Decay Estimates for Fourth Order Wave Equations

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1 Introduction

The equation

$$u_{tt} + \Delta^2 u + u = f(u) \qquad \text{on} \qquad \mathbb{R}^{n+1} \tag{1.1}$$

may be thought of as a nonlinear beam equation. In this paper we obtain both L^p-L^q estimates and space-time integrability estimates on solutions to the linear equation. We also use these estimates to study the local existence and asymptotic behavior of solutions to the nonlinear equation, for nonlinear terms which grow like a certain power of u.

The main L^p-L^q estimate (Theorem 2.1) states that solutions of the linear equation with initial data $(u(0), u_t(0))$ in $W^{2,q'}(\mathbb{R}^n) \oplus L^{q'}(\mathbb{R}^n)$ are bounded in $L^q(\mathbb{R}^n) \oplus W^{-2,q}(\mathbb{R}^n)$ for $2 \le q \le 2^{**}$ for all time and that their norm in this space decays at the (optimal) rate $t^{\frac{n}{2q}-\frac{n}{4}}$. Here and throughout

$$2^{**} = \begin{cases} \frac{2n}{n-4} & \text{for} \quad n \ge 5 \\ \infty & \text{for} \quad 1 \le n \le 4 \end{cases} \tag{1.2}$$

denotes the critical exponent q in the Sobolev embedding $H^2(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$. The space-time integrability estimate (Theorem 3.1) states that solutions of the linear equation with initial data in the energy space $X = H^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ lie in the space $L^q(\mathbb{R}^{n+1}) \oplus W^{-2,q}(\mathbb{R}^{n+1})$ for all q between $2 + 8/n$ and $2 + 12/(n-4)$.

The local well-posedness result (Theorem 4.1) states that for nonlinearities $f(u)$ which satisfy

$$|f'(s)| \le C|s|^{p-1}. \tag{1.3}$$

with $1 < p < 2^{**} - 1$, there exist strongly continuous finite energy solutions of (1.1) which exist locally in time with arbitrary initial data in X. Finally, in Theorem 5.1 we prove that, for nonlinear terms $f(u)$ which satisfy (1.3) with $p > 1 + 8/n$, low energy scattering states exist. That is, given $g_- \in X$ small enough, there exists a solution $w(t) = (u(t), u_t(t))$ of (1.1) and $g_+ \in X$ such that $\|U_0(t)g_\pm - w(t)\|_X \to 0$ as $t \to \pm\infty$, where U_0 is the solution operator for the linear equation.

We remark here that the local well-posedness result verifies an important assumption used in [5] in the stability analysis of solitary wave solutions of (1.1). Using the method of concentrated compactness [6] it was shown that traveling wave solutions of (1.1) exist. The variational characterization of the traveling waves was then used to show that there is a function $d(c)$ of the wave velocity c such that, modulo local well- posedness, the stability of the traveling waves is determined by the convexity of $d(c)$. In particular there exist stable solitary waves of arbitrarily small norm in X when $p < 1 + 4/n$. We also remark that the low energy scattering result is complementary to the stability result, in the sense that it excludes the possibility of small stable solitary waves for these values of p.

In Section 2 we prove the L^p-L^q estimate following the ideas used by Marshall, Strauss and Wainger [9] to obtain estimates for the Klein-Gordon equation. We find estimates on an analytic family of operators and use complex interpolation ([14],[2]). The main difference is that the oscillatory term in the kernel (2.28) is $\sin(tr_2)$, where $r_2 = (1 + r^4)^{1/2}$, while in [9] the oscillatory term is $\sin(tr_1)$, with $r_1 = (1 + r^2)^{1/2}$. The difference in the behavior of the derivatives of r_1 and r_2, for both large and small r, affects both the decay rate and the values of q for which they hold. While the decay rates for (1.1) are slower than those for the Klein-Gordon equation, the range of q for which an $L^{q'}$-L^q estimate holds includes the interval $[2, 2^{**}]$.

