Stability and Weak Rotation Limit of Solitary Waves of the Ostrovsky Equation

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Abstract

In this paper we study several aspects of solitary wave solutions of the Ostrovsky equation. Using variational methods, we show that as the rotation parameter goes to zero, ground state solitary waves of the Ostrovsky equation converge to solitary waves of the Korteweg-deVries equation. We also investigate the properties of the function $d(c)$ which determines the stability of the ground states. Using an important scaling identity, together with numerical approximations of the solitary waves, we are able to numerically approximate $d(c)$. These calculations suggest that d is convex everywhere, and therefore all ground state solitary waves of the Ostrovsky equation are stable.

1 Introduction

The equation

$$\left(u_t - \beta u_{xxx} + \left(u^2\right)_x\right)_x = \gamma u, \qquad x \in \mathbf{R} \tag{1.1}$$

was derived by Ostrovsky [10] as a model for the propagation of small-amplitude internal waves in a rotating fluid, where the parameter $\gamma > 0$ measures the effect of rotation. The parameter β determines the type of dispersion. For $\beta < 0$ (negative dispersion), the equation models surface and internal waves in the ocean and surface waves in a shallow channel with uneven bottom [2], while for $\beta > 0$ (positive dispersion), it models capillary waves on the surface of a liquid and magneto-acoustic waves in a plasma [4, 5].

Liu and Varlamov [14] considered the Cauchy problem for the Ostrovsky equation. They showed that equation (1.1) is well-posed in the space

$$X_s = \{f \in H^s(\mathbf{R}) \mid \mathcal{F}^{-1}\left(\hat{f}(\xi)/\xi\right) \in H^s(\mathbf{R})\}$$

with norm

$$\|f\|_{X_s} = \|f\|_{H^s} + \left\|\mathcal{F}^{-1}\left(\frac{\hat{f}(\xi)}{\xi}\right)\right\|_{H^s}$$

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for $s > 3/2$, where $\mathcal{F}$ denotes the Fourier transform. Equation (1.1) has the conserved quantities

$$E(u) = \int \frac{1}{2}\beta u_x^2 + \frac{1}{2}\gamma |D_x^{-1}u|^2 + \frac{1}{3}u^3 \, dx, \qquad V(u) = \frac{1}{2}\int u^2 \, dx.$$

Solitary wave solutions of the form $u(x,t) = \varphi(x - ct)$ satisfy the stationary equation

$$\beta\varphi_{xx} + c\varphi + \gamma D_x^{-2}\varphi = \varphi^2. \tag{1.2}$$

Liu and Varlamov [9] showed using the Concentration Compactness Lemma that there exist solutions of (1.2) in the space X_1 provided $c < 2\sqrt{\beta\gamma}$. Moreover, these *ground state* solutions are characterized as minimizers of

$$I(u; \beta, c, \gamma) = \int \beta u_x^2 - cu^2 + \gamma(D_x^{-1}u)^2 \, dx, \tag{1.3}$$

subject to a constraint of the form

$$K(u) = -\int u^3 \, dx = \lambda > 0. \tag{1.4}$$

The set of all such solutions will be denoted by $G(\beta, c, \gamma)$. In this paper, we are concerned with two aspects of these solutions.

First, we are interested in their behavior as the rotation parameter γ vanishes. If we set $\gamma = 0$ in equation (1.1) and integrate, we obtain the Korteweg-deVries (KdV) equation

$$u_t - \beta u_{xxx} + \left(u^2\right)_x = 0. \tag{1.5}$$

Thus it is natural to wonder whether the solitary waves of the Ostrovsky equation converge to those of the KdV equation as γ approaches zero. We prove in Theorem 3.1 that the solitary waves of the Ostrovsky equation converge strongly in the space $H^1(\mathbf{R})$ to those of the KdV equation as γ goes to zero. The proof uses the variational characterization of both the Ostrovsky and KdV solitary waves.

We are also interested in the stability of solitary wave solutions of (1.1). Following the notation of Liu and Varlamov [9], we make the following definition.

Definition 1.1. *A set $S \subset X$ is X-*stable *with respect to equation* (1.1) *if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $u_0 \in X \cap X_s$, $s > 3/2$, with*

$$\inf_{v \in S} \|u_0 - v\|_X < \delta$$

the solution $u(t)$ of (1.1) *satisfying the initial condition $u(0) = u_0$ can be extended to a solution in $C([0,\infty), X \cap X_s)$ and*

$$\sup_{0 \le t < \infty} \inf_{v \in S} \|u(t) - v\|_X < \epsilon.$$

*Otherwise we say S is X-*unstable.

Using the methods of [3], [6] and [13], it was shown by Liu and Varlamov [9] that the function

$$d(c) = E(\varphi) - cV(\varphi), \qquad \varphi \in G(\beta, c, \gamma) \tag{1.6}$$

determines the stability of the set of ground states. The main result is the following.

Theorem 1.2. *Let $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$. If d is strictly convex in a neighborhood of c, then the set of ground states $G(\beta, c, \gamma)$ is X_1-stable.*

Unfortunately, there is no known closed form expression for d, and thus it is difficult to determine whether or not d is convex. As in [6], Liu and Varlamov obtained bounds on d which enabled them to conclude that d must be convex for c near $2\sqrt{\beta\gamma}$. By entirely different methods, which avoid the use of the function $d(c)$, Liu [8] has shown that solitary waves are stable for small $\gamma > 0$. Here, we investigate in detail the function $d(c)$, both analytically and numerically. A scaling property of d allows us to reduce the problem of computing d from the unbounded domain $c < 2\sqrt{\beta\gamma}$ to two finite intervals. Numerical calculations of d appear to indicate that d is convex everywhere.

The paper is organized as follows. In Section 2 we establish several important analytical properties of d. In Section 3 we use the behavior of d at the boundary $\gamma = 0$ to discuss the relationship between the solitary waves of the Ostrovsky equation and those of the Korteweg deVries equation. Finally, in Section 4 we present the numerical computations of d which imply its convexity.

