |\widehat{\varphi}|(\xi) &\leq (m-1)! p^{m-1} \|\mathcal{T}_{(m+1)(p-1)+1}(|\widehat{\varphi}|)\|_{L^\infty(\mathbb{R})} \\ &\leq (m-1)! p^{m-1} \|\mathcal{T}_{(m+1)(p-1)}(|\widehat{\varphi}|)\|_{L^2(\mathbb{R})} \|\widehat{\varphi}\|_{L^2(\mathbb{R})} \\ &\leq (m-1)! p^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R})}^{(m+1)(p-1)-1} \|\widehat{\varphi}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Let

$$a_m = \frac{p^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R})}^{(m+1)(p-1)-1} \|\widehat{\varphi}\|_{L^2(\mathbb{R})}^2}{m},$$

then it is clear that

$$\frac{a_{m+1}}{a_m} \longrightarrow (p+1) \|\widehat{\varphi}\|_{L^1(\mathbb{R})}^{p-1},$$

as $m \rightarrow +\infty$. Therefore the series $\sum_{m=0}^{\infty} \zeta^m r^m |\widehat{\varphi}|(\xi)/m!$ converges uniformly in $L^\infty(\mathbb{R})$, if $0 < \zeta < \kappa = \frac{1}{(p)} \|\widehat{\varphi}\|_{L^1(\mathbb{R})}^{-p+1}$. Hence $e^{\zeta r} \widehat{\varphi}(\xi, \eta) \in L^\infty(\mathbb{R})$, for $\zeta < \kappa$.

We define the function

$$\psi(z) = \int_{\mathbb{R}} e^{i\xi z} \widehat{\varphi}(\xi) \, d\xi.$$

By the Paley-Wiener Theorem, ψ is well defined and analytic in \mathcal{H}_κ ; and by Plancherel's Theorem, we have $\psi(x) = \varphi(x)$ for all $x \in \mathbb{R}$. This proves the theorem. \square

Theorem 2.4 *Let β, γ and c be as in Theorem 2.1. Then any solution φ of (2.1) satisfies $|x|^\ell \varphi^{(k)}(x) \in L^q(\mathbb{R})$, for $1 \leq q \leq \infty$, $k \in \{-2, -1, 0\}$ and $\ell \in [0, 4+k]$. Furthermore,*

$$|x|^\ell \varphi^{(n)}(x) \in L^q(\mathbb{R}), \quad \text{for } 1 \leq q \leq \infty, n \in \mathbb{N}, \ell \in [0, 5]. \quad (2.16)$$

Proof. First a straightforward calculation reveals that $\widehat{h} \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^4(\mathbb{R})$. Moreover $\partial_\xi^j \widehat{h} \in L^q(\mathbb{R})$, for $q \in [1, \infty]$ and $1 \leq j \leq 4$. This implies that $\widehat{h} \in H^4(\mathbb{R})$. Hence by [12, Corollary 3.1.3], we see that $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $|x|^\ell \varphi(x) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for $\ell \in [0, 4]$. Now the elementary inequality

$$|x|^\ell |\varphi| \leq \| |x|^\ell h * g(\varphi) \| + \| h * |x|^\ell g(\varphi) \|$$

and the Young inequality imply that $|x|^\ell \varphi(x) \in L^1(\mathbb{R})$, for $\ell \in [0, 4]$.

Analogously, by using (2.1), one can show for $k = -2, -1$ that $|x|^\ell \varphi^{(k)}(x) \in L^q(\mathbb{R})$, for any $1 \leq q \leq \infty$ and $\ell \in [0, 4 + k]$.

To prove (2.16), first we note that the fact $\varphi \in L^\infty(\mathbb{R})$, the inequality

$$|x|^\ell |\varphi'| \leq \| |x|^\ell h * (g(\varphi))_x \| + \| h * |x|^\ell (g(\varphi))_x \|$$

and the Young inequality implies that $|x|^\ell \varphi'(x) \in L^q(\mathbb{R})$, for any $1 \leq q \leq \infty$ and $\ell \in [0, 4]$. On the other hand, a straightforward computation shows that $h' \in L^1(\mathbb{R})$ and $|x|^\ell h' \in L^q(\mathbb{R})$ for any $\ell \in [1, 5]$ and $1 \leq q \leq \infty$. Therefore combining the inequality

$$|x|^5 |\varphi'| \leq \| |x|^5 h' * g(\varphi) \| + \| h' * |x|^5 g(\varphi) \|,$$

the identity $|x|^5 |\varphi|^p = |x| |\varphi| (|x|^{4/(p-1)} |\varphi|)^p$ and the Young inequality yields that $|x|^5 \varphi'(x) \in L^q(\mathbb{R})$, for any $1 \leq q \leq \infty$. Finally a bootstrapping argument proves (2.16). \square

Proposition 2.1 *Let $\beta > 0$, $\gamma > 0$ and $c < c_*$ be as in Theorem (2.1), then there exists $C \in \mathbb{R}$, $C \neq 0$, such that any solution of (2.1) satisfies*

$$\lim_{|x| \rightarrow +\infty} |x|^6 \varphi(x) = C. \quad (2.17)$$

Proof. The kernel h in (2.12) can be written in $h(x) = -\frac{d^2}{dx^2} \mathbb{K}(x)$, where

$$\widehat{\mathbb{K}}(\xi) = \frac{1}{\beta |\xi|^3 - c\xi^2 + \gamma}.$$

Since \mathbb{K} is an even function, hence

$$\mathbb{K}(x) = \int_{\mathbb{R}} \frac{e^{ix\xi}}{\beta |\xi|^3 - c\xi^2 + \gamma} d\xi = \int_0^{+\infty} \frac{\cos(|x|\xi)}{\beta \xi^3 - c\xi^2 + \gamma} d\xi. \quad (2.18)$$

Then by using the residue theorem, there holds that

$$\begin{aligned} \mathbb{K}(x) &= \int_0^{+\infty} \frac{-\beta y^3 e^{-y|x|}}{(\gamma + cy^2)^2 + \beta^2 y^6} dy + 2\pi \operatorname{Re} \left(\frac{ie^{b|x| - a|x|}}{3\beta(b^2 - a^2) - 2cb + 2ai(3b\beta - 2c)} \right) \\ &= \int_0^{+\infty} \frac{-\beta y^3 e^{-y|x|}}{(\gamma + cy^2)^2 + \beta^2 y^6} dy \\ &\quad + 2\pi \frac{e^{-a|x|} [2a(3b\beta - c) \cos(b|x|) + \sin(b|x|)(2cb - 3\beta(b^2 - a^2))]}{(3\beta(b^2 - a^2) - 2cb)^2 + 4a^2(c - 3b\beta)^2}, \end{aligned} \quad (2.19)$$

where $b + ia$ is the complex root of $\beta |\xi|^3 - c\xi^2 + \gamma$, with $a, b > 0$. Therefore $\mathbb{K} \in C^\infty(\mathbb{R} \setminus \{0\})$. It is therefore concluded for $c < c_*$ that

$$\begin{aligned} h(x) &= \int_0^{+\infty} \frac{\beta y^5 e^{-y|x|}}{(\gamma + cy^2)^2 + \beta^2 y^6} dy \\ &\quad - 2\pi a(3b\beta - c) \frac{e^{-a|x|} (2ab \sin(b|x|) + (a^2 - b^2) \cos(b|x|))}{(3\beta(b^2 - a^2) - 2cb)^2 + 4a^2(c - 3b\beta)^2} \\ &\quad - 2\pi(2cb - 3\beta(b^2 - a^2)) \frac{e^{-a|x|} ((a^2 - b^2) \sin(b|x|) - 2ab \cos(b|x|))}{(3\beta(b^2 - a^2) - 2cb)^2 + 4a^2(c - 3b\beta)^2}. \end{aligned} \quad (2.20)$$

Now the change of variable $\eta = xy$ in the first term of the right-hand side of (2.20) reveals that

$$\lim_{|x| \rightarrow +\infty} |x|^6 h(x) = \frac{\beta}{\gamma^2}.$$

Applying Theorem 3.1.5 in [12], it transpires that there exists $C \neq 0$ such that (2.17) holds. \square

Remark 2.1 *By Theorem 2.4 and Proposition 2.1, one can see that the solitary waves of (2.1) decay faster than the solitary waves of (1.3) (see (5.2)).*

Remark 2.2 *By (2.20), it seems that the solitary wave φ of (2.1) does not decay exponentially.*

3 Nonexistence

In this section we present conditions on the parameters β , c , γ and the nonlinearity $f(u)$ that guarantee equation (1.1) has no solitary wave solutions in the space \mathcal{X} . These conditions follow from the following functional identities.

Lemma 3.1 *Suppose $\varphi \in \mathcal{X}$ is a solution of equation (2.1). Then*

$$\begin{aligned} \int_{\mathbb{R}} \beta (D_x^{1/2} \varphi)^2 dx - c \int_{\mathbb{R}} \varphi^2 dx + \gamma \int_{\mathbb{R}} (\partial_x^{-1} \varphi)^2 dx &= -(p+1) \int_{\mathbb{R}} F(\varphi) dx \\ -c \int_{\mathbb{R}} \varphi^2 dx + 3\gamma \int_{\mathbb{R}} (\partial_x^{-1} \varphi)^2 dx &= -2 \int_{\mathbb{R}} F(\varphi) dx \end{aligned} \quad (3.1)$$

Proof. These relations follow by multiplying equation (2.1) by φ and $x\varphi_x$, respectively and integrating over \mathbb{R} . To see that the β term vanishes in the second relation, first observe that since φ_x has zero mass, it follows that $\mathcal{H}(x\varphi_x) = x\mathcal{H}(\varphi_x)$. Then, using the anti-commutative property of \mathcal{H} we have

$$\int_{\mathbb{R}} \mathcal{H} \varphi_x \cdot x\varphi_x dx = - \int_{\mathbb{R}} \varphi_x \cdot \mathcal{H}(x\varphi_x) dx = - \int_{\mathbb{R}} \varphi_x \cdot x\mathcal{H} \varphi_x dx.$$

This completes the proof. \square

Theorem 3.1 *Equation (2.1) has no solution in \mathcal{X} provided any of the following conditions hold.*

(i) $\beta < 0$, $\gamma > 0$ and $c^3 < \frac{27(3p+1)\gamma\beta^2}{(p-1)^3}$.

(ii) $\beta > 0$, $\gamma < 0$ and $c^3 > \frac{27(3p+1)\gamma\beta^2}{(p-1)^3}$.

(iii) $f(u) = |u|^{p-1}u$, $\beta > 0$ and $\gamma < 0$.

(iv) $f(u) = -|u|^{p-1}u$, $\beta < 0$ and $\gamma > 0$.

Proof. Eliminating the terms on the right hand sides of (3.1), we find that

$$-2\beta \int_{\mathbb{R}} (D_x^{1/2} \varphi)^2 dx - (p-1)c \int_{\mathbb{R}} \varphi^2 dx + (3p+1)\gamma \int_{\mathbb{R}} (\partial_x^{-1} \varphi)^2 dx = 0. \quad (3.2)$$

Now suppose $\beta < 0$ and $\gamma > 0$. Then since the expression

$$(3p+1)\gamma|\xi|^{-2} - 2\beta|\xi|$$

has minimum value $-3\beta(3p+1)^{1/3}(-\gamma/\beta)^{1/3} > 0$ it follows that

$$-2\beta \int_{\mathbb{R}} (D_x^{1/2}\varphi)^2 dx + (3p+1)\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx \geq -3\beta(3p+1)^{1/3}(-\gamma/\beta)^{1/3} \int_{\mathbb{R}} \varphi^2 dx,$$

so if c satisfies the inequality in (i), the left hand side of (3.2) will be negative, a contradiction. This proves statement (i). Statement (ii) follows similarly.

Next, subtracting the two relations in (3.2), we have

$$-\beta \int_{\mathbb{R}} (D_x^{1/2}\varphi)^2 dx + 2\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx = (p-1) \int_{\mathbb{R}} F(u) dx.$$

The right and left hand sides of this equation have opposite signs when either condition (iii) or condition (iv) holds. \square

4 Ground States and Variational Characterizations

A ground state of (2.1) is a solitary wave of (1.1) which minimizes the action $S(u) = E(u) - cQ(u)$ among all nonzero solutions of (2.1), where $E(u)$ and $Q(u)$ are defined in (1.4) and (1.5), respectively. Recall that a solitary wave of (1.1) corresponds to a critical point of $S(u)$, that is, $S'(u) = 0$. Thus, the set of ground states may be characterized as

$$\mathcal{G}(\beta, c, \gamma) = \{\varphi \in \mathcal{X}; S'(\varphi) = 0, S(\varphi) \leq S(\psi) \text{ for all } \psi \in \mathcal{X} \text{ satisfying } S'(\psi) = 0\}. \quad (4.1)$$

The theorem below finds a ground state of (2.1) as a minimizer for $S(\varphi)$ under a new constraint. Our result is related to that in [24].

Theorem 4.1 *If β , c and γ are as in Theorem 2.1, then $\mathcal{G}(\beta, c, \gamma)$ is nonempty and $\varphi \in \mathcal{G}(\beta, c, \gamma)$ if and only if $S(\varphi)$ solves the minimization problem*

$$J = \inf\{S(u); \psi \in \mathcal{X}, \psi \neq 0, P(\psi) = 0\}, \quad (4.2)$$

where

$$P(\psi) = \int_{\mathbb{R}} (-c\psi^2 + \beta(D_x^{1/2}\psi)^2 + \gamma(\partial_x^{-1}\psi)^2 + (p+1)F(\psi)) dx.$$

Proof. First, we prove that there is a nontrivial minimizer for (4.2) which is a solution of (2.1).