In Section 3 we prove the space-time integrability estimate. The proof is based on the ideas of Strichartz [20], with one major exception. In [20] an explicit formula for the Fourier transform of a quadratic form is used to obtain the estimate, while in the case of equation (1.1) the Fourier transform of a *quartic* form must be estimated. Without an explicit formula available this is quite difficult. However, as (1.1) is still second order in time, the symbol of the differential operator is quadratic in τ and, since the symbol is radial in ξ, the explicit relation for quadratic forms may be used to reduce the problem to estimating a one dimensional integral very similar to the integral (2.28) associated with the time decay estimate. The difference is that $\sin(tr_2)$ is replaced with a Bessel function of complex order. Thus the result follows using techniques similar to those in Section 2, along with a careful analysis of the dependence of Bessel functions of complex order.

In Section 4 we prove the local well-posedness theorem. We remark that the difficulty lies in proving well-posedness for p up to the critical number $2^{**} - 1$. For $1 < p < 2^{**}/2$ nonlinearities satisfying (1.3) are locally Lipschitz on X and therefore well-posedness is a consequence of standard semi-group arguments. For higher values of p we first use the time decay estimates to establish existence in a weaker space $X_3 = L^{p+1}(\mathbb{R}^n) \oplus W^{-2,p+1}(\mathbb{R}^n)$. Approximating the nonlinearity by Lipschitz functions produces a sequence of solutions in X which converges to the solution of (1.1). Using the decay estimates again, we show that these solutions are uniformly Hölder continuous in time with values in X_3. Together with arguments from [16] and [17] we then show that the original solution is in fact in X.

In Section 5 we prove the existence of low energy scattering states. The key ingredients in the proof are the local existence result in X, along with the time decay and space-time integrability estimates on $U_0(t)$ established in Sections 2 and 3. Once these are established, the result is an application of the abstract framework developed by Strauss [17].

Notation

$\mathcal{F}$	–	Fourier transform on $\mathbb{R}^{n+1}$ (ξ and τ)
$\hat{}$	–	Fourier transform on $\mathbb{R}^{n}$ (ξ)
$\mathcal{F}_\tau$	–	Fourier transform on $\mathbb{R}$ (τ)
$*$	–	convolution on $\mathbb{R}^{n}$

2 Time Decay Estimates

In this section we establish time decay estimates from $L^{q'}$ to L^q on solutions to the linear equation

$$u_{tt} + \Delta^2 u + u = 0. \tag{2.1}$$

The main result is the following Theorem.

Theorem 2.1 *For any n let q satisfy $2 \le q \le 2^{**}$, where 2^{**} is defined as in (1.2). Let $u(t)$ be the solution of (2.1) with initial data $u(0) = u_0$ and $u_t(0) = v_0$. Then*

$$\|u(t)\|_{L^q(\mathbb{R}^n)} \le Ct^{\frac{n}{2q}-\frac{n}{4}}\left(\|u_0\|_{W^{2,q'}(\mathbb{R}^n)} + \|v_0\|_{L^{q'}(\mathbb{R}^n)}\right) \qquad \text{for} \qquad t > 0. \tag{2.2}$$

Furthermore, if $u_0 = 0$ then

$$\|u(t)\|_{L^q(\mathbb{R}^n)} \le Ct^{1+\frac{n}{q}-\frac{n}{2}}\|v_0\|_{L^{q'}(\mathbb{R}^n)} \qquad \text{for} \qquad t < 1 \tag{2.3}$$

where $1/q' + 1/q = 1$. The rates of decay in (2.2) and (2.3) are optimal.