2 Analytical Properties of d.

In this section we establish some important regularity and scaling properties of the function d. We begin by relating d to the minimization problem that characterizes the ground states. First, we define

$$m(\beta, c, \gamma) = \inf_{u \in X_1, K(u) \neq 0} \frac{I(u; \beta, c, \gamma)}{K(u)^{2/3}}.$$

For $\beta > 0$, $\gamma > 0$ and $c < 2\sqrt{\beta\gamma}$ we have

$$I(u; \beta, c, \gamma) \geq M \int u_x^2 + |D_x^{-1} u|^2 \, dx,$$

where

$$M = \left\{ \begin{array}{ccc} \frac{4\beta\gamma - c^2}{2(\beta+\gamma+\sqrt{(\beta-\gamma)^2+c^2})} & \text{for} & 0 < c < 2\sqrt{\beta\gamma} \\ \min\{\beta, \gamma\} & \text{for} & c \leq 0 \end{array} \right\} > 0. \tag{2.1}$$

Thus, since $\|u\|_{L^3} \leq C\|u\|_{X_1}$ for some constant C, it follows that

$$m(\beta, c, \gamma) > 0 \quad \text{for} \quad \beta > 0, \gamma > 0, c < 2\sqrt{\beta\gamma}.$$

The ground state solutions of (1.2) achieve this minimum, and if we multiply (1.2) by φ and integrate, we find that

$$I(\varphi; \beta, c, \gamma) = K(\varphi)$$

so

$$m(\beta, c, \gamma) = I(\varphi; \beta, c, \gamma)^{1/3} = K(\varphi)^{1/3}.$$

Next, using the definition (1.6) of d, which we will now consider as a function of β, c and γ, we have

$$d(\beta, c, \gamma) = \frac{1}{2} I(\varphi; \beta, c, \gamma) - \frac{1}{3} K(\varphi) = \frac{1}{6} I(\varphi; \beta, c, \gamma) = \frac{1}{6} K(\varphi) = \frac{1}{6} m(\beta, c, \gamma)^3. \tag{2.2}$$

This in fact proves that d is well-defined, independently of the choice of $\varphi \in G(\beta, c, \gamma)$ in equation (1.6). Using equation (2.2), we may very concisely characterize the set of ground states is as

$$G(\beta, c, \gamma) = \{\varphi \in X_1 \mid I(\varphi; \beta, c, \gamma) = K(\varphi) = 6d(\beta, c, \gamma)\}. \tag{2.3}$$

The following lemma restates in terms of d the relative compactness, up to translation, of minimizing sequences. See the proof of Theorem 2.3 in [9].

Lemma 2.1. *Let $\psi_k \in X_1$ satisfy*

$$\lim_{k\to\infty} I(\psi_k; \beta, c, \gamma) = \lim_{k\to\infty} K(\psi_k) = 6d(\beta, c, \gamma).$$

Then there exists some $\varphi \in G(\beta, c, \gamma)$, a subsequence, renamed ψ_k, and a sequence $y_k \in \mathbf{R}$ such that $\psi_k(\cdot - y_k) \to \varphi$ strongly in X_1.

Our next lemma establishes the scaling properties of m, and hence of d.

Lemma 2.2. *Let $\beta > 0$, $\gamma > 0$ and $c < 2\sqrt{\beta\gamma}$. For any $r > 0$ and $s > 0$ we have*

$$m(rs^2\beta, rc, rs^{-2}\gamma) = rs^{1/3} m(\beta, c, \gamma).$$

Proof. Let $u \in X_1$ with $K(u) \neq 0$. For any $r > 0$ we have

$$I(u; r\beta, rc, r\gamma) = rI(u; \beta, c, \gamma),$$

so $m(r\beta, rc, r\gamma) = rm(\beta, c, \gamma)$. Next let $v(x) = u(sx)$ for $s > 0$. Then

$$I(v; \beta, c, \gamma) = \frac{1}{s} I(u; s^2\beta, c, s^{-2}\gamma) \qquad K(v) = \frac{1}{s} K(u)$$

so

$$\frac{I(v; \beta, c, \gamma)}{K(v)^{2/3}} = s^{-1/3} \frac{I(u; s^2\beta, c, s^{-2}\gamma)}{K(u)^{2/3}}$$

and consequently

$$m(s^2\beta, c, s^{-2}\gamma) = s^{1/3} m(\beta, c, \gamma).$$

$\square$

Corollary 2.3. *Let $\beta > 0$, $\gamma > 0$ and $c < 2\sqrt{\beta\gamma}$. For any $r > 0$ and $s > 0$ we have*

$$d(rs^2\beta, rc, rs^{-2}\gamma) = r^3 s d(\beta, c, \gamma).$$

Since the stability condition in Theorem 1.2 involves the convexity of d, it would be desirable to work with $d''(c)$. However, it is not generally known whether or not d is twice differentiable. The following lemmas summarize the regularity properties of d.

Lemma 2.4. *The function d is continuous on the domain $\beta > 0$, $\gamma > 0$, $c < 2\sqrt{\gamma\beta}$. Furthermore, d is strictly increasing in γ and β and strictly decreasing in c.*

Proof. First, fix c and $\gamma > 0$ and consider $\beta_1 > \beta_2 > c_+^2/4\gamma$, where $c_+ = \max\{0, c\}$. Let φ_{β_1} and φ_{β_2} be ground states with $\beta = \beta_1$ and $\beta = \beta_2$, respectively. Then

$$\begin{aligned}
m(\beta_2, c, \gamma) &\leq \frac{I(\varphi_{\beta_1}; \beta_2, c, \gamma)}{K(\varphi_{\beta_1})^{2/3}} \\
&= \frac{I(\varphi_{\beta_1}; \beta_1, c, \gamma) + (\beta_2 - \beta_1) \int (\varphi_{\beta_1})_x^2 \, dx}{K(\varphi_{\beta_1})^{2/3}} \\
&= \frac{I(\varphi_{\beta_1}; \beta_1, c, \gamma)}{K(\varphi_{\beta_1})^{2/3}} + (\beta_2 - \beta_1) \frac{\int (\varphi_{\beta_1})_x^2 \, dx}{K(\varphi_{\beta_1})^{2/3}} \\
&= m(\beta_1, c, \gamma) + (\beta_2 - \beta_1) \frac{\int (\varphi_{\beta_1})_x^2 \, dx}{K(\varphi_{\beta_1})^{2/3}} \\
&< m(\beta_1, c, \gamma),
\end{aligned}$$