By (2.7), one can easily observe that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for every nontrivial function $\varphi \in \mathcal{N}$, we have $\|\varphi\|_{\mathcal{X}} \geq \varepsilon_1$ and $S(\varphi) \geq \varepsilon_2$, where $\mathcal{N} = \{\psi \in \mathcal{X}; \psi \neq 0, P(\psi) = 0\}$.

Now, let $\{\varphi_n\} \subset \mathcal{N}$ be a minimizing sequence of (4.2). Then $\|\varphi_n\|_{\mathcal{X}} \geq \varepsilon_1$ and

$$S(\varphi_n) = \frac{p-1}{2(p+1)} I(\varphi_n) \cong \frac{p-1}{2(p+1)} \|\varphi_n\|_{\mathcal{X}}^2,$$

so that $\{\varphi_n\}_n$ is bounded in \mathcal{X} . To show that there is a convergent subsequence, with a limit $\varphi \in \mathcal{X}$, similar to [1, 6], we use again the concentration-compactness lemma [23], applied to the sequence

$$\rho_n = |D_x^{1/2}\varphi_n|^2 + |\partial_x^{-1}\varphi_n|^2.$$

First similar to Theorem 2.1, the evanescence case is excluded. To rule out the dichotomy case, one shows that

$$J < J_\sigma := \inf \left\{ S(\psi) - \frac{1}{2}P(\psi); P(\psi) = \sigma \right\},$$

for all $\sigma < 0$. Now if the dichotomy would occur, i.e. φ_n splits into a sum $\varphi_n^1 + \varphi_n^2$ and the distance of the supports of these functions tends to $+\infty$, then one shows that $P(\varphi_n^1) \rightarrow \sigma$, $P(\varphi_n^2) \rightarrow -\sigma$, $\sigma \in \mathbb{R}$ and $J \geq J_\sigma + J_{-\sigma} > J$ which is a contradiction. Therefore the sequence φ_n concentrates and the limit φ satisfies $P(\varphi) \leq 0$. The case $P(\varphi) < 0$ can be excluded by the same reason as above, and we see that $\varphi \in \mathcal{X}$ is a minimizer for (4.2).

Now since φ is a minimizer for (4.2), there exists a Lagrange multiplier θ such that $S'(\varphi) = \theta P'(\varphi)$. Since $\langle S'(\varphi), \varphi \rangle = 0$ and

$$\langle P'(\varphi), \varphi \rangle = 2I(\varphi) - (p+1)K(\varphi) = (1-p)I(\varphi) < 0,$$

we see that $\theta = 0$, i.e. φ is a solution of (2.1).

Our next task is to show that $\varphi \in \mathcal{G}(\beta, c, \gamma)$. But since $P(u) = \langle S'(u), u \rangle_{L^2(\mathbb{R})}$ for any $u \in \mathcal{X}$, it follows that $P(v) = 0$ for any solitary wave $v \in \mathcal{X}$ of (2.1). Hence $S(\varphi) = J$ asserts that $S(\varphi) \leq S(v)$.

Finally we show that a ground state of (2.1) achieves the minimum J in (4.2). Let $u \in \mathcal{X}$ satisfy $u \neq 0$, $S'(u) = 0$ and $S(u) \leq S(v)$ for any $v \in \mathcal{X}$ satisfying $S'(v) = 0$. Since $S'(v) = 0$ implies $P(v) = \langle S'(v), v \rangle_{L^2(\mathbb{R})} = 0$, it follows that $S(u) \leq S(v)$ for any $v \in \mathcal{X}$ with $P(v) = 0$. That is, v is a minimizer for J . This completes the proof. \square

The following proposition proves that minima for M_λ in (2.4) are exactly the ground states of (2.1).

Proposition 4.1 *There is a positive real number λ^* such that the following statements are equivalent:*

- (i) $K(\varphi) = \lambda^*$ and φ is a minimizer of M_{λ^*} in (2.4);
- (ii) φ is a ground state;
- (iii) $P(\varphi) = 0$ and $K(\varphi) = \inf\{K(u); u \in \mathcal{X}, u \neq 0, P(u) = 0\}$;
- (iv) $P(\varphi) = 0 = \inf\{P(u); u \in \mathcal{X}, u \neq 0, K(u) = K(\varphi)\}$.

Proof. (ii) \Rightarrow (i). Let φ be a ground state of (2.1). Since $P(\varphi) = I(\varphi) - K(\varphi) = 0$ and $S(\varphi) = \frac{1}{2}I(\varphi) - \frac{1}{p+1}K(\varphi)$, it follows that φ minimizes I among solutions of (2.1). Set $\lambda^* = K(\varphi) = I(\varphi)$.

Let v be a minimizer for M_{λ^*} . That is, $K(v) = \lambda^*$ and $I(v) = M_{\lambda^*}$ minimizes $I(u)$ among $K(u) = \lambda^*$. In particular $M_{\lambda^*} = I(v) \leq I(\varphi) = \lambda^*$. From variational considerations, v satisfies

$$-cv + \beta \mathcal{H}v_x - \gamma \partial_x^{-2}v = -\theta f(v),$$

for some $\theta \in \mathbb{R}$. Multiplication of the above by v and integration by parts yields $I(v) = \theta K(v)$. Since $I(v) = M_{\lambda^*}$, this implies $\theta \leq 1$. On the other hand, since $w = \theta^{\frac{1}{p-1}}v$ is a solution of (2.1), one obtains $\theta^{\frac{2}{p-1}}I(v) = I(w) \geq I(\varphi)$. Since $I(v) = \theta \lambda^*$ and $I(\varphi) = K(\varphi) = \lambda^*$, this implies $\theta \geq 1$. Therefore, $\theta = 1$ and $I(\varphi) = M_{\lambda^*}$.

(i) \Rightarrow (iii). Suppose $K(\varphi) = \lambda^*$ and $\varphi \in \mathcal{X}$ is a minimizer for M_{λ^*} . Note that $P(\varphi) = 0$ and $I(\varphi) = K(\varphi) = \lambda^*$. Let $u \in \mathcal{X}$ be such that $u \neq 0$ and $P(u) = 0$. Then $K(u) \neq \lambda^*$ so we may define $b = (K(\varphi)/K(u))^{1/(p+1)}$. We show that $b \leq 1$.

Straightforward calculations yield that $P(bu) = b^2(1 - b^{p-1})I(u)$. Since $K(bu) = b^{p+1}K(u) = K(\varphi) = \lambda^*$, it follows that $I(\varphi) \leq I(bu)$, and consequently $0 = P(\varphi) = I(\varphi) - K(\varphi) \leq I(bu) - K(bu) = b^2(1 - b^{p-1})I(u)$. Hence the assertion follows.

(ii) \Leftrightarrow (iii) is a direct consequence of Theorem 4.1.

(iii) \Rightarrow (iv). Let $u \in \mathcal{X}$, $u \neq 0$ with $K(u) = K(\varphi)$, where $\varphi \in \mathcal{X}$ satisfies (iv). We prove that $P(u) \geq 0$. Suppose on the contrary that $P(u) < 0$. Note that $P(\tau u) > 0$ for $\tau \in (0, 1)$ sufficiently small. Correspondingly, $K(u) > 0$ must hold and $P(\tau_0 u) = 0$ for some $\tau_0 \in (0, 1)$. This however contradicts (iii) since $K(\tau_0 u) < K(u) = K(\varphi)$. Therefore, $P(u) \geq 0$. The assertion then follows since $P(\varphi) = 0$.

(iv) \Rightarrow (iii). Let $u \in \mathcal{X}$, $u \neq 0$ with $P(u) = 0$. We show that $K(u) \geq K(\varphi)$, where $\varphi \in \mathcal{X}$ satisfies (iv). Assume the opposite inequality. Similarly as in the previous argument, a scaling consideration shows that $K(\tau_0 u) = \tau_0^{p+1} K(u) = K(\varphi)$, for some $\tau_0 > 1$. This contradicts (iii) since $P(\tau_0 u) < 0 = P(\varphi)$. This completes the proof. \square

5 Weak Rotation Limit

In this section, we show that the solitary waves of the RGBO equation (1.1) converge to those of the generalized Benjamin-ono equation (1.3). We remark that such a relationship is somewhat surprising since the solitary waves of (1.1) have zero mass, as can be seen by integrating (1.1) with respect to x , while it is well-known (see [1, 2, 3, 4, 5] and references therein) that the solitary waves of (1.3) are strictly negative functions and do not have zero mass.

In order to precisely state the convergence result, it is worth noting that for each $c < 0$ and $\beta > 0$ the GBO equation (1.3) possesses a nontrivial solitary wave φ and it satisfies

$$-c\varphi + \beta\mathcal{H}\varphi' + f(\varphi) = 0. \quad (5.1)$$

The uniqueness of solitary waves of the GBO equation for $p > 1$ is unknown; however Amick and Toland [7] showed that the solitary wave solutions of the classical Benjamin-Ono ($p = 1$) are unique (up to translation). The explicit solution was found by Benjamin [11]:

$$\varphi(\xi) = \frac{4c\beta^2}{\beta^2 + c^2\xi^2}. \quad (5.2)$$

One can see that contrary to the unique solitary wave of the KdV equation, the solitary wave of the Benjamin-Ono equation does not decay exponentially.

Theorem 5.1 *For $\beta > 0$ and $c < 0$ fixed, let a sequence $\gamma_n \rightarrow 0^+$ as $n \rightarrow \infty$, and let ψ_n any element of $G(\beta, c, \gamma_n)$. Then there exists a subsequence (still denoted as γ_n), translations y_n and a solitary wave $\psi \in H^{1/2}(\mathbb{R})$ of (5.1) so that $\psi_n(\cdot + y_n) \rightarrow \psi$ in $H^{1/2}(\mathbb{R})$, as $\gamma_n \rightarrow 0^+$. That is, the solitary waves of the GBO equation are the limits in $H^{1/2}(\mathbb{R})$ of solitary waves of the RGBO equation.*

To prove this result, we first note for $\beta > 0$ and $c < 0$ that solutions of (5.1) satisfies in a variational problem of the type of Theorem 2.1. More precisely, ground states of (5.1) achieve the minimum

$$m(\beta, c, 0) = \inf \left\{ \frac{I(u; \beta, c, 0)}{K(u)^{\frac{2}{p+1}}}; u \in H^{1/2}(\mathbb{R}), K(u) > 0 \right\},$$

where $I(u; \beta, c, 0) = \int_{\mathbb{R}} (\beta(D_x^{1/2}u)^2 - cu^2)dx$. Analogous to Theorem 2.1, one can show that for a given sequence ψ_n in $H^{1/2}(\mathbb{R})$ satisfying

$$\lim_{n \rightarrow \infty} I(\psi_n; \beta, c, 0) = \lim_{n \rightarrow \infty} K(\psi_n) = (m(\beta, c, 0))^{\frac{p+1}{2}},$$

there exists a subsequence, renamed ψ_n , scalars $y_n \in \mathbb{R}$ and $\varphi_0 \in H^{1/2}(\mathbb{R})$ such that $\psi_n(\cdot + y_n) \rightarrow \varphi_0$ in $H^{1/2}(\mathbb{R})$.

The proof of Theorem 5.1 is approached via the following lemmas.

Lemma 5.1 *The function m is continuous on the domain $\beta > 0$, $\gamma > 0$, $c < c_*$. Furthermore, m is strictly increasing in γ and β and strictly decreasing in c .*

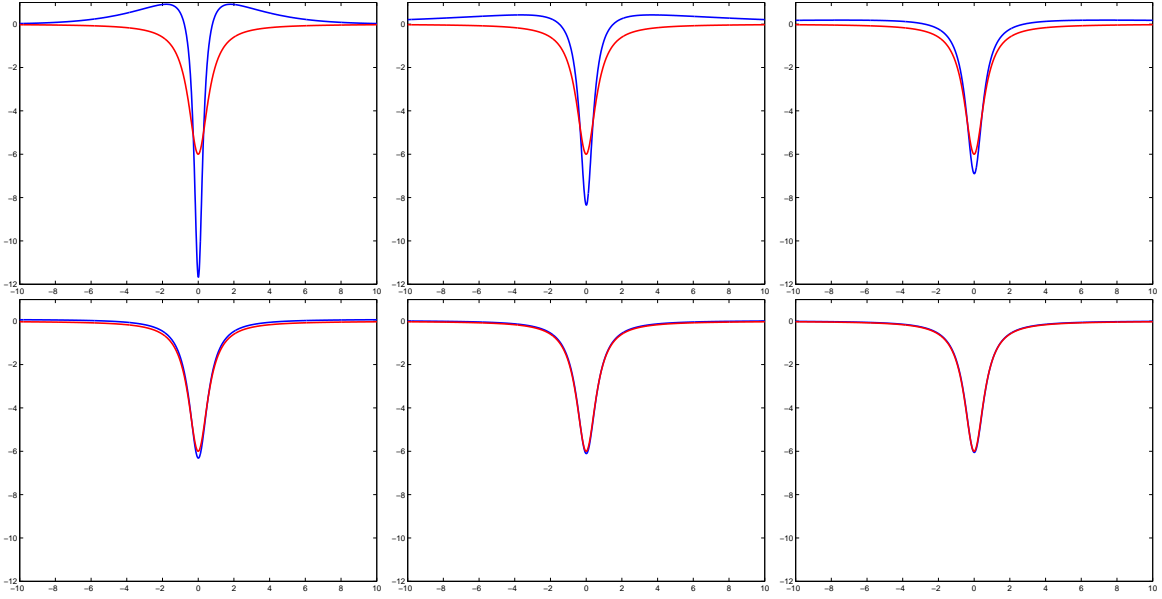


Figure 2: Solitary waves of the rotation generalized Benjamin-Ono equation with $f(u) = u^2$, $\beta = 2$, $c = -3$ and $\gamma = 1, 0.1, 0.01, 0.0001, 0.00001$ are shown in blue. The exact solitary wave solution of the Benjamin-Ono given by (5.2) is shown in red.