To prove these results we introduce the following notation. Taking the Fourier transform of (2.1) we obtain

$$\hat{u}_{tt} + (1 + |\xi|^4)\hat{u} = 0 \tag{2.4}$$

which has solution

$$\hat{u}(\xi, t) = \hat{u}_0(\xi)\cos(1 + |\xi|^4)^{1/2}t + \hat{v}_0(\xi)\frac{\sin(1 + |\xi|^4)^{1/2}t}{(1 + |\xi|^4)^{1/2}}. \tag{2.5}$$

The solution of (2.1) is therefore given by

$$u(t) = K^c(t) * (1 + \Delta^2)^{1/2}u_0 + K^s(t) * v_0, \tag{2.6}$$

where

$$\widehat{K^c}(\xi, t) = \frac{\cos(1 + |\xi|^4)^{1/2}t}{(1 + |\xi|^4)^{1/2}} \qquad \widehat{K^s}(\xi, t) = \frac{\sin(1 + |\xi|^4)^{1/2}t}{(1 + |\xi|^4)^{1/2}}. \tag{2.7}$$

We next embed the operators $T^{c,s}(t) = K^{c,s}(t)*$ in families of operators $T^{c,s}_\alpha(t) = K^{c,s}_\alpha(t)*$ where

$$\widehat{K^s_\alpha(t)}(\xi) = \frac{\sin(1 + |\xi|^4)^{1/2}t}{(1 + |\xi|^4)^{\alpha/2}}, \qquad \widehat{K^c_\alpha(t)}(\xi) = \frac{\cos(1 + |\xi|^4)^{1/2}t}{(1 + |\xi|^4)^{\alpha/2}}. \tag{2.8}$$

We denote by $\|\cdot\|_{p,q}$ the operator norm on the space of bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Theorem 2.1 is equivalent to showing that the operators $T^{c,s}(t) = T_1^{c,s}(t)$ satisfy

$$\begin{aligned} \|T^s(t)\|_{q',q} &\leq C \min\left(t^{\frac{n}{2q}-\frac{n}{4}}, t^{1+\frac{n}{q}-\frac{n}{2}}\right) \\ \|T^c(t)\|_{q',q} &\leq C t^{\frac{n}{2q}-\frac{n}{4}}. \end{aligned} \tag{2.9}$$

It can be seen immediately that for all $t > 0$ we have

$$\|T_\alpha^s(t)\|_{2,2} \leq 1 \qquad \text{and} \qquad \|T_\alpha^c(t)\|_{2,2} \leq 1 \tag{2.10}$$

for $Re(\alpha) = 0$ and

$$\|T^s(t)\|_{2,2} \leq \min(1, t) \qquad \text{and} \qquad \|T^c(t)\|_{2,2} \leq 1. \tag{2.11}$$

We will establish (2.9) using Stein's interpolation theorem, which we state here. See [14].

Theorem 2.2 *(Stein) Let S be the strip $0 \leq Re(\alpha) \leq 1$ and let T_α be an analytic family of linear operators satisfying*

$$\|T_{iIm(\alpha)}\|_{p_0,q_0} \leq M_0(Im(\alpha)) \qquad \|T_{1+iIm(\alpha)}\|_{p_1,q_1} \leq M_1(Im(\alpha)) \tag{2.12}$$

where $1 \leq p_j, q_j \leq \infty$ for $j = 0, 1$ and

$$\sup_{-\infty < Im(\alpha) < \infty} e^{-b|Im(\alpha)|} \log M_j(Im(\alpha)) < \infty \tag{2.13}$$

for some $b < \pi$. Then if $0 \leq s \leq 1$ there is a constant M_s so that $\|T_s\|_{p_s,q_s} \leq M_s$ for

$$\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1} \qquad \frac{1}{q_s} = \frac{1-s}{q_0} + \frac{s}{q_1}. \tag{2.14}$$

Furthermore we may replace $q_1 = \infty$ with BMO.

The heart of the proof of Theorem 2.1 lies in the following estimates.