so m is strictly increasing in β. On the other hand,

$$\begin{aligned}
m(\beta_1, c, \gamma) &\leq \frac{I(\varphi_{\beta_2}; \beta_1, c, \gamma)}{K(\varphi_{\beta_2})^{2/3}} \\
&= \frac{I(\varphi_{\beta_2}; \beta_2, c, \gamma) + (\beta_1 - \beta_2) \int (\varphi_{\beta_2})_x^2 \, dx}{K(\varphi_{\beta_2})^{2/3}} \\
&= m(\beta_2, c, \gamma) + (\beta_1 - \beta_2) \frac{\int (\varphi_{\beta_2})_x^2 \, dx}{K(\varphi_{\beta_2})^{2/3}},
\end{aligned}$$

so

$$0 \leq m(\beta_1, c, \gamma) - m(\beta_2, c, \gamma) \leq (\beta_1 - \beta_2) \frac{\int (\varphi_{\beta_2})_x^2 \, dx}{K(\varphi_{\beta_2})^{2/3}}.$$

Now since

$$I(\varphi_{\beta_2}; \beta_2, c, \gamma) \geq M \int (\varphi_{\beta_2})_x^2 \, dx,$$

where M is defined by (2.1), it follows that

$$|m(\beta_1, c, \gamma) - m(\beta_2, c, \gamma)| \leq M^{-1} m(\beta_2, c, \gamma)(\beta_1 - \beta_2)$$

so m is locally Lipschitz continuous in β. By similar reasoning it follows that m is decreasing in c, increasing in γ, and locally Lipschitz in c and γ. Since $d = \frac{1}{6} m^3$, the same conclusions hold for d. $\square$

Lemma 2.5. *For each fixed c and $\gamma > 0$, the partial derivative $\partial d/\partial\beta(\beta, c, \gamma)$ exists for all but countably many $\beta > \min\{0, c^2/4\gamma\}$. Similarly, $\partial d/\partial c$ and $\partial d/\partial\gamma$ exist for all but countably many c and γ, respectively. At points where the partials exist,*

$$\frac{\partial d}{\partial \beta} = \frac{1}{2}\int (\varphi_x)^2\,dx$$
$$\frac{\partial d}{\partial c} = -\frac{1}{2}\int \varphi^2\,dx$$
$$\frac{\partial d}{\partial \gamma} = \frac{1}{2}\int (D_x^{-1}\varphi)^2\,dx.$$

Proof. Since d is continuous and monotone with respect to each variable, it follows that the partial derivatives exist at all but countable many points. To verify the formulas above, first fix c and γ. Then by the inequalities in the proof of Lemma 2.4,

$$\frac{\int(\varphi_{\beta_1})_x^2\,dx}{K(\varphi_{\beta_1})^{2/3}} \le \frac{m(\beta_1, c, \gamma) - m(\beta_2, c, \gamma)}{\beta_1 - \beta_2} \le \frac{\int(\varphi_{\beta_2})_x^2\,dx}{K(\varphi_{\beta_2})^{2/3}}$$

for $\beta_1 > \beta_2 > c_+^2/4\gamma$. Let

$$g_s(\beta, c, \gamma) = \sup\left\{\int(\varphi_\beta)_x^2\,dx \mid \varphi_\beta \in G(\beta, c, \gamma)\right\}$$
$$g_i(\beta, c, \gamma) = \inf\left\{\int(\varphi_\beta)_x^2\,dx \mid \varphi_\beta \in G(\beta, c, \gamma)\right\}$$

Then, for $\beta_1 > \beta_2 > c_+^2/4\gamma$,

$$\frac{g_s(\beta_1, c, \gamma)}{m(\beta_1, c, \gamma)^2} \le \frac{m(\beta_1, c, \gamma) - m(\beta_2, c, \gamma)}{\beta_1 - \beta_2} \le \frac{g_i(\beta_2, c, \gamma)}{m(\beta_2, c, \gamma)^2}$$

We now claim that

$$\lim_{\beta\to\beta_0} \sup g_i(\beta, c, \gamma) \le g_s(\beta_0, c, \gamma).$$

To see this, choose any $\beta_k \to \beta_0$ and $\varphi_k \in G(\beta_k, c, \gamma)$. The continuity of m implies that $I(\varphi_k) \to 6d(\beta_0, c, \gamma)$ and $K(\varphi_k) \to 6d(\beta_0, c, \gamma)$. Then by Lemma 2.1, there is a translated subsequence φ_{k_j} which converges in X_1 to some function φ in $G(\beta_0, c, \gamma)$. Hence

$$\lim_{j\to\infty} \sup g_i(\beta_j, c, \gamma) \le \int (\varphi_x)^2\,dx \le g_s(\beta_0, c, \gamma).$$

Consequently

$$\frac{\partial m}{\partial \beta}(\beta^-, c, \gamma) = \frac{g_s(\beta, c, \gamma)}{m(\beta, c, \gamma)^2}.$$

Now, as $d = \frac{1}{6}m^3$, this implies

$$\frac{\partial d}{\partial \beta}(\beta^-, c, \gamma) = \frac{1}{2} g_s(\beta, c, \gamma)$$

Likewise,

$$\frac{\partial d}{\partial \beta}(\beta^+, c, \gamma) = \frac{1}{2} g_i(\beta, c, \gamma)$$

So at points where the partial derivative exists, we must have $g_s(\beta, c, \gamma) = g_i(\beta, c, \gamma)$, and the first formula above follows. The proof of the other formulas is similar. □

Remark 2.6. *If the ground state solutions of* (1.2) *are unique up to translation, so that each* $G(\beta, c, \gamma)$ *consists only of translations of a single profile, it would then follow that* d *is differentiable everywhere.*

For the remainder of this section we fix $\beta = 1$ and we make the following assumption.

Assumption 2.7. *The function* $d(c, \gamma)$ *is twice differentiable in the region* $\{(c, \gamma) : \gamma > 0, c < 2\sqrt{\beta\gamma}\}$.