Proof. The proof is similar to [20, Lemma 2.3], [21, Lemma 2.4] and [25, Lemma 3.3] by using the following inequality

$$I(u; \beta, c, \gamma) \geq (c_* - c) \int_{\mathbb{R}} u^2 dx, \quad (5.3)$$

where c_* is defined in Theorem 2.1. □

Lemma 5.2 *The space \mathcal{X} is dense in $H^{1/2}(\mathbb{R})$.*

Proof. For any $u \in H^{1/2}(\mathbb{R})$ and $\delta > 0$, let us define u_δ as $\widehat{u}_\delta(\xi) = \widehat{u}(\xi)\chi_{|\xi|>\delta}(\xi)$. By Parseval's identity follow that

$$\|\partial_x^{-1} u_\delta\|_{L^2(\mathbb{R})}^2 = \|\xi^{-1} \widehat{u}_\delta\|_{L^2(\mathbb{R})}^2 = \int_{|\xi|>\delta} \xi^{-2} |\widehat{u}(\xi)|^2 d\xi < \delta^{-2} \|u\|_{L^2(\mathbb{R})}^2 < +\infty.$$

Since $\|u_\delta\|_{L^2(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})} < +\infty$ and since $\|D_x^{1/2} u_\delta\|_{L^2(\mathbb{R})} \leq \|D_x^{1/2} u\|_{L^2(\mathbb{R})} < +\infty$, it follows that $u_\delta \in \mathcal{X}$. In view of the definition of u_δ and $u \in H^{1/2}(\mathbb{R})$ then the inequality

$$\|u_\delta - u\|_{H^{1/2}(\mathbb{R})}^2 = \int_{|\xi|<\delta} (1 + |\xi|) |\widehat{u}(\xi)|^2 d\xi \leq \|u\|_{H^{1/2}(\mathbb{R})}^2 < +\infty$$

holds true. Hence from continuity we may choose $\delta > 0$ sufficiently small so that

$$\|u_\delta - u\|_{H^{1/2}(\mathbb{R})}^2 = \int_{|\xi|<\delta} (1 + |\xi|) |\widehat{u}(\xi)|^2 d\xi < \epsilon,$$

which completes the proof. □

Proof of Theorem 5.1. For $\beta > 0$ and $c < 0$ let $\{\psi_n\}$ be a sequence in \mathcal{X} of the ground states of (2.1) with $\gamma = \gamma_n$, where $\gamma_n \rightarrow 0^+$ as $n \rightarrow \infty$. It is immediate that $I(\psi_n; \beta, c, \gamma) = K(\psi_n) = m(\beta, c, \gamma_n)^{\frac{p+1}{p-1}}$ holds for each n . Below we prove the continuity of $m(\beta, c, \gamma)$ at $\gamma = 0$, that is, $\lim_{\gamma \rightarrow 0^+} m(\beta, c, \gamma) = m(\beta, c, 0)$. The assertion then follows from

$$I(\psi_n; \beta, c, 0) = I(\psi_n; \beta, c, \gamma_n) - \gamma_n \|\partial_x^{-1} \psi_n\|_{L^2(\mathbb{R})}^2 \leq I(\psi_n; \beta, c, \gamma_n) = m(\beta, c, \gamma_n)^{\frac{p+1}{p-1}} \longrightarrow m(\beta, c, 0)^{\frac{p+1}{p-1}}$$

and

$$K(\psi_n) = m(\beta, c, \gamma_n)^{\frac{p+1}{p-1}} \longrightarrow m(\beta, c, 0)^{\frac{p+1}{p-1}}.$$

We now claim that $\lim_{\gamma \rightarrow 0^+} m(\beta, c, \gamma) = m(\beta, c, 0)$. By the monotonicity of $m(\beta, c, \gamma)$ in γ , it suffices to show that $m(\beta, c, \gamma_n) \rightarrow m(\beta, c, 0)$ for some sequence $\{\gamma_n\}$ with $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\varphi \in H^{1/2}(\mathbb{R})$ be a ground state of (5.1). For each n a positive integer it follows from lemma 5.2 that there is a function $\psi_n \in \mathcal{X}$ with $\|\psi_n - \varphi\|_{H^{1/2}(\mathbb{R})} < 1/n$. Let

$$\gamma_n = \min \left\{ \frac{1}{n}, \frac{1}{n} \|\partial_x^{-1} \psi_n\|_{L^2(\mathbb{R})}^{-2} \right\}.$$

Then $\gamma_n \rightarrow 0$ and

$$m(\beta, c, \gamma_n) \leq \frac{I(\psi_n; \beta, c, \gamma)}{K(\psi_n)^{\frac{2}{p+1}}} = \frac{I(\psi_n; \beta, c, 0) + \gamma_n \|\partial_x^{-1} \psi_n\|_{L^2(\mathbb{R})}^2}{K(\psi_n)^{\frac{2}{p+1}}} \leq \frac{I(\psi_n; \beta, c, 0) + 1/n}{K(\psi_n)^{\frac{2}{p+1}}}.$$

Since both $I(\cdot; \beta, c, 0)$ and K are continuous on $H^{1/2}(\mathbb{R})$, it follows that

$$\lim_{n \rightarrow \infty} m(\beta, c, \gamma_n) \leq \frac{I(\varphi; \beta, c, 0)}{K(\varphi)^{\frac{2}{p+1}}} = m(\beta, c, 0).$$

On the other hand, since $m(\beta, c, \gamma)$ is strictly increasing in γ , it follows that

$$\lim_{n \rightarrow \infty} m(\beta, c, \gamma_n) = m(\beta, c, 0).$$

This proves the claim. The proof is complete. \square

6 Stability

In this section we investigate the stability of the set $\mathcal{G}(\beta, c, \gamma)$ of ground state solitary waves. We first state precisely our definition of stability.

Definition 6.1 A set $\Omega \in \mathcal{X}$ is \mathcal{X} -stable with respect to (1.1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in \mathcal{X} \cap X_s$, $s > 3/2$, with

$$\inf_{v \in \Omega} \|u_0 - v\|_{\mathcal{X}} < \delta,$$

then the solution $u(t)$ of (1.1) with initial value $u(0) = u_0$ can be extended to a solution in the space $C([0, +\infty), \mathcal{X} \cap X_s)$ and satisfies

$$\sup_{t \geq 0} \inf_{v \in \Omega} \|u(t) - v\|_{\mathcal{X}} < \varepsilon.$$

Otherwise we say that Ω is \mathcal{X} -unstable.

Since ground state solitary waves minimize the action $S(u) = E(u) - cQ(u)$, it is natural to consider the function

$$d(c) = d(\beta, c, \gamma) = E(\varphi) - cQ(\varphi), \quad (6.1)$$

where φ is any element of $\mathcal{G}(\beta, c, \gamma)$. The fact that d is well-defined follows from the relation

$$d(\beta, c, \gamma) = \frac{1}{2}I(\varphi) - \frac{1}{p+1}K(\varphi) = \frac{p-1}{2(p+1)}K(\varphi) = \frac{p-1}{2(p+1)}m(\beta, c, \gamma)^{\frac{p+1}{p-1}}. \quad (6.2)$$

Together with Lemma 5.1, this relation also implies that d is continuous on the domain $\beta > 0$, $\gamma > 0$, $c < c_*$, strictly increasing in γ and β and strictly decreasing in c . It can also be shown as in [20] and [21] that d has the following differentiability properties.

Lemma 6.1 *For each fixed $\beta > 0$ and $\gamma > 0$, the partial derivative $d_c(\beta, c, \gamma)$ exists for all but countably many c . For fixed c and γ , $d_\beta(\beta, c, \gamma)$ exists for all but countably many β and for fixed β and c , $d_\gamma(\beta, c, \gamma)$ exists for all but countably many γ . At points of differentiability, we have*

$$\begin{aligned} d_\beta(\beta, c, \gamma) &= \frac{1}{2} \int (D_x^{1/2} \varphi)^2 dx \\ d_c(\beta, c, \gamma) &= -\frac{1}{2} \int \varphi^2 dx = -Q(\varphi) \\ d_\gamma(\beta, c, \gamma) &= \frac{1}{2} \int (\partial_x^{-1} \varphi) dx \end{aligned}$$

For the remainder of this section we shall regard $\beta > 0$ and $\gamma > 0$ as fixed and denote $d(c) = d(\beta, c, \gamma)$, $d'(c) = d_c(\beta, c, \gamma)$ and $d''(c) = d_{cc}(\beta, c, \gamma)$. The main stability result is that the stability of the set of ground states is determined by the sign of $d''(c)$.

Theorem 6.1 *Let $\beta > 0$, $\gamma > 0$, $c < c_*$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$. If $d''(c) > 0$, then the set of ground states $\mathcal{G}(\beta, c, \gamma)$ is \mathcal{X} -stable.*

Define the ϵ -neighborhood of the set of ground states defined by

$$U_\epsilon = \{u \in \mathcal{X}; \inf_{\varphi \in \mathcal{G}(\beta, c, \gamma)} \|u - \varphi\|_{\mathcal{X}} < \epsilon\}.$$

Since d is strictly decreasing in c and K is continuous on \mathcal{X} , we may define

$$c(u) = d^{-1} \left(\frac{p-1}{2(p+1)} K(u) \right)$$

for $u \in U_\epsilon$ for sufficiently small $\epsilon > 0$.

Lemma 6.2 *If $d''(c) > 0$, then there is some $\epsilon > 0$ such that for any $u \in U_\epsilon$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$, we have*

$$E(u) - E(\varphi) - c(u)(Q(u) - Q(\varphi)) \geq \frac{1}{4}d''(c)|c(u) - c|^2. \quad (6.3)$$

Proof. Since $d'(c) = -Q(\varphi)$, it follows from Taylor's theorem that

$$d(c_1) = d(c) - Q(\varphi)(c_1 - c) + \frac{1}{2}(c_1 - c)^2 + o(|c_1 - c|^2),$$

for c_1 near c . Using the continuity of $c(u)$ and choosing $\epsilon > 0$ sufficiently small we get that

$$d(c(u)) \geq d(c) - Q(\varphi)(c(u) - c) + \frac{1}{4}d''(c)(c(u) - c)^2 = E(\varphi) - c(u)Q(\varphi) + \frac{1}{4}(c(u) - c)^2,$$

for $u \in U_\epsilon$. Next, if $\varphi_{c(u)} \in \mathcal{G}(\beta, c(u), \gamma)$; then $K(\varphi_{c(u)}) = 2(p+1)d(c(u))/(p-1) = K(u)$ and $\varphi_{c(u)}$ minimizes $I(\cdot; \beta, c(u), \gamma)$ subject to this constraint, so

$$E(u) - c(u)Q(u) = \frac{1}{2}I(u; \beta, c(u), \gamma) - \frac{1}{p+1}K(u) \geq \frac{1}{2}I(\varphi_{c(u)}; \beta, c(u), \gamma) - \frac{1}{p+1}K(\varphi_{c(u)}) = d(c(u)).$$

This concludes the proof of Lemma 6.2. \square

Proof of Theorem 6.1. Assume that $\mathcal{G}(\beta, c, \gamma)$ is \mathcal{X} -unstable with regard to the flow of the RGBO equation. Then there exists a sequence of the initial data $u_k(0)$ such that

$$\inf_{\varphi \in \mathcal{G}(\beta, c, \gamma)} \|u_k(0) - \varphi\|_{\mathcal{X}} < \frac{1}{k}.$$

Let $u_k(t)$ be the solution of (1.1) with initial data $u_k(0)$. We can also choose $\delta > 0$ and a sequence of times t_k such that

$$\inf_{\varphi \in \mathcal{G}(\beta, c, \gamma)} \|u_k(t) - \varphi\|_{\mathcal{X}} = \delta. \quad (6.4)$$

Moreover we can find $\varphi \in \mathcal{G}(\beta, c, \gamma)$ such that

$$\lim_{k \rightarrow \infty} \|u_k(0) - \varphi_k\|_{\mathcal{X}} = 0.$$

Since E and V are conserved by the flow of (1.1),

$$\lim_{k \rightarrow \infty} E(u_k(t_k)) - E(\varphi_k) = \lim_{k \rightarrow \infty} E(u_k(0)) - E(\varphi_k) = 0 \quad (6.5)$$

and

$$\lim_{k \rightarrow \infty} Q(u_k(t_k)) - Q(\varphi_k) = \lim_{k \rightarrow \infty} Q(u_k(0)) - Q(\varphi_k) = 0. \quad (6.6)$$