Lemma 2.3 *There exists a constant C so that for $n \geq 4$ we have*

$$\|K_{n/4}^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/4} \qquad \textit{for} \qquad t > 0 \tag{2.15}$$

$$\|K_{n/4}^c(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/4} \qquad \textit{for} \qquad t > 0 \tag{2.16}$$

and for $n < 4$ we have

$$\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/4} \qquad \textit{for} \qquad t > 1 \tag{2.17}$$

$$\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{1-n/2} \qquad \textit{for} \qquad t < 1 \tag{2.18}$$

$$\|K^c(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/4} \qquad \textit{for} \qquad t > 0. \tag{2.19}$$

Proof of Theorem 2.1 For $n < 4$ it follows from (2.17) and (2.18) that

$$\|T^s(t)\|_{1,\infty} \leq C \min(t^{-n/4}, t^{1-n/2}) \tag{2.20}$$

for all $t > 0$ and therefore interpolation between (2.20) and (2.11) yields the first estimate in (2.9). By (2.19) we have

$$\|T^c(t)\|_{1,\infty} \leq Ct^{-n/4} \tag{2.21}$$

and interpolation between (2.21) and (2.11) proves the second estimate in (2.9). In dimension $n \geq 4$ we set $Re(\alpha) = n/4$ and write

$$K^s_\alpha(t) = K^s_{n/4}(t) M_\alpha \qquad \text{where} \qquad \widehat{M_\alpha}(\xi) = (1 + |\xi|^4)^{-iIm(\alpha)/2}. \tag{2.22}$$

It follows from ([2] p151, Cor.2) that $\|M_\alpha\|_{BMO,BMO} \leq C(1 + |Im(\alpha)|^k)$ for any integer $k > n/4$. Together with (2.15) this implies that for $Re(\alpha) = n/4$

$$\|T^s_\alpha(t)\|_{1,BMO} \leq Ct^{-n/4}(1 + |Im(\alpha)|^k). \tag{2.23}$$

By applying Theorem 2.2 with $p_0 = q_0 = 2$, $p_1 = 1$, $q_1 = BMO$ and $s = 4/n$ to (2.10) and (2.23), it follows that

$$\|T^s(t)\|_{q',q} \leq Ct^{-1} \qquad \text{where} \qquad q = 2^{**}. \tag{2.24}$$

Similarly, (2.16) may be used to establish

$$\|T^c(t)\|_{q',q} \leq Ct^{-1} \qquad \text{where} \qquad q = 2^{**}. \tag{2.25}$$

Interpolating between (2.11) and (2.24),(2.25) yields (2.9) for $n \geq 4$. ∎

The remainder of this section is devoted to the proof of Lemma 2.3. We consider only the estimates on K^s_α, since the estimates for K^c_α may be obtained by applying the identical arguments with $\cos(tr_2)$ instead of $\sin(tr_2)$. We first write the formula for the Fourier transform of a radial function

$$\hat{f}(R) = (2\pi)^n R^{1-n/2} \int_0^\infty f(r) J_{n/2-1}(Rr) r^{n/2} dr \tag{2.26}$$

where $R = |x|$, $r = |\xi|$, and $J_\nu(s)$ is the Bessel function of order ν. We shall denote

$$\tilde{J}_\nu(s) = \frac{J_\nu(s)}{s^\nu}. \tag{2.27}$$

In terms of $\tilde{J}_\nu$ we can use (2.26) to express the kernel $K^s_\alpha(t)$ as

$$K^s_\alpha(t)(R) = (2\pi)^n \int_0^\infty \sin(tr_2) r_2^{-\alpha} r^{n-1} \tilde{J}_{n/2-1}(Rr) dr \tag{2.28}$$

where $r_2 = (1 + r^4)^{1/2}$.

The following properties of Bessel functions of complex order ν are proved in the appendix.

Lemma 2.4 *If $Re(\mu)$ is fixed, then*

$$|\tilde{J}_\mu(s)| \leq Ce^{\pi|Im(\mu)|} \qquad