In order to be able to draw conclusions about the concavity of d for all c, we need to use its scaling properties to reduce the scope of the problem. Using Corollary 2.3 with $s = r^{-1/2}$, we have

$$d(rc, r^2\gamma) = r^{5/2} d(c, \gamma) \tag{2.4}$$

Consequently, the value of d at any point along a curve of the form $c = a\sqrt{\gamma}$ determines the values of d at all other points on that curve. We therefore need only compute d along some set of paths which crosses every such curve. We make the following choice. Let $\Gamma_1 = \{(c, 1/4) \mid -1 \leq c < 1\}$ and $\Gamma_2 = \{(-1, \gamma) \mid 0 < \gamma \leq 1/4\}$. It is clear from Figure 1 that every curve of the form $c = a\sqrt{\gamma}$ within the domain of d ($c < 2\sqrt{\gamma}$) passes through either Γ_1 or Γ_2. We now consider d_{cc} along each path.

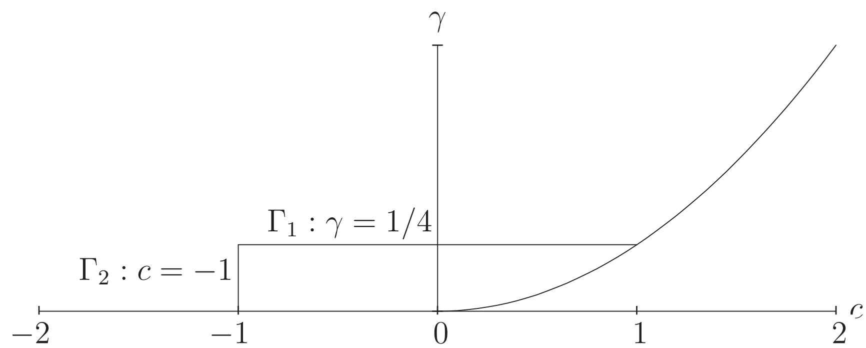

Figure 1: Domain of d and the curves Γ_1 and Γ_2.

Along Γ_1. Fix $\gamma > 0$. By equation (2.4) we have $d(c, \gamma) = (4\gamma)^{5/4} d(c/\sqrt{4\gamma}, 1/4)$ for $c \leq 2\sqrt{\gamma}$ and thus

$$d_{cc}(c, \gamma) = (4\gamma)^{1/4} d_{cc}(c/\sqrt{\gamma}, 1/4). \tag{2.5}$$

Thus, it suffices to determine the sign of $d_{cc}(c, 1/4) > 0$ for $-1 < c < 1$.
Along Γ_2. Now fix $c < 0$. Using equation (2.4) we have $d(c, \gamma) = (-c)^{5/2} d(-1, \gamma/c^2)$. The partials with respect to c are given by

$$d_c(c, \gamma) = 2\gamma(-c)^{-1/2} d_\gamma(-1, \gamma/c^2) - \frac{5}{2}(-c)^{3/2} d(-1, \gamma/c^2)$$

and

$$d_{cc}(c,\gamma) = 4\gamma^2(-c)^{-7/2}d_{\gamma\gamma}(-1,\gamma/c^2) - 4\gamma(-c)^{-3/2}d_\gamma(-1,\gamma/c^2) + \frac{15}{4}(-c)^{1/2}d(-1,\gamma/c^2)$$

If we let $r = \gamma/c^2$, then this becomes

$$\begin{aligned} d_{cc}(c,\gamma) &= \frac{1}{4}(-c)^{1/2}\left(16r^2 d_{\gamma\gamma}(-1,r) - 16r d_\gamma(-1,r) + 15d(-1,r)\right) \\ &= (-c)^{1/2}d_{cc}(-1,r). \end{aligned}$$

Thus it suffices to determine the sign of

$$d_{cc}(-1,\gamma) = \frac{1}{4}\left(16\gamma^2 d_{\gamma\gamma}(-1,\gamma) - 16\gamma d_\gamma(-1,\gamma) + 15d(-1,\gamma)\right) \tag{2.6}$$

for $0 < \gamma \leq 1/4$. The numerical results presented in Section 4 suggest that the quantities are positive for all $\gamma > 0$ and $c < 2\sqrt{\gamma}$.

3 Weak Rotation Limit

In this section, we show that the solitary waves of the Ostrovsky equation (1.1) converge to those of the KdV equation (1.5). We remark that such a relationship is somewhat surprising since the Ostrovsky solitary waves have zero mass, as can be seen by integrating (1.1) with respect to x, while the KdV solitary waves given by equation (3.2) below clearly do *not* have zero mass. Figure 2 illustrates that the positive part of the Ostrovsky solitary waves translates off to infinity as γ goes to zero. These images were actually the inspiration for Lemma 3.3.

In order to precisely state the convergence result, we first recall that solitary waves of the KdV equation satisfy

$$\beta\varphi_{xx} + c\varphi = \varphi^2 \tag{3.1}$$

and that for each $\beta > 0$ and $c < 0$ the unique (up to translation) ground state solution of (3.1) is given explicitly by

$$\varphi_0(x) = \frac{3}{2}c\,\mathrm{sech}^2\left(\frac{1}{2}x\sqrt{\frac{|c|}{\beta}}\right). \tag{3.2}$$

Theorem 3.1. *Fix $\beta > 0$ and $c < 0$ and consider any sequence $\gamma_k \to 0^+$. Denote by φ_k any element of $G(\beta, c, \gamma_k)$ and let φ_0 be given as in equation (3.2). Then there exists a subsequence (renamed γ_k) and translations y_k so that*

$$\varphi_k(\cdot - y_k) \to \varphi_0$$

in H^1. That is, the KdV solitary waves are the limits in H^1 of solitary waves of the Ostrovsky equation.