By using Lemma 6.2, we have for δ sufficiently small that

$$E(u_k(t_k)) - E(\varphi_k) - c(u_k(t_k))(Q(u_k(t_k)) - Q(\varphi_k)) \geq \frac{1}{4}d''(c)|c(u_k(t_k)) - c|^2. \quad (6.7)$$

By (6.4) there is some $\psi_k \in \mathcal{G}(\beta, c, \gamma)$ such that $\|u_k(t_k)\|_{\mathcal{X}} < 2\delta$, and by using the fact $I(u) = I(u; \beta, c, \gamma) \geq C\|u\|_{\mathcal{X}}^2$, we obtain

$$\|u_k(t_k)\|_{\mathcal{X}} \leq \|\psi_k\|_{\mathcal{X}} + 2\delta \leq C^{-1}I(\psi_k; \beta, c, \gamma) + 2\delta = \frac{2(p+1)}{C(p-1)}d(c) + 2\delta < \infty.$$

Thus since K is Lipschitz continuous on \mathcal{X} and d^{-1} is continuous, it follows that $c(u_k(t_k))$ is uniformly bounded in k . Thus by (6.5)-(6.7) it follows that $\lim_{k \rightarrow \infty} c(u_k(t_k)) = c$; and therefore

$$\lim_{k \rightarrow \infty} K(u_k(t_k)) = \lim_{k \rightarrow \infty} \frac{2(p+1)}{(p-1)}d(c(u_k(t_k))) = \frac{2(p+1)}{(p-1)}d(c). \quad (6.8)$$

This implies that

$$\begin{aligned} \frac{1}{2}I(u_k(t_k)) &= E(u_k(t_k)) - cQ(u_k(t_k)) + \frac{1}{p+1}K(u_k(t_k)) \\ &= d(c) + E(u_k(t_k)) - E(\varphi_k) - c(Q(u_k(t_k)) - Q(\varphi_k)) + \frac{1}{p+1}K(u_k(t_k)). \end{aligned}$$

Hence it follows from (6.5), (6.6) and (6.8) that $\lim_{k \rightarrow \infty} I(u_k(t_k)) = 2(p+1)d(c)/(p-1)$. Thus $u_k(t_k)$ is a minimizing sequence and therefore has a subsequence which converges in \mathcal{X} to some $\varphi \in \mathcal{G}(\beta, c, \gamma)$. This contradicts (6.4), so the proof of the theorem is complete. \square

7 Instability

In this section we present conditions that imply orbital instability of ground state solitary waves. Given $\varphi \in \mathcal{G}(\beta, c, \gamma)$ and $\epsilon > 0$, we define

$$\Omega_{\varphi, \epsilon} = \left\{ u \in \mathcal{X}; \inf_{v \in \mathcal{O}_{\varphi}} \|v - u\|_{\mathcal{X}} < \epsilon \right\}.$$

Theorem 7.1 *Let $\beta > 0$, $\gamma > 0$, $c < c_*$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$. Suppose there exists $\psi \in L^2(\mathbb{R})$ such that $\psi' \in X_s$, $s > 3/2$, $\psi'' \in \mathcal{X}$, and the following two conditions hold.*

$$\begin{aligned} \langle \psi', \varphi \rangle &= 0, \\ \langle S''(\varphi)\psi', \psi' \rangle &< 0. \end{aligned} \tag{7.1}$$

Then \mathcal{O}_{φ} is \mathcal{X} -unstable.

Lemma 7.1 *Let $c < c_*$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$ be fixed. There are an $\epsilon_0 > 0$ and a unique C^2 map $\alpha : \Omega_{\varphi, \epsilon_0} \rightarrow \mathbb{R}$ such that $\alpha(\varphi) = 0$, and for all $v \in \Omega_{\varphi, \epsilon_0}$ and any $r \in \mathbb{R}$,*

$$(i) \quad \langle \tau_{\alpha(v)}\varphi', v \rangle = 0,$$

$$(ii) \quad \alpha(\tau_r v) = \alpha(v) + r,$$

$$(iii) \quad \alpha'(v) = -\frac{1}{\langle v, \varphi''(\cdot + \alpha(v)) \rangle} \varphi'(\cdot + \alpha(v)), \text{ and}$$

$$(iv) \quad \langle \alpha'(v), v \rangle = 0 \text{ and } \alpha'(v) = \|\varphi'\|_{L^2(\mathbb{R})}^{-2} v', \text{ if } v \in \mathcal{O}_{\varphi}.$$

Proof. The proof follows the line of reasoning laid down in Theorem 3.1 in [17] and Lemma 3.8 in [26]. \square

Let ψ be as in Theorem 7.1. Define another vector field B_{ψ} by

$$B_{\psi}(u) = \tau_{\alpha(u)}\psi' - \frac{\langle u, \tau_{\alpha(u)}\psi' \rangle}{\langle u, \tau_{\alpha(u)}\varphi'' \rangle} \tau_{\alpha(u)}\varphi'',$$

for $u \in \Omega_{\varphi, \epsilon}$. Geometrically, B_{ψ} can be interpreted as the derivative of the orthogonal component of $\tau_{\alpha(\cdot)}\psi$ with regard to $\tau_{\alpha(\cdot)}\varphi'$.

Lemma 7.2 *Let ψ be as in Theorem 7.1. Then the map $B_{\psi} : \Omega_{\varphi, \epsilon_0} \rightarrow \mathcal{X}$ is C^1 with bounded derivative. Moreover,*

$$(i) \quad B_{\psi} \text{ commutes with translations,}$$

$$(ii) \quad \langle B_{\psi}(u), u \rangle = 0, \text{ if } u \in \Omega_{\varphi, \epsilon_0},$$

$$(iii) \quad B_{\psi}(\varphi) = \psi', \text{ if } \langle \varphi, \psi' \rangle = 0.$$

Proof. The proof follows the same lines from the proof of Lemma 3.5 in [8], Lemma 3.3 in [9] or Lemma 4.7 in [20]. \square

Proof of Theorem 7.1. First we claim that there exist $\epsilon_3 > 0$ and $\sigma_3 > 0$ such that for each $u_0 \in \Omega_{\varphi, \epsilon_3}$,

$$S(\varphi) \leq S(u_0) + \mathcal{P}(u_0)s, \tag{7.2}$$

for some $s \in (-\sigma_3, \sigma_3)$, where $\mathcal{P}(u) = \langle S'(u), B_\psi(u) \rangle$.

We consider $u_0 \in \Omega_{\varphi, \epsilon_0}$, where ϵ_0 is given in Lemma 7.1, the initial value problem

$$\begin{aligned} \frac{d}{ds}u(s) &= B_\psi(u(s)) \\ u(0) &= u_0. \end{aligned} \quad (7.3)$$

By Lemma 7.2, we have that (7.3) admits for each $u_0 \in \Omega_{\varphi, \epsilon_0}$ a unique maximal solution $u \in C^2((-\sigma, \sigma); \Omega_{\varphi, \epsilon_0})$, where $\sigma \in (0, +\infty]$. Moreover for each $\epsilon_1 < \epsilon$, there exists $\sigma_1 > 0$ such that $\sigma(u_0) \geq \sigma_1$, for all $u_0 \in \Omega_{\varphi, \epsilon_1}$. Hence we can define for fixed ϵ_1, σ_1 , the following dynamical system

$$\begin{aligned} \mathcal{U} : (-\sigma_1, \sigma) \times \Omega_{\varphi, \epsilon_1} &\longrightarrow \Omega_{\varphi, \epsilon_0} \\ (s, u_0) &\mapsto \mathcal{U}(s)u_0, \end{aligned}$$

where $s \rightarrow \mathcal{U}(s)u_0$ is the maximal solution of (7.3) with initial data u_0 . It is also clear from Lemma 7.2 that \mathcal{U} is a C^1 -function, also we have that for each $u_0 \in \Omega_{\varphi, \epsilon_1}$, the function $s \rightarrow \mathcal{U}(s)u_0$ is C^2 for each $s \in (-\sigma_1, \sigma_1)$, and the flow $s \rightarrow \mathcal{U}(s)u_0$ commutes with translations. One can also observe from the relation

$$\mathcal{U}(t)\varphi = \varphi + \int_0^t \tau_{\alpha(\mathcal{U}(s)\varphi)}\psi' ds - \int_0^t \rho(s)\tau_{\alpha(\mathcal{U}(s)\varphi)}\varphi'' ds$$

that $\mathcal{U}(s)\varphi \in X_r$, $r > 3/2$, for all $s \in (-\sigma_1, \sigma_1)$, where

$$\rho(s) = \frac{\langle \mathcal{U}(s)\varphi, \tau_{\alpha(\mathcal{U}(t)\varphi)}\psi' \rangle}{\langle \mathcal{U}(t)\varphi, \tau_{\alpha(\mathcal{U}(t)\varphi)}\varphi'' \rangle}.$$

Now we get from Taylor's theorem that there is $\varrho \in (0, 1)$ such that

$$S(\mathcal{U}(s)u_0) = S(u_0) + \mathcal{P}(u_0)s + \frac{1}{2}R(\mathcal{U}(\varrho s)u_0)s^2,$$

where $R(u) = \langle S''(u)B_\psi, B_\psi(u) \rangle + \langle S''(u), B'_\psi(u)(B_\psi(u)) \rangle$. Since R and \mathcal{P} are continuous, $S'(\varphi) = 0$ and $R(\varphi) < 0$, then there exists $\epsilon_2 \in (0, \epsilon_1]$ and $\sigma_2 \in (0, \sigma_1]$ such that (7.2) holds for $u_0 \in B(\varphi, \epsilon_2)$ and $s \in (-\sigma_2, \sigma_2)$. On the other hand, it is straightforward to verify that

$$P(\mathcal{U}(s)u_0)|_{(u_0, s)=(\varphi, 0)} = 0 \quad \text{and} \quad \frac{d}{ds}P(\mathcal{U}(s)u_0)|_{(u_0, s)=(\varphi, 0)} = \langle P'(\varphi), \psi' \rangle,$$

where P is defined in Theorem 4.1. We show that $\langle P'(\varphi), \psi' \rangle \neq 0$. Otherwise, ψ' would be tangent to \mathcal{N} at φ , is defined in Theorem 4.1. Hence, $\langle S''(\varphi)\psi', \psi' \rangle \geq 0$, since φ minimizes S on \mathcal{N} by Theorem 4.1. But this contradicts (7.1). Therefore, by the implicit function theorem, there exist $\epsilon_3 \in (0, \epsilon_2)$ and $\sigma_3 \in (0, \sigma_2)$ such that for all $u_0 \in B\varphi, \epsilon_3$, there exists a unique $s = s(u_0) \in (-\sigma_3, \sigma_3)$ such that $P(\mathcal{U}(s)u_0) = 0$. Then applying (7.2) to $(u_0, s(u_0)) \in B\varphi, \epsilon_3 \times (-\sigma_3, \sigma_3)$ and using the fact φ minimizes S on \mathcal{N} , we have that for $u_0 \in B\varphi, \epsilon_3$ there exists $s \in (-\sigma_3, \sigma_3)$ such that $S(\varphi) \leq S(\mathcal{U}(s)u_0) \leq S(u_0) + \mathcal{P}(u_0)s$. This inequality can be extended to $\Omega_{\varphi, \epsilon_3}$ from from the gauge invariance.

Since $\mathcal{U}(s)u_0$ commutes with τ_r , it follows by replacing u_0 with $\mathcal{U}(s)u_0$ in (7.2) and then $\delta = -s$ that

$$S(\varphi) \leq S(\mathcal{U}(\delta)\varphi) - \mathcal{P}(\mathcal{U}(\delta)\varphi)\delta, \quad (7.4)$$

for all $\delta \in (-\sigma_3, \sigma_3)$. Moreover, using Taylor's theorem again and the fact $\mathcal{P}(\varphi) = 0$, it follows that the map $\delta \mapsto S(\mathcal{U}(\delta)\varphi)$ has a strict local maximum at $\delta = 0$. Hence, we obtain

$$S(\mathcal{U}(\delta)\varphi) < S(\varphi), \quad \delta \neq 0, \delta \in (-\sigma_4, \sigma_4), \quad (7.5)$$

where $\sigma_4 \in (0, \sigma_3]$. Thus it follows from (7.4) that

$$\mathcal{P}(\mathcal{U}(\delta)\varphi) < 0, \quad \delta \in (0, \sigma_4). \quad (7.6)$$

Let $\delta_j \in (0, \sigma_4)$ such that $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. Consider the sequences of initial data $u_{0,j} = \mathcal{U}(\delta_j)\varphi$. It is clear to see that $u_{0,j} \in X_s$, $s > 3/2$ for all positive integers j and $u_{0,j} \rightarrow \varphi$ in \mathcal{X} as $j \rightarrow \infty$.

Now we need only verify that the solution $u_j(t) = \mathcal{U}(t)u_{0,j}$ of (1.1) with $u_j(0) = u_{0,j}$ escapes from $\Omega_{\varphi, \epsilon_3}$, for all positive integers j in finite time. Define

$$T_j = \sup\{t' > 0; u_j(t) \in \Omega_{\varphi, \epsilon_3}, \forall t \in (0, t')\}$$

and

$$\mathcal{D} = \{u \in \Omega_{\varphi, \epsilon_3}; S(u) < S(\varphi), \mathcal{P}(u) < 0\}.$$

Hence it follows from (7.2) that for all $j \in \mathbb{N}$ and $t \in (0, T_j)$, there exists $s = s_j(t) \in (-\sigma_3, \sigma_3)$ satisfying $S(\varphi) \leq S(u_{0,j}) + \mathcal{P}(u_j(t))s$. By (7.5) and (7.6), $u_{0,j} \in \mathcal{D}$; and therefore $u_j(t) \in \mathcal{D}$ for all $t \in [0, T_j]$. Indeed, if $\mathcal{P}(u_j(t_0)) > 0$ for some $t_0 \in [0, T_j]$, then the continuity of \mathcal{P} implies that there exists some $t_1 \in [0, T_j]$ satisfying $\mathcal{P}(u_j(t_1)) = 0$, and consequently $S(\varphi) \leq S(u_{0,j})$, which contradicts $u_{0,j} \in \mathcal{D}$. Hence, \mathcal{D} is bounded away from zero and

$$-\mathcal{P}(u_j) \geq \frac{S(\varphi) - S(u_{0,j})}{\sigma_3} - \eta_j > 0, \quad \forall t \in [0, T_j]. \quad (7.7)$$

Now suppose that for some j , $T_j = +\infty$. Then we define a Liapunov function

$$A(t) = \int_{\mathbb{R}} \psi(x + \alpha(u_j))u_j(x, t)dx, \quad t \in [0, T_j].$$

Then by the Cauchy-Schwarz inequality,

$$|A(t)| \leq \|\psi\|_{L^2(\mathbb{R})} \|u_j(t)\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(\mathbb{R})} \|u_{0,j}\|_{L^2(\mathbb{R})} < \infty, \quad t \in [0, T_j].$$

On the other hand, since $\frac{du_j}{dt} = -\partial_x E'(u_j)$, then we have

$$\begin{aligned} \frac{dA}{dt} &= \left\langle \alpha'(u_j(t)), \frac{du_j}{dt} \right\rangle \langle \tau_{\alpha(u_j(t))} \psi', u_j(t) \rangle + \left\langle \tau_{\alpha(u_j(t))}, \frac{du_j}{dt} \right\rangle \\ &= \langle \tau_{\alpha(u_j(t))} \psi', u_j(t) \rangle \partial_x \alpha'(u_j(t)) + \tau_{\alpha(u_j(t))} \psi', E'(u_j(t)) \rangle \\ &= \langle B_\psi(u_j(t)), S'(u_j(t)) \rangle + c \langle B_\psi(u_j(t)), u_j(t) \rangle = \mathcal{P}(u_j(t)), \end{aligned}$$

for $t \in [0, T_j]$. Therefore it is deduced from (7.7) that

$$-\frac{dA}{dt} \geq \eta_j > 0, \quad \forall t \in [0, T_j].$$

This contradicts the boundedness of $A(t)$. Consequently $T_j < +\infty$ for all j , which means that u_j eventually leaves $\Omega_{\varphi, \epsilon_3}$. This completes the proof. \square

Theorem 7.2 Fix $\beta > 0$, $\gamma > 0$ and assume there exists a C^2 map $c \mapsto \varphi_c \in \mathcal{G}(\beta, c, \gamma)$ for $c < c_*$. If $d''(c) < 0$, then \mathcal{O}_{φ_c} is \mathcal{X} -unstable.

Proof. It suffices to show that there exists a function ψ that satisfies the conditions of Theorem 7.1. Define

$$\psi(x) = \int_{-\infty}^x \varphi_c(y) - \frac{2d'(c)}{d''(c)} \frac{d}{dc} \varphi_c(y) dy.$$

Then since $\varphi_c \in \mathcal{X}$ and $\frac{d}{dc} \varphi_c \in \mathcal{X}$ it follows that $\psi' \in \mathcal{X}$, and thus $\psi \in L^2$. Since $w = \frac{d}{dc} \varphi_c$ satisfies the linear equation

$$\beta \mathcal{H}(w_x) - cw - \gamma \partial_x^{-1} w - f'(\varphi)w = \varphi$$

it follows as in the proof of Theorem 2.2 that $w \in H^\infty$ and $\partial_x^{-1}w \in H^\infty$. Hence $\psi' \in X_s$ and $\psi'' \in \mathcal{X}$. Now since $d'(c) = -\frac{1}{2}\langle \varphi_c, \varphi_c \rangle$, we have

$$\langle \psi', \varphi_c \rangle = \langle \varphi_c, \varphi_c \rangle - \frac{2d'(c)}{d''(c)} \frac{1}{2} \frac{d}{dc} \langle \varphi_c, \varphi_c \rangle = -2d'(c) + \frac{2d'(c)}{d''(c)} d''(c) = 0.$$

Next we compute

$$\langle S''(\varphi)\psi', \psi' \rangle = \langle S''(\varphi_c)\varphi_c, \varphi_c \rangle - \frac{4d'(c)}{d''(c)} \left\langle S''(\varphi_c)\varphi_c, \frac{d}{dc}\varphi_c(y) \right\rangle + \frac{4d'(c)^2}{d''(c)^2} \left\langle S''(\varphi_c) \frac{d}{dc}\varphi_c(y), \frac{d}{dc}\varphi_c(y) \right\rangle$$

For any $\varphi \in \mathcal{G}(\beta, c, \gamma)$ we have $S''(\varphi)\varphi = (p-1)f(\varphi)$, so it follows that

$$\langle S''(\varphi)\varphi, \varphi \rangle = (1-p)K(\varphi). \quad (7.8)$$

Since $d(c) = \frac{p-1}{2(p+1)}K(\varphi_c)$ we have

$$\left\langle S''(\varphi)\varphi_c, \frac{d}{dc}\varphi_c(y) \right\rangle = \left\langle (p-1)f(\varphi_c), \frac{d}{dc}\varphi_c(y) \right\rangle = -\frac{p-1}{p+1} \frac{d}{dc} K(\varphi_c) = -2d'(c).$$

Finally, since $S''(\varphi_c) \frac{d}{dc}\varphi_c(y) = \varphi_c$ we have

$$\left\langle S''(\varphi_c) \frac{d}{dc}\varphi_c(y), \frac{d}{dc}\varphi_c(y) \right\rangle = \left\langle \varphi_c, \frac{d}{dc}\varphi_c(y) \right\rangle = -d''(c).$$

Altogether this implies

$$\langle S''(\varphi)\psi', \psi' \rangle = (1-p)K(\varphi_c) + \frac{4(d'(c))^2}{d''(c)}.$$

Since $p > 1$ and $d''(c) < 0$, both terms on the right hand side are negative. Thus ψ' satisfies all of the conditions of Theorem 7.1. \square

8 Applications of the Stability and Instability Theorems

In this section we apply the stability and instability conditions in Theorems 6.1, 7.1 and 7.2 to determine conditions on p, β, c and γ that imply stability or instability. We first apply Theorem 7.1 with $\psi' = \varphi + 2x\varphi'$.

Lemma 8.1 *Let $c < c_*$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$. Define*

$$\psi(x) = \int_{-\infty}^x \varphi(y) + 2y\varphi'(y) dy.$$

Then ψ satisfies the assumptions of Theorem 7.1 and

$$\langle S''(\varphi)\psi', \psi' \rangle = \frac{(p-1)(3-p)}{p+1} K(\varphi) + 12\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx.$$

Proof. The first part of the lemma is clear by using the fact $\langle \varphi, \psi' \rangle = 0$ and Theorems 2.2 and 2.4. Now we estimate the quantity $\langle S''(\varphi)\psi', \psi' \rangle$. First by (2.1), we note that $S'' = \beta \mathcal{H} \partial_x - \gamma \partial_x^{-2} - c - f'(\varphi)$. Next, using (3.1), we see that

$$\langle S''(\varphi), \varphi \rangle = (1-p)K(\varphi). \quad (8.1)$$

Next using again (3.1) an the facts $F' = f$ and $pf(\varphi) = f'(\varphi)\varphi$, it yields that

$$\langle S''(\varphi), x\varphi' \rangle = \int_{\mathbb{R}} (p-1)x\varphi' f(\varphi) dx = \frac{p-1}{1+p} K(\varphi). \quad (8.2)$$

Finally we show that $\langle S''(x\varphi'), x\varphi' \rangle = 3\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx$.

First we observe from (2.1) and (3.1) that

$$\begin{aligned} S''(x\varphi') &= \beta \mathcal{H}(x\varphi')_x - \gamma \partial_x^{-2}(x\varphi') - cx\varphi' - x\varphi' f'(\varphi) \\ &= \beta \mathcal{H}\varphi' + x(\beta \mathcal{H}\varphi' - c\varphi - \gamma \partial_x^{-2}\varphi + f(\varphi))_x + 2\gamma \partial_x^{-2}\varphi \\ &= \beta \mathcal{H}\varphi' + 2\gamma \partial_x^{-2}\varphi = 3\gamma \partial_x^{-2}\varphi + c\varphi - f(\varphi); \end{aligned}$$

and by using (3.1) again, we obtain

$$\begin{aligned} \langle S''(x\varphi'), x\varphi' \rangle &= \int_{\mathbb{R}} (3\gamma \partial_x^{-2}\varphi + c\varphi - f(\varphi)) x\varphi' dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (9(\partial_x^{-1}\varphi)^2 - c\varphi^2) dx - \frac{1}{p+1} K(\varphi) = 3\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx. \end{aligned} \quad (8.3)$$

Therefore we deduce from (8.1), (8.2) and (8.3) that

$$\begin{aligned} \langle S''(\psi'), \psi' \rangle &= \langle S''(\varphi), \varphi \rangle + 4\langle S''(\varphi), x\varphi' \rangle + \langle S''(x\varphi'), x\varphi' \rangle \\ &= (1-p)K(\varphi) + \frac{4(p-1)}{p+1} K(\varphi) + 12\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx \\ &= \frac{(p-1)(3-p)}{p+1} K(\varphi) + 12\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx. \end{aligned}$$

□

Theorem 8.1 *Let $\beta > 0$, $\gamma > 0$, $c < c_* = 3(\beta^2\gamma/4)^{1/3}$ and $\varphi \in \mathcal{G}(\beta, c, \gamma)$. Then the orbit \mathcal{O}_φ is \mathcal{X} -unstable if one the following cases occurs:*

- (i) $c < 0$, $p > 3$ and γ is sufficiently small,
- (ii) $p > 5$ and $c < \left(\frac{p-5}{p-1}\right) c_*$

Proof. By Theorem 7.1 and Lemma 8.1, we only need to check condition (7.1) for ψ defined in Lemma 8.1.

First we note that $\lim_{\gamma \rightarrow 0} \gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx = 0$. Indeed, we already know from (3.1) that

$$\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx = \int_{\mathbb{R}} c\varphi^2 - \beta(D_x^{1/2}\varphi)^2 dx + (m(\beta, c, \gamma))^{\frac{p+1}{p-1}}.$$

Applying Theorem 5.1, it transpires that

$$\lim_{\gamma \rightarrow 0} \gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx = \int_{\mathbb{R}} c\phi^2 - \beta(D_x^{1/2}\phi)^2 dx + (m(\beta, c, 0))^{\frac{p+1}{p-1}} = -I(\phi; \beta, c, 0) + (m(\beta, c, \gamma))^{\frac{p+1}{p-1}} = 0,$$

where ϕ is a ground state of (1.3) with $c < 0$. Applying Theorem 5.1 once more we see that

$$\lim_{\gamma \rightarrow 0^+} \frac{(p-1)(3-p)}{p+1} K(\varphi) + 12\gamma \int_{\mathbb{R}} (\partial_x^{-1}\varphi)^2 dx = \frac{(p-1)(3-p)}{p+1} m(\beta, c, 0)^{\frac{p+1}{p-1}} < 0$$

since $p > 3$. Therefore by Lemma 8.1 one has $\langle S''(\varphi)\psi', \psi' \rangle < 0$ for $\gamma > 0$ sufficiently small. This which proves (i).

Attention is now given to the proof of (ii). Suppose $p > 5$. By Lemma 8.1 and equation (3.1) we have

$$\langle S''(\psi'), \psi' \rangle = (5 - p)K(\varphi) + 4c \int_{\mathbb{R}} \varphi^2 dx.$$

This is clearly negative when $c \leq 0$. Now for $c > 0$, a straightforward calculation reveals that

$$\int_{\mathbb{R}} \varphi^2 dx \leq \frac{1}{c_* - c} I(\varphi)$$

and thus

$$\langle S''(\psi'), \psi' \rangle \leq \left(5 - p + \frac{4c}{c_* - c} \right) K(\varphi).$$

The term on the right hand side is negative when $c < \left(\frac{p-5}{p-1} \right) c_*$. This completes the proof. \square

Remark 8.1 Notice that as $p \rightarrow \infty$, $\left(\frac{p-5}{p-1} \right) c_* \rightarrow c_*$, so the region of instability approaches the entire domain of existence.

We now investigate what conclusions may be drawn from Theorems 6.1 and 7.2, which state that stability is determined by the sign of $d''(c)$. Although no explicit formula for d is available, it is possible to determine the behavior of $d''(c)$ for small $\gamma > 0$. The following scaling property is the main ingredient in this analysis.