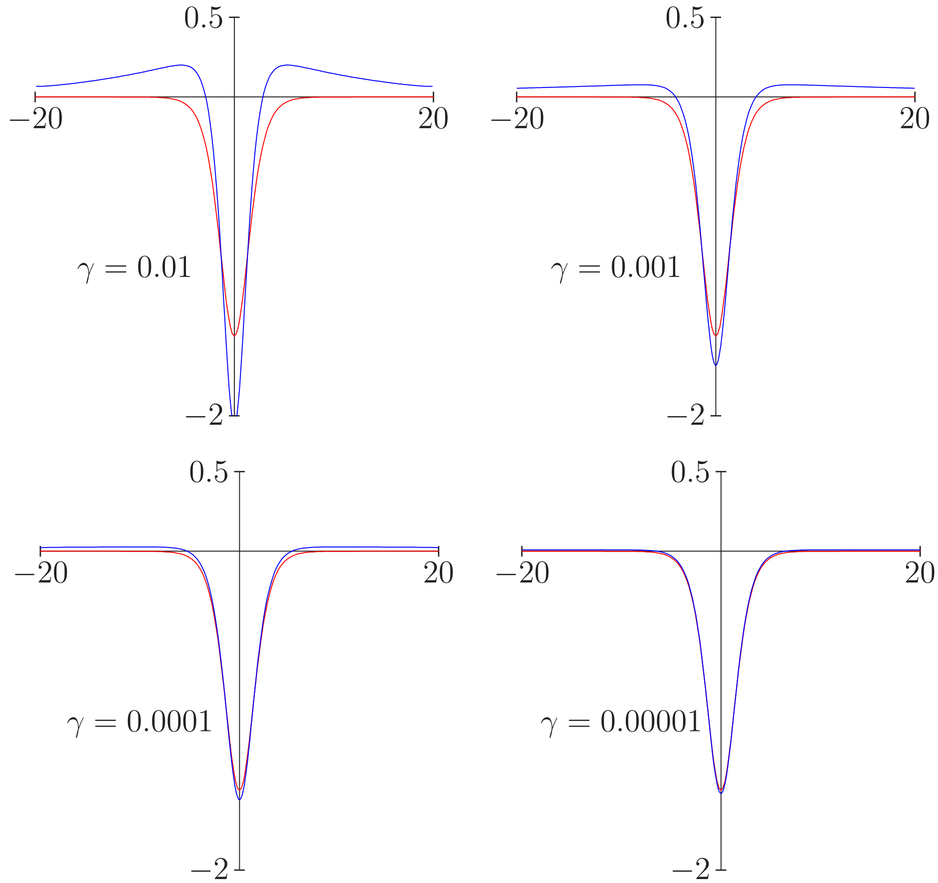

Figure 2: Convergence to KdV solitary waves as $\gamma \to 0$. Here $\beta = 1$ and $c = -1$.

To prove this result, we first note that the solutions (3.2) satisfy the same variational principle as do the Ostrovsky solitary waves, but with $\gamma = 0$ and in the space $H^1(\mathbb{R})$. That is, they achieve the minimum

$$m(\beta, c, 0) = \inf_{u \in H^1, K(u) \neq 0} \frac{I(u; \beta, c, 0)}{K(u)^{2/3}}$$

where

$$I(u; \beta, c, 0) = \int \beta u_x^2 - cu^2 \, dx,$$

and K is defined as before by equation (1.4). We may extend the definition of $d(\beta, c, \gamma)$ to the boundary $\gamma = 0$ (see Figure 1), as

$$d(\beta, c, 0) = \frac{1}{6} K(\varphi_0) = -\frac{9c^3}{16} \int_{-\infty}^{\infty} \operatorname{sech}^6 \left(\frac{1}{2} x \sqrt{\frac{|c|}{\beta}} \right) dx = \frac{6}{5} (-c)^{5/2} \sqrt{\beta}.$$

Thus, equation (2.3) again characterizes the set of these ground states, and we have the following analog of Lemma 2.1

Lemma 3.2. *Let $\psi_k \in H^1(\mathbf{R})$ satisfy*

$$\lim_{k\to\infty} I(\psi_k; \beta, c, 0) = \lim_{k\to\infty} K(\psi_k) = 6d(\beta, c, 0).$$

Then there exists some $\varphi \in G(\beta, c, 0)$, a subsequence, renamed ψ_k, and a sequence $y_k \in \mathbf{R}$ such that $\psi_k(\cdot - y_k) \to \varphi$ strongly in $H^1(\mathbf{R})$.

The idea of the proof of Theorem 3.1 is to show that the Ostrovsky solitary waves form a minimizing sequence for the KdV variational problem as γ vanishes. We first need to show that d is continuous up to the boundary $\gamma = 0$. The following lemma is needed.

Lemma 3.3. *The space X_1 is dense in $H^1(\mathbf{R})$.*

Proof. Let $\varphi \in H^1$. Then by the Sobolev embedding, we may assume φ is continuous. Now, given $\epsilon > 0$, choose $R > 0$ so large that

$$\int_{|x|>R} \varphi_x^2 + \varphi^2 \, dx < \epsilon$$

Next, choose $R_1 > R$ such that $|\varphi(R_1)| < \min(1, \epsilon)$, and let

$$M = \int_0^{R_1} \varphi(x)\, dx$$

Without loss of generality, suppose $\varphi(R_1) \geq 0$ and $M \geq 0$. We now define a function $\psi \in X_1$. First we set

$$\psi(x) = \varphi(x), \qquad x \in [0, R_1]$$

Next we extend ψ to be linear with slope -1 on $I_1 = [R_1, R_2]$ with $\psi(R_2) = -h$, where $h \equiv \frac{\epsilon}{M+2}$. We now define ψ to be the constant $-h$ on the interval $I_2 = [R_2, R_3]$, of length w, to be determined. Finally, extend ψ linearly with slope 1 on $I_3 = [R_3, R_4]$ so that $\psi(R_4) = 0$, and set $\psi(x) = 0$ for all $x > R_4$. See Figure 3. We now have

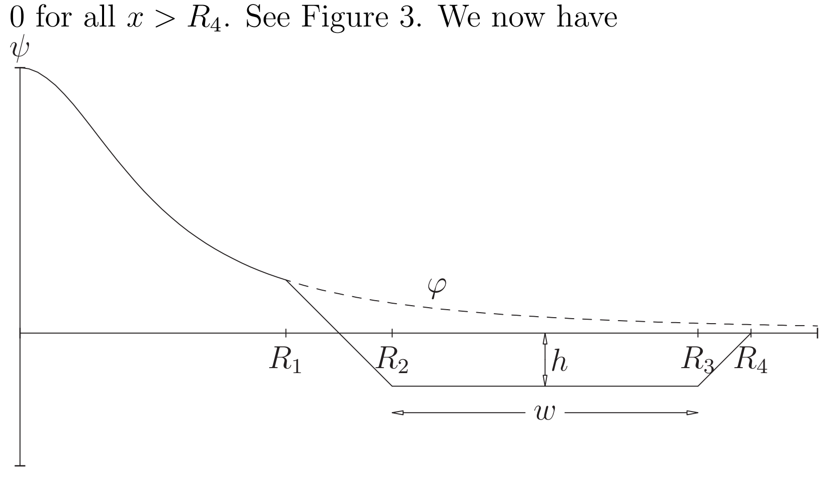

Figure 3: X_1 approximation of an H^1 function.

$$\int_0^\infty \psi(x)\, dx = M + \frac{1}{2}\left(\varphi(R_1)^2 - h^2\right) - wh - \frac{1}{2}h^2$$

This equals zero when

$$w = \frac{1}{h}\left(M + \frac{1}{2}\varphi(R_1)^2 - h^2\right)$$

We define ψ similarly for $x \leq 0$. Since ψ is continuous, piecewise C^1 and compactly supported, it is in H^1. Furthermore, since

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0,$$

it follows that

$$(D_x^{-1}\psi)(x) = \int_{-\infty}^{x} \psi(y)\, dy$$

is also compactly supported. Thus both ψ and $D_x^{-1}\psi$ are in H^1, so ψ is in X_1. We now show that ψ is close to φ in H^1. First, since ψ and φ agree on $[0, R_1]$, we have

$$\begin{aligned}
\|\psi - \varphi\|^2_{H^1(0,\infty)} &= \|\psi - \varphi\|^2_{H^1(R_1,\infty)} \\
&\leq \|\varphi\|^2_{H^1(|x|>R)} + \int_{R_1}^{\infty} \psi_x^2 + \psi^2\, dx \\
&= \|\varphi\|^2_{H^1(|x|>R)} + \int_{I_1 \cup I_2 \cup I_3} \psi_x^2 + \psi^2\, dx
\end{aligned}$$

The first term is less than ϵ by our choice of R. We now calculate the second term explicitly. On I_1 and I_3, $\psi_x^2 = 1$, so

$$\int_{I_1 \cup I_3} \psi_x^2 + \psi^2\, dx = \varphi(R_1) + h + \frac{1}{3}(\varphi(R_1)^3 + h^3) + h + \frac{1}{3}h^3 < 3\epsilon + \epsilon^3$$

On the interval I_2, we have $\psi_x = 0$, so

$$\int_{I_2} \psi_x^2 + \psi^2\, dx = wh^2 = h\left(M + \frac{1}{2}\varphi(R_1)^2 - h^2\right) < \epsilon$$

We estimate the integral over $(-\infty, 0)$ similarly. This completes the proof of the lemma. $\square$

Lemma 3.4. *Fix $\beta > 0$ and $c < 0$. Then $\lim_{\gamma \to 0^+} m(\beta, c, \gamma) = m(\beta, c, 0)$.*

Proof. By the monotonicity of m in γ, it suffices to show that $m(\beta, c, \gamma_k) \to m(\beta, c, 0)$ for some sequence $\gamma_k \to 0$. For each positive integer k, choose a function ψ_k in X_1 with

$$\|\psi_k - \varphi_0\|_{H^1} < \frac{1}{k}$$

and let

$$\gamma_k = \min\left(\frac{1}{k}, \frac{1}{k \int |D_x^{-1}\psi_k|^2\, dx}\right)$$

Then

$$\begin{aligned}
m(\beta, c, \gamma_k) &\leq \frac{I(\psi_k; \beta, c, \gamma_k)}{K(\psi_k)^{2/3}} \\
&= \frac{I(\psi_k; \beta, c, 0) + \gamma_k \int |D_x^{-1}\psi_k|^2\, dx}{K(\psi_k)^{2/3}} \\
&\leq \frac{I(\psi_k; \beta, c, 0) + \frac{1}{k}}{K(\psi_k)^{2/3}}
\end{aligned}$$

Since $I(\cdot; \beta, c, 0)$ and K are both continuous on H^1, we therefore have

$$\lim_{k\to\infty} m(\beta, c, \gamma_k) \leq \frac{I(\varphi_0; \beta, c, 0)}{K(\varphi_0)^{2/3}} = m(\beta, c, 0)$$

On the other hand, since m is strictly increasing in γ, we have $m(\beta, c, \gamma_k) > m(\beta, c, 0)$, so

$$\lim_{k\to\infty} m(\beta, c, \gamma_k) = m(\beta, c, 0)$$

which proves the lemma. □

Proof of Theorem 3.1. By continuity of m at $\gamma = 0$,

$$\begin{aligned}
I(\varphi_k; \beta, c, 0) &= I(\varphi_k; \beta, c, \gamma_k) - \gamma_k \int (D_x^{-1}\varphi_k)^2\, dx \\
&\leq I(\varphi_k; \beta, c, \gamma_k) \\
&= m(\beta, c, \gamma_k)^3 \to m(\beta, c, 0)^3 = 6d(\beta, c, 0)
\end{aligned}$$

and

$$K(\varphi_k) = m(\beta, c, \gamma_k)^3 \to m(\beta, c, 0)^3 = 6d(\beta, c, 0).$$