Lemma 8.2 Let $\beta > 0$, $\gamma > 0$ and $c < c_*$. For any $r > 0$ and $s > 0$ we have

$$d(r\beta, rcs^{-1}, rs^{-3}\gamma) = r^{\frac{p+1}{p-1}} s^{\frac{-2}{p-1}} d(\beta, c, \gamma).$$

Proof. The lemma follows from (6.2) once we show that

$$m(r\beta, rcs^{-1}, rs^{-2}\gamma) = rs^{\frac{-2}{p+1}} m(\beta, c, \gamma).$$

Let $u \in \mathcal{X}$ with $K(u) > 0$. For any $r > 0$ we have

$$I(u; r\beta, rc, r\gamma) = rI(u; \beta, c, \gamma),$$

so $m(r\beta, rc, r\gamma) = rm(\beta, c, \gamma)$. Next let $v(x) = u(sx)$ for $s > 0$. Then

$$I(v; \beta, c, \gamma) = I(u; \beta, cs^{-1}, s^{-3}\gamma) \quad K(v) = \frac{1}{s} K(u)$$

so

$$\frac{I(v; \beta, c, \gamma)}{K(v)^{\frac{2}{p+1}}} = s^{\frac{2}{p+1}} \frac{I(u; \beta, cs^{-1}, s^{-3}\gamma)}{K(u)^{\frac{2}{p+1}}}$$

and consequently

$$m(\beta, cs^{-1}, s^{-3}\gamma) = s^{\frac{-2}{p+1}} m(\beta, c, \gamma).$$

\square

Setting $r = 2/\beta$ and $s^3 = 2\gamma/\beta$ gives

$$d\left(2, c\left(\frac{4}{\gamma\beta^2}\right)^{1/3}, 1\right) = \left(\frac{2\gamma}{\beta}\right)^{\frac{-2}{3(p-1)}} \left(\frac{2}{\beta}\right)^{\frac{p+1}{p-1}} d(\beta, c, \gamma) \quad (8.4)$$

Hence for any constant k , the values of d along the surface $c^3 = k\gamma\beta^2$ are determined by the value of d at any single point on that surface. Next, setting $r = 1$ and $s = -c/3$ gives

$$d(\beta, -3, \gamma(-3/c)^3) = (-c/3)^{\frac{-2}{p-1}} d(\beta, c, \gamma). \quad (8.5)$$

or equivalently

$$d(\beta, c, \gamma) = (-c/3)^{\frac{2}{p-1}} d(\beta, -3, \gamma(-3/c)^3).$$

Next we set $q = \frac{2}{p-1}$ and assume that d is twice differentiable. Then differentiating with respect to c gives

$$d_c(\beta, c, \gamma) = \left(-\frac{1}{3}q(-c/3)^{q-1}d + \gamma(-c/3)^{q-4}d_\gamma \right) \Big|_{(\beta, -3, -27\gamma/c^3)}$$

and

$$d_{cc}(\beta, c, \gamma) = \left(\frac{1}{9}q(q-1)(-c/3)^{q-2}d - \frac{1}{3}\gamma(2q-4)(-c/3)^{q-5}d_\gamma + \gamma^2(-c/3)^{q-8}d_{\gamma\gamma} \right) \Big|_{(\beta, -3, -27\gamma/c^3)}. \quad (8.6)$$

Theorem 8.2 *Assume d is twice differentiable on the domain $c < c_*$.*

(i) *Fix $1 < p < 3$, $\beta > 0$ and $c < 0$. Then there exist $\gamma_k \rightarrow 0^+$ such that $d_{cc}(\beta, c, \gamma_k) > 0$.*

(ii) *Fix $p > 3$, $\beta > 0$ and $c < 0$. Then there exist $\gamma_k \rightarrow 0^+$ such that $d_{cc}(\beta, c, \gamma_k) < 0$.*

Proof. First observe that

$$\lim_{\gamma \rightarrow 0^+} \frac{1}{9}q(q-1)(-c/3)^{q-2}d(\beta, -3, -27\gamma/c^3) = \frac{1}{9} \frac{2(3-p)}{(p-1)^2} (-c/3)^{(4-2p)/(p-1)} \frac{p-1}{2(p+1)} m(\beta, -3, 0)^{\frac{p+1}{p-1}}.$$

This is positive when $1 < p < 3$ and negative when $p > 3$. As shown in the proof of Theorem 8.1, the term

$$\gamma d_\gamma = \gamma \int_{\mathbb{R}} (\partial_x^{-1} \varphi)^2 dx$$

vanishes as γ approaches zero. It therefore remains to show that the term $\gamma^2 d_{\gamma\gamma}$ vanishes as well. To do so, define

$$g(\gamma) = \begin{cases} \gamma^2 d_\gamma & \gamma > 0 \\ 0 & \gamma = 0 \end{cases}$$

Then since $\gamma d_\gamma \rightarrow 0$ as $\gamma \rightarrow 0^+$, g defines a continuous function for $\gamma \geq 0$. Furthermore by the assumption that d is differentiable, it follows that g is differentiable for $\gamma > 0$. By the Mean Value Theorem, for each integer k there exists $\gamma_k \in (0, 1/k)$ such that $g(1/k) - g(0) = \frac{1}{k}g'(\gamma_k)$, and thus

$$g'(\gamma_k) = kg \left(\frac{1}{k} \right) = \frac{1}{k} d_\gamma \left(\frac{1}{k} \right) \rightarrow 0$$

as $k \rightarrow \infty$. Now

$$g'(\gamma_k) = 2\gamma_k d_\gamma(\gamma_k) + \gamma_k^2 d_{\gamma\gamma}(\gamma_k),$$

so we have

$$\lim_{k \rightarrow \infty} \gamma_k^2 d_{\gamma\gamma}(\gamma_k) = \lim_{k \rightarrow \infty} g'(\gamma_k) - 2\gamma_k d_\gamma(\gamma_k) = 0.$$

□

We next consider the behavior of d for c near $c_* = 3(\beta^2\gamma/4)^{1/3}$. Using appropriately chosen trial functions, we obtain upper bounds on d as c approaches c_* for the nonlinearities $f(u) = |u|^p$ and

$f(u) = -|u|^{p-1}u$. In both cases, for any $p \geq 2$, these bounds imply that $d(c) \rightarrow 0$ as $c \rightarrow c_*$. In the case of the odd nonlinearity $f(u) = -|u|^{p-1}u$, the bound implies that d is convex (and hence $\mathcal{G}(\beta, c, \gamma)$ is stable) for c near c_* .

Our choice of trial function is $u = w_x$, where $w(x) = e^{-a|x|} \sin(bx)$ for appropriately chosen $a > 0$ and $b \neq 0$. It is clear that $u \in \mathcal{X}$, and we have

$$I(u) = \int_{\mathbb{R}} (\beta|\xi|^3 - c|\xi|^2 + \gamma)|\hat{w}|^2 d\xi.$$

Since

$$\hat{w}(\xi) = \frac{-4iab\xi}{(\xi^2 - (a^2 + b^2))^2 + 4a^2\xi^2},$$

we have

$$I(u) = 32a^2b^2 \int_0^\infty \frac{\beta\xi^5 - c\xi^4 + \gamma\xi^2}{((\xi^2 - (a^2 + b^2))^2 + 4a^2\xi^2)^2} d\xi.$$

This integral may be evaluated explicitly using Maple to obtain

$$I(u) = \frac{1}{ab(a^2 + b^2)} \left(2\beta \arctan\left(\frac{b}{a}\right) (a^2 + b^2)^3 + \pi(\gamma b^3 - cb^5 - ca^2b^3) + \beta(2b^5a - 2a^5b) \right). \quad (8.7)$$

For $c < c_*$, the cubic $\beta r^3 - cr^2 + \gamma$ has one real root and two complex roots. Let $b \pm ai$ denote the complex roots. Then by the cubic formula, we have

$$a = \frac{\sqrt{3}}{6\beta} \left(\frac{D}{2} - \frac{2c^2}{D} \right)$$

and

$$b = -\frac{D}{12\beta} - \frac{c^2}{3\beta D} + \frac{c}{3\beta},$$

where

$$D = (8c^3 - 108\gamma\beta^2 + 12\beta\sqrt{3\gamma(27\gamma\beta^2 - 4c^3)})^{1/3}.$$

As $c \rightarrow c_* = 3(\beta^2\gamma/4)^{1/3}$, we have $D \rightarrow -2c_*$ and thus

$$\begin{aligned} \lim_{c \rightarrow c_*} a &= 0 \\ \lim_{c \rightarrow c_*} b &= \frac{2c_*}{3\beta}. \end{aligned}$$

Moreover,

$$D + 2c_* = \frac{D^3 + 8c_*^3}{D^2 - 2c_*D + 4c_*^2} = 8(c^3 - c_*^3) + 12\beta\sqrt{12\gamma(c_*^3 - c^3)} = O(\sqrt{c_* - c}),$$

$$\begin{aligned} a &= \frac{\sqrt{3}(D^2 - 4c^2)}{12\beta D} = \frac{\sqrt{3}(D - 2c)(D^3 + 8c^3)}{12\beta D(D^2 - 2cD + 4c^2)} \\ &= \frac{\sqrt{3}(D - 2c)}{12\beta D(D^2 - 2cD + 4c^2)} \left(16(c^3 - c_*^3) + 12\beta\sqrt{12\gamma(c_*^3 - c^3)} \right) \\ &= O(\sqrt{c_* - c}) \end{aligned}$$

and

$$\begin{aligned} b - \frac{2c_*}{3\beta} &= \frac{c - c_*}{3\beta} - \frac{D^2 + 4c^2 + 4c_*D}{12\beta D} \\ &= \frac{c - c_*}{3\beta} - \frac{4(c^2 - c_*^2) + (D + 2c_*)^2}{12\beta D} \\ &= O(c - c_*) \end{aligned}$$

as $c \rightarrow c_*$.

Lemma 8.3 Suppose that $d(c)$ is differentiable for $c < c_*$. Then for $c < c_*$ it holds that

$$d(c) \geq d(0) \left(1 - \frac{c}{c_*}\right)^{\frac{p+1}{p-1}}. \quad (8.8)$$

Proof. By (5.3), (6.2) and Lemma 6.1, it follows that

$$d(c) = \frac{p-1}{2(p+1)} I(\varphi) \geq \frac{p-1}{2(p+1)} (c_* - c) \int_{\mathbb{R}} \varphi^2 dx = -\frac{p-1}{p+1} (c_* - c) d'(c).$$

Hence, we obtain that

$$\frac{d'(c)}{d(c)} \geq \frac{p+1}{(p-1)(c-c_*)},$$

and therefore (8.8) follows. \square

Lemma 8.4 For u , a and b as chosen above, we have

$$I(u) = O(\sqrt{c_* - c})$$

as $c \rightarrow c_*$.

Proof. Since $a = O(\sqrt{c_* - c})$ and $b = O(1)$ as $c \rightarrow c_*$, it suffices to show that the term in parentheses in expression (8.7) is $O(c_* - c)$. Using the expansion

$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - x + O(x^2)$$

which holds for small $x > 0$, we have

$$2\beta \arctan\left(\frac{b}{a}\right) = \beta\pi - 2\beta\frac{a}{b} + O(a^2/b^2)$$

and thus

$$2\beta \arctan\left(\frac{b}{a}\right) (a^2 + b^2)^3 = \beta\pi b^6 - 2\beta ab^5 + O(a^2).$$

Combining this with the other two terms in equation (8.7) we are left with

$$\pi b^3(\beta b^3 - cb^2 + \gamma) + O(a^2) = \pi b^3(\beta b^3 - cb^2 + \gamma) + O(c_* - c).$$

Finally, since $b = \frac{2c_*}{3\beta} + O(c_* - c)$, it follows that

$$\begin{aligned} \beta b^3 - cb^2 + \gamma &= \beta \left(\frac{2c_*}{3\beta}\right)^3 - c \left(\frac{2c_*}{3\beta}\right)^2 + \gamma + O(c_* - c) \\ &= (c_* - c) \left(\frac{2c_*}{3\beta}\right)^2 + O(c_* - c) \\ &= O(c_* - c). \end{aligned}$$

\square

This bound on $I(u)$, together with a lower bound on $K(u)$, leads to an upper bound on $m(\beta, c, \gamma)$. The lower bound on $K(u)$ depends on the nonlinear term $f(u)$. For even nonlinearities we have the following bound.

Lemma 8.5 Suppose $f(u) = \pm|u|^p$. Fix $\beta > 0$ and $\gamma > 0$. Then

$$d(c) = O(\sqrt{c_* - c})$$

as c approaches c_* .