Thus, the result follows by Lemma 3.2. □

4 Numerical Results

We now present some numerical results that imply the convexity of d in c for all c. We first outline the numerical method employed to compute the solutions. Letting $z = D_x^{-1}\varphi$, equation (1.2) may be rewritten

$$\beta z'''' + cz'' + \gamma z = 2z'z'' \tag{4.1}$$

Multiplying (4.1) by z' and integrating gives

$$\beta z'''z' - \frac{\beta}{2}(z'')^2 + \frac{c}{2}(z')^2 + \frac{\gamma}{2}z^2 = \frac{2}{3}(z')^3,$$

where we have used the fact that z and its derivatives decay to zero as $|x| \to \infty$. We assume without loss of generality that φ has a local minimum at $x = 0$, so that $z''(0) = 0$. The shooting parameters are chosen to be $a_1 = z(0)$ and $a_2 = z'(0)$, and the equation above gives $z'''(0) = \left(\frac{2}{3}a_2^3 - \frac{c}{2}a_2^2 - \frac{\gamma}{2}a_1^2\right)/\beta a_2$. We therefore need to solve the system

$$
\begin{aligned}
z_1' &= z_2 \\
z_2' &= z_3 \\
z_3' &= z_4 \\
z_4' &= (2z_2z_3 - cz_3 - \gamma z_1)/\beta
\end{aligned}
$$

with initial data $(a_1, a_2, 0, \left(\frac{2}{3}a_2^3 - \frac{c}{2}a_2^2 - \frac{\gamma}{2}a_1^2\right)/\beta a_2)$ over an interval $[-X, X]$. Since we are looking for a solution that decays to zero as $x \to \pm\infty$, it must lie in the intersection of the stable and unstable manifolds of the system above. Let E_s and E_u denote the stable and unstable subspaces of the corresponding linear system

$$
\mathbf{z}' = M\mathbf{z}, \qquad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma/\beta & 0 & -c/\beta & 0 \end{bmatrix}.
$$

For any $\mathbf{z} \in \mathbf{R}^4$, we may write $\mathbf{z} = \mathbf{z}_s + \mathbf{z}_u$ where $\mathbf{z}_s \in E_s$ and $\mathbf{z}_u \in E_u$. Furthermore, there exists matrices A_s and A_u, which depend only on β, γ and c, such that $\mathbf{z}_s = A_s\mathbf{z}$ and $\mathbf{z}_u = A_u\mathbf{z}$. Setting $\mathbf{v}_u^+ = A_u\mathbf{z}(X)$ and $\mathbf{v}_s^- = A_s\mathbf{z}(-X)$, we then define the shooting function by $S(a_1, a_2) = (\|\mathbf{v}_u^+\|^2, \|\mathbf{v}_s^-\|^2)$. To implement Newton's method on S, derivatives of the solution with respect to α_1 and α_2 are required, so we also solve the auxiliary system

$$
\begin{aligned}
w_1' &= w_2 \\
w_2' &= w_3 \\
w_3' &= w_4 \\
w_4' &= (2w_2z_3 + 2w_3z_2 - cw_3 - \gamma w_1)/\beta
\end{aligned}
$$

with initial data $\left(1, 0, 0, -\frac{\gamma a_1}{\beta a_2}\right)$ and $\left(0, 1, 0, (\frac{4}{3}a_2 - \frac{1}{2}c + \frac{1}{2}\gamma\frac{a_1^2}{a_2^2})/\beta\right)$. After solving these systems using a Runge-Kutta-Fehlberg solver, the method then proceeds by applying Newton's method and incrementing X until the tail of the solution is sufficiently small. MATLAB routines and data are available at http://mathcs.holycross.edu/˜spl/papers/ostrovsky/.

Typical profiles are shown in Figure 4 and Figure 2. For $c > -2\sqrt{\beta\gamma}$ they have exponentially decaying oscillatory tails, while for $c \le -2\sqrt{\beta\gamma}$ the tails are exponentially decaying, but not oscillatory. All profiles appear to be even functions of x. However, we do not have a proof of this fact, nor do we assume this in the numerical computations.

We now fix $\beta = 1$ and recall from Section 2 that it suffices to compute d_{cc} along Γ_1 and Γ_2 (see Figure 1) and that d_{cc} is given by equation (2.6) along Γ_2. Using the numerically computed solitary waves, we can compute $K(\varphi)$, and use formula (2.2) to compute d. We can also use Lemma 2.5 to compute d_c and d_γ. We then have two methods of approximating d_{cc}; by using a second difference of d, or by using a first difference of its derivatives. The graph of d_{cc} along Γ_1 is shown in Figure 5. It is clear that $d_{cc}(c, 1/4) > 0$ for all $c \in (-1, 1)$. Figure 6 shows a blowup of the minimum, which occurs at approximately $c = 0.964$. The curve appears to be smooth there. The graph of d_{cc} along Γ_2 is shown in Figure 7. Again,

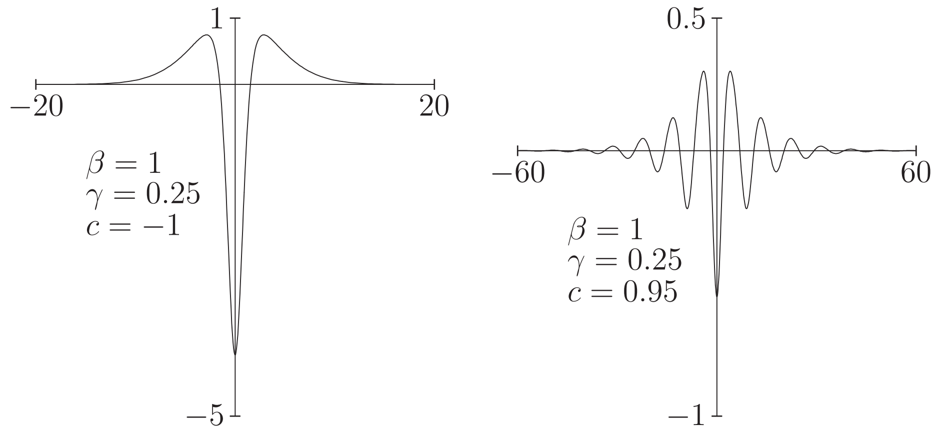

Figure 4: Solitary waves of the Ostrovsky equation.