Proof. It suffices to prove that $K(u) \geq C\sqrt{c_* - c}$ for some constant C independent of c . For then

$$m(\beta, c, \gamma) \leq \frac{I(u)}{K(u)^{\frac{2}{p+1}}} \leq \frac{C(c_* - c)^{1/2}}{(c_* - c)^{\frac{1}{p+1}}} = O(c_* - c)^{\frac{p-1}{2(p+1)}}$$

and it follows from (6.2) that

$$d(c) = \frac{p-1}{2(p+1)} m(\beta, c, \gamma)^{\frac{p+1}{p-1}} = O(c_* - c)^{\frac{1}{2}}.$$

To obtain the lower bound on $K(u)$, first write

$$K(u) = \int_{\mathbb{R}} |u|^p u \, dx = 2 \int_0^{\infty} e^{-a(p+1)x} |b \cos(bx) - a \sin(bx)|^p (b \cos(bx) - a \sin(bx)) \, dx.$$

Rewriting $b \cos(bx) - a \sin(bx) = \sqrt{a^2 + b^2} \cos(bx + \phi)$ where $\phi = \arctan(a/b)$ this becomes

$$2(a^2 + b^2)^{\frac{p+1}{2}} \int_0^{\infty} e^{-a(p+1)x} |\cos(bx + \phi)|^p \cos(bx + \phi) \, dx,$$

and after the change of variable $y = bx + \phi$ this becomes

$$\frac{2e^{a(p+1)\phi/b}}{b} (a^2 + b^2)^{\frac{p+1}{2}} \int_{\phi}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^p \cos(y) \, dy.$$

As c approaches c_* the term outside the integral approaches $2(\gamma/\beta)^{p/4} > 0$, so we will henceforth ignore this term. We now break up the integral as

$$\int_{\phi}^0 e^{-a(p+1)y/b} |\cos(y)|^p \cos(y) \, dy + \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} e^{-a(p+1)y/b} |\cos(y)|^p \cos(y) \, dy.$$

The first term is negative, but bounded below by

$$-\phi = -\arctan(a/b) \geq -a/b.$$

In each term of the summation we make the change of variable $z = y - k\pi$ to obtain

$$\sum_{k=0}^{\infty} \int_0^{\pi} e^{-a(p+1)(z+k\pi)/b} |\cos(z)|^p (-1)^k \cos(z) \, dz$$

which, after summing the geometric series, can be rewritten as

$$\frac{1}{1 + e^{-a(p+1)\pi/b}} \int_0^{\pi} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z) \, dz.$$

The remaining integral we rewrite as

$$\int_0^{\pi/2} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z) \, dz + \int_{\pi/2}^{\pi} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z) \, dz$$

and make the change of variable $y = \pi - z$ in the second integral to obtain

$$\int_0^{\pi/2} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z) dz + \int_{\pi/2}^0 e^{-a(p+1)(\pi-y)/b} |\cos(y)|^p \cos(y) dy.$$

Combining these, we have

$$\int_0^{\pi/2} (e^{-a(p+1)z/b} - e^{-a(p+1)(\pi-z)/b}) \cos(z)^{p+1} dz.$$

Since

$$\lim_{a \rightarrow 0} \frac{e^{-a(p+1)z/b} - e^{-a(p+1)(\pi-z)/b}}{a} = \frac{p+1}{b} \cdot (\pi - 2z)$$

uniformly in x in $[0, \pi/2]$ the integral approaches

$$\frac{a}{b} \int_0^{\pi/2} (p+1)(\pi - 2z) |\cos(z)|^p \cos(z) dz$$

as $c \rightarrow c_*$. Since

$$\begin{aligned} \int_0^{\pi/2} (p+1)(\pi - 2z) \cos(z)^{p+1} dz &= 2(p+1) \int_0^{\pi/2} x \sin(x)^{p+1} dx \\ &\geq 2(p+1) \int_0^{\pi/2} x (2x/\pi)^{p+1} dx \\ &= \frac{(p+1)\pi^2}{2(p+3)} \\ &> \frac{\pi^2}{4} \end{aligned}$$

for all $p > 1$, it follows that as $c \rightarrow c_*$ we have

$$\frac{1}{1 + e^{-a(p+1)\pi/b}} \int_0^{\pi} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z) dz \geq \frac{1}{2} \cdot \frac{1}{4} \pi^2 \cdot \frac{a}{b},$$

and therefore

$$\int_{\phi}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^p \cos(y) dy \geq \frac{1}{2} \left(\frac{1}{4} \pi^2 - 2 \right) \frac{a}{b},$$

which implies that

$$K(u) \geq O(a) = O(\sqrt{c_* - c})$$

as desired. □

While the bound in the previous lemma shows that $d \rightarrow 0$ as $c \rightarrow c_*$, unfortunately it does not provide any information about the sign of $d''(c)$. For the odd nonlinearity $f(u) = -|u|^{p-1}u$, however, the integrand of the functional K is nonnegative, and we have the following stronger bound.

Lemma 8.6 *Suppose $f(u) = -|u|^{p-1}u$. Fix $\beta > 0$ and $\gamma > 0$. Then*

$$d(c) = O(c_* - c)^{\frac{p+3}{2(p-1)}}$$

as c approaches c_* .

Proof. It suffices to prove that $K(u) \geq C(c_* - c)^{-1/2}$ for some constant C independent of c . For then

$$m(\beta, c, \gamma) \leq \frac{I(u)}{K(u)^{\frac{2}{p+1}}} \leq \frac{C(c_* - c)^{1/2}}{(c_* - c)^{-\frac{1}{p+1}}} = O(c_* - c)^{\frac{p+3}{2(p+1)}}$$

and the lemma follows from (6.2). Now, using the calculations from the previous lemma, we have

$$\begin{aligned} K(u) &= \int_{\mathbb{R}} |u|^{p+1} dx = \frac{2e^{a(p+1)\phi/b}}{b} (a^2 + b^2)^{\frac{p+1}{2}} \int_{\phi}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^{p+1} dy \\ &\geq \frac{2e^{a(p+1)\phi/b}}{b} (a^2 + b^2)^{\frac{p+1}{2}} \int_{\pi/2}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^{p+1} dy. \end{aligned}$$

Writing the integral as

$$\sum_{k=1}^{\infty} \int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} e^{-a(p+1)y/b} |\cos(y)|^{p+1} dy,$$

and making the change of variable $z = y - k\pi$, this becomes

$$\sum_{k=1}^{\infty} e^{-a(p+1)\pi k/b} \int_{-\pi/2}^{\pi/2} e^{-a(p+1)z/b} \cos(z)^{p+1} dz = \frac{e^{a(p+1)\pi/b}}{e^{a(p+1)\pi/b} - 1} \int_{-\pi/2}^{\pi/2} e^{-a(p+1)z/b} \cos(z)^{p+1} dz.$$

For small a this is approximately

$$\frac{b}{a(p+1)\pi} \int_{-\pi/2}^{\pi/2} \cos(z)^{p+1} dz \geq C'a^{-1} = O(c_* - c)^{-1/2}.$$

□

Theorem 8.3 Suppose $f(u) = -|u|^{p-1}u$ where $1 < p < 5$. Fix $\beta > 0$ and $\gamma > 0$. Then there exist c arbitrarily close to c_* for which $\mathcal{G}(\beta, c, \gamma)$ is \mathcal{X} -stable.

Proof. For $1 < p < 5$ the function $(c_* - c)^{\frac{p+3}{2(p-1)}}$ is convex and vanishes at $c = c_*$. Since d is positive and is bounded above by a multiple of this convex function, its second derivative must be positive at points c arbitrarily close to c_* . □

9 Numerical Studies

In this section we present numerical results which illustrate the behavior of the solitary waves as the parameters c and γ are varied, and provide insight into the nature of the function $d(c)$ whose concavity determines the stability of the solitary waves. To obtain the numerical approximations we use a spectral method due to Petviashvili. First observe that the solitary wave equation (2.1) may be written

$$\beta \mathcal{H} \varphi_{xxx} - c \varphi_{xx} + f(\varphi)_{xx} = \gamma \varphi.$$

Writing $\psi_{xx} = \varphi$ this becomes

$$-\beta \mathcal{H} \psi_{xxx} + c \psi_{xx} + \gamma \psi = f(\varphi)$$

so taking the Fourier transform yields

$$(\beta |\xi|^3 - c \xi^2 + \gamma) \widehat{\psi} = \widehat{f(\varphi)}.$$

Thus a natural iterative scheme is the following:

$$\begin{aligned}\widehat{\psi}_{n+1} &= \frac{\widehat{f(\varphi_n)}}{\beta|\xi|^3 - c\xi^2 + \gamma} \\ \varphi_{n+1} &= (\psi_n)_{xx}.\end{aligned}$$

Unfortunately, the algorithm has poor convergence properties. However, the algorithm

$$\begin{aligned}\widehat{\psi}_{n+1} &= M_n^\alpha \frac{\widehat{f(\varphi_n)}}{\beta|\xi|^3 - c\xi^2 + \gamma} \\ \varphi_{n+1} &= (\psi_n)_{xx},\end{aligned}$$

with stabilizing factor M_n defined by

$$M_n = \frac{\int (\beta|\xi|^3 - c\xi^2 + \gamma) |\widehat{\psi}_n|^2 d\xi}{\int \widehat{\psi}_n \widehat{f(\varphi_n)} d\xi}$$

has much better convergence properties. It was shown in [29] that this algorithm converges for $1 < \alpha < (p+1)/(p-1)$ and the rate of converges is fastest when $\alpha = \alpha^* = p/(p-1)$. This algorithm was implemented in MATLAB using a large spatial domain to compute the solitary waves for a range of parameter values (β, c, γ) . Figures 1 and 2 show several numerically computed solitary waves for the nonlinearity $f(u) = u^2$. Figure 1 illustrates the oscillatory tails that develop as c approaches c_* , while Figure 2 illustrates the convergence to the exact solitary wave solution of the Benjamin-Ono equation as γ approaches zero.

Once a solitary wave $\varphi \in \mathcal{G}(\beta, c, \gamma)$ is computed, the values of $d(\beta, c, \gamma)$, $d_c(\beta, c, \gamma)$ and $d_\gamma(\beta, c, \gamma)$ are found by using relation (6.2) and Lemma 6.1. The domain of $d(\beta, c, \gamma)$ is the region $\{(\beta, c, \gamma) : \beta > 0, \gamma > 0, c^3 < 27\beta^2\gamma/4\}$, shown in Figure 3. By the scaling relation (8.4), it suffices to compute $d(\beta, c, \gamma)$ at a single point (β, c, γ) on each surface of the form $c^3 = k\gamma\beta^2/4$ for $k < 27$. The segments $S_1 = \{\beta = 2, \gamma = 1, -3 \leq c < 3\}$ and $S_2 = \{\beta = 2, c = -3, 0 < \gamma \leq 1\}$ cross all of these surfaces. Along the segment S_1 , d_{cc} is computed numerically using the computed values of d_c , while along S_2 , relation (8.6) is used to compute d_{cc} in terms of the numerically values of d , d_γ and $d_{\gamma\gamma}$.

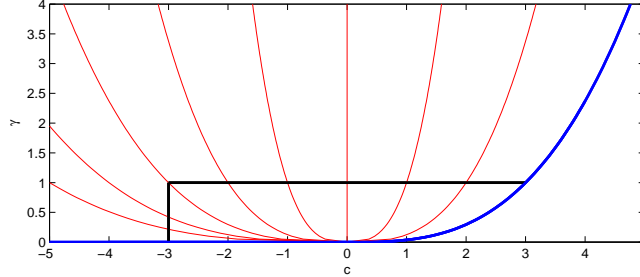


Figure 3: For $\beta = 2$, the domain of d is $\{(c, \gamma) : \gamma > 0, c^3 < 27\gamma\}$. The numerical computations were performed along the segments $\{-3 \leq c < 3, \gamma = 1\}$ and $\{c = -3, 0 < \gamma \leq 1\}$. Every curve of the form $c^3 = k\gamma$ within the domain of d passes through one of these segments.

These computations were performed for two families of nonlinearities, even nonlinearities of the form $f(u) = |u|^p$ and odd nonlinearities of the form $f(u) = -|u|^{p-1}u$. The results for the even nonlinearity $f(u) = |u|^p$ are shown in Figures 4 and 5 and summarized in Table 1. For $p = 2$ and $p = 2.2$ we have $d_{cc} > 0$ for all $c < c_*$. However, when $p = 2.4$ there is a small interval of speeds for which $d_{cc} < 0$. As p increases this interval grows, and when $p = 4$ we have $d_{cc} < 0$ for all $c < c_*$. The

behavior for small $\gamma > 0$ agrees with the results of Theorems 8.1 and 8.2 in that when $p < 3$ we have $d_{cc} > 0$ for small γ and when $p > 3$ we have $d_{cc} < 0$ for small γ . We note that in the case $p = 3$, to which these theorems do not apply, we have $d_{cc} > 0$ for small γ . The behavior for c near c_* is rather interesting. It appears that, for $p < 3$, $d_{cc} \rightarrow +\infty$ as $c \rightarrow c_*$, while for $p > 3$, $d_{cc} \rightarrow -\infty$ as $c \rightarrow c_*$. When $p = 3$, d_{cc} appears to approach some finite negative value.

Table 1: Sign of d_{cc} for $f(u) = |u|^p$.

| p | Regions where $d_{cc} > 0$. |
|-----|-----------------------------------|
| 2 | $c < c_*$ |
| 2.2 | $c < c_*$ |
| 2.4 | $c < 0.980c_*$ and $c > 0.991c_*$ |
| 2.6 | $c < 0.976c_*$ and $c > 0.994c_*$ |
| 2.8 | $c < 0.972c_*$ and $c > 0.996c_*$ |
| 3 | $c < 0.968c_*$ |
| 3.2 | $-1.287c_* < c < 0.962c_*$ |
| 3.4 | $-0.023c_* < c < 0.954c_*$ |
| 3.6 | $0.465c_* < c < 0.942c_*$ |
| 3.8 | $0.738c_* < c < 0.915c_*$ |
| 4 | empty |

The results for the odd nonlinearity $f(u) = -|u|^{p-1}u$ are shown in Figures 6 and 7 and summarized in Table 2. When $p \leq 3$ we have $d_{cc} > 0$ for all $c < c_*$. On the other hand, when $p \geq 5$ we have $d_{cc} < 0$ for all $c < c_*$. When $3 < p < 5$ it appears that there exists some speed c_p such that $d_{cc} < 0$ for $c < c_p$ and $d_{cc} > 0$ for $c_p < c < c_*$. Once again, the behavior for small $\gamma > 0$ agrees with the results of Theorems 8.1 and 8.2. The behavior for c near c_* is similar to that of the even nonlinearity, only the critical exponent appears to be $p = 5$ in this case, in agreement with Theorem 8.3.