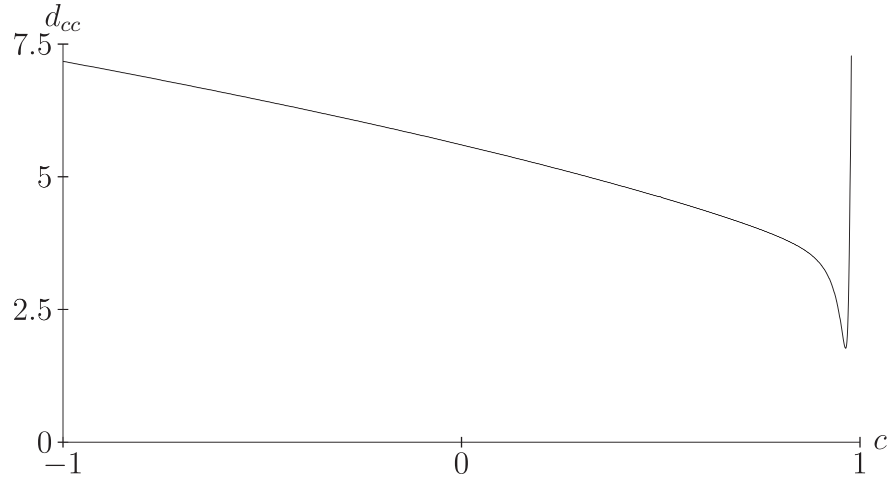

Figure 5: $d_{cc}(c, \gamma)$ for $\gamma = 1/4$, $-1 \leq c < 1$.

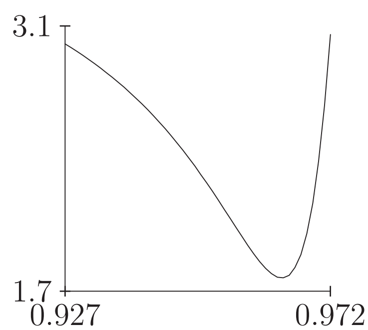

Figure 6: Blowup of the minimum in Figure 5.

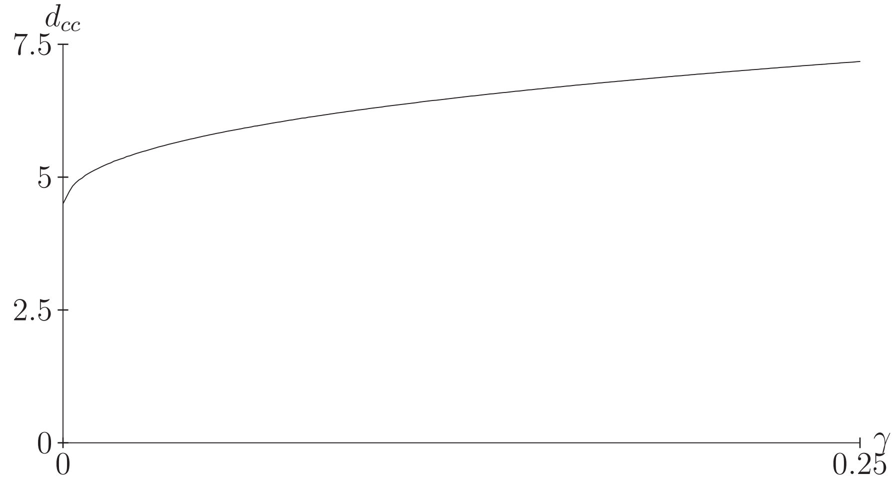

Figure 7: Graph of $d_{cc}(c,\gamma)$ for $c=-1$ and $0<\gamma<1/4$.

it is clear that d_{cc} is positive. We also note that, as $\gamma \to 0$, $d_{cc}(-1,\gamma) \to 9/2$, the value of $d''(-1)$ for the KdV solitary waves.

These calculations suggest that d is convex in c for all $\beta > 0$, $\gamma > 0$ and $c < 2\sqrt{\beta\gamma}$, from which it would follow that all ground states of the Ostrovsky equation are stable.

Acknowledgements

We wish to thank the referee for providing valuable comments and suggestions.

References

[1] C.J. Amick and J.F. Toland, Global uniqueness of homoclinic orbits for a class of fourth order equations. *Z. angew. Math. Phys.* **43** (1992)

[2] E.S. Benilov, On the surface waves in a shallow channel with an uneven bottom, *Stud. Appl. Math.*, **87** (1992), 1-14.

[3] T. Cazenave and P.L. Lions (1982). Orbital stability of standing waves for some nonlinear Schrodinger equations *Commun. Math. Phys.* **85** , 549-561

[4] V.N. Galkin and Y.A. Stepanyants, On the existence of stationary solitary waves in a rotating fluid, *J. Appl. Math. Mech.*, **55**(6) (1991), 939-943.

[5] O.A. Gilman, R. Grimshaw and Y.A. Stepanyants, Approximate and numerical solutions of the stationary Ostrovsky equation, *Stud. Apl. Math.*, **95** (1995), 115-126.

[6] S.P. Levandosky (1998). Stability and instability of fourth-order solitary waves. *J. Dynam. Diff. Eq.* **10**, no. 1, 151-188

[7] P.L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Parts 1 and 2. *Ann. Inst. H. Poincaré, Anal. Nonlinéare*, **1** (1984), 109-145, 223-283.

[8] Y. Liu, Stability of solitary waves for the Ostrovsky equation with weak rotation. Preprint (2004).

[9] Y. Liu and V. Varlamov. Stability of Solitary Waves and Weak Rotation Limit for the Ostrovsky Equation. *J. Differential Equations*, **203**, no 1 (2004), 159-183.

[10] L.A. Ostrovsky, Nonlinear internal waves in a rotating ocean, *Okeanologia*, **18**, (1978), 181-191.

[11] A. Pankov and K. Pflüger, Periodic and solitary traveling wave solutions for the generalized Kadomtsev-Petviashvili equation. *Math. Methods Appl. Sci.*, **22** no. 9 (1999), 733-752.

[12] A. Pankov and K. Pflüger, On ground-traveling waves for the generalized Kadomtsev-Petviashvili equations. *Math. Phys. Anal. Geom.*, **3** no. 1 (2000), 33-47.

[13] J. Shatah (1983). Stable standing waves of nonlinear Klein-Gordon equations. *Comm. Math. Phys.* **91** no. 3, 313-327

[14] V. Varlamov and Y. Liu, Cauchy problem for the Ostrovsky equation. *Discrete Cont. Dynam. Systems*, **10** (3) (2004), 731-753.