Table 2: Sign of d_{cc} for $f(u) = -|u|^{p-1}u$.

| p | Regions where $d_{cc} > 0$. |
|-----|------------------------------|
| 2 | $c < c_*$ |
| 2.2 | $c < c_*$ |
| 2.4 | $c < c_*$ |
| 2.6 | $c < c_*$ |
| 2.8 | $c < c_*$ |
| 3 | $c < c_*$ |
| 3.2 | $-1.262c_* < c < c_*$ |
| 3.4 | $0.033c_* < c < c_*$ |
| 3.6 | $0.589c_* < c < c_*$ |
| 3.8 | $0.918c_* < c < c_*$ |
| 4 | $0.944c_* < c < c_*$ |
| 4.2 | $0.959c_* < c < c_*$ |
| 4.4 | $0.970c_* < c < c_*$ |
| 4.6 | $0.978c_* < c < c_*$ |
| 4.8 | $0.987c_* < c < c_*$ |
| 5 | empty |

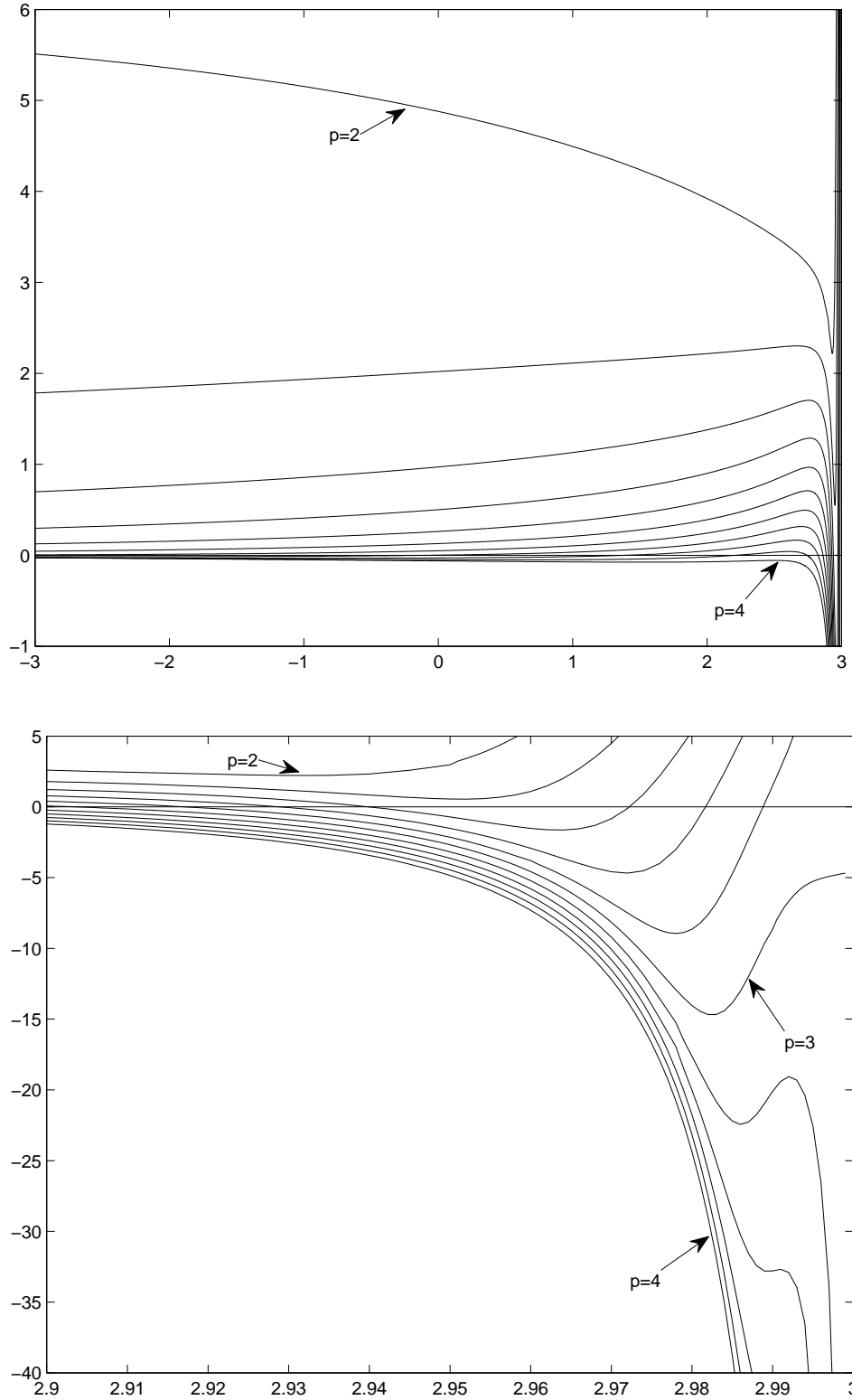


Figure 4: Plots of d_{cc} for $f(u) = |u|^p$ with $\beta = 2$, $\gamma = 1$, $-3 \leq c < 3$ and $p = 2, 2.2, 2.4, \dots, 4$. The second plot is a blowup of the first, illustrating the behavior for c near $c_* = 3$

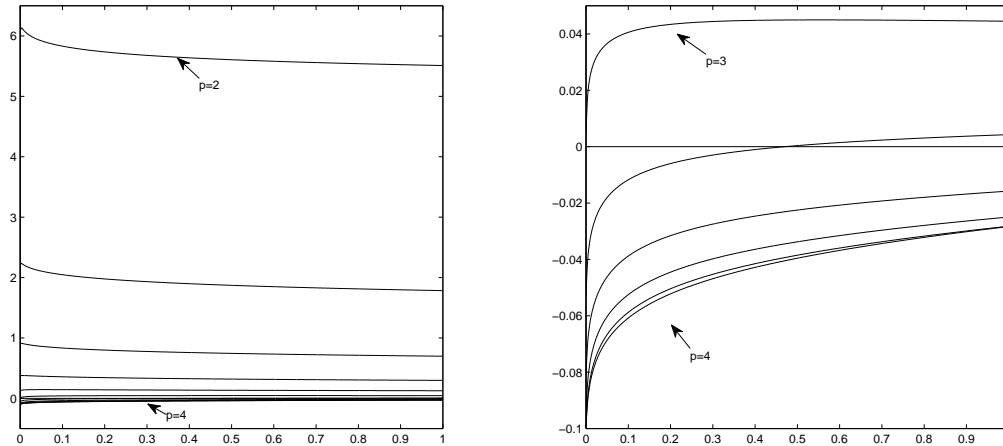


Figure 5: Plots of d_{cc} for $f(u) = |u|^p$ with $\beta = 2$, $c = -3$, $0 < \gamma \leq 1$ and $p = 2, 2.2, 2.4, \dots, 4$. The second plot is a blowup of the first, and better illustrates the plots for $3 \leq p \leq 4$.

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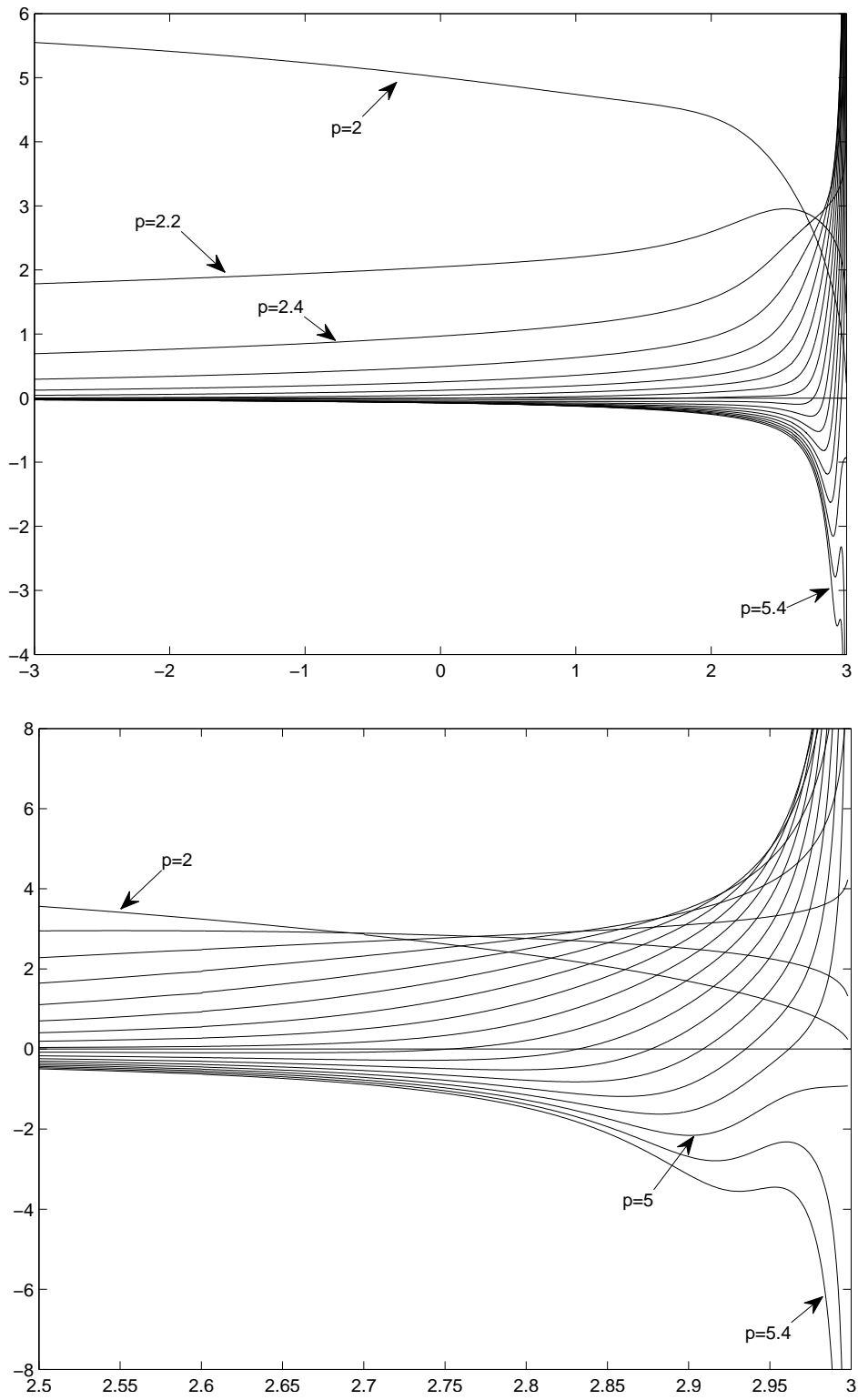


Figure 6: Plots of d_{cc} for $f(u) = |u|^{p-1}u$ with $\beta = 2$, $\gamma = 1$, $-3 \leq c < 3$ and $p = 2, 2.2, 2.4, \dots, 5.4$.

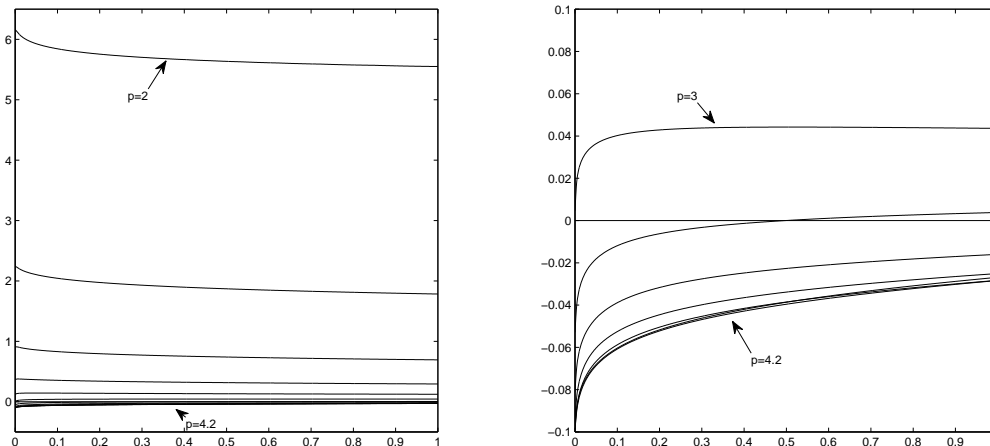


Figure 7: Plots of d_{cc} for $f(u) = |u|^{p-1}u$ with $\beta = 2$, $c = -3$, $0 < \gamma \leq 1$ and $p = 2, 2.2, 2.4, \dots, 4.2$. The second plot is a blowup of the first, and better illustrates the plots for $3 \leq p \leq 4$.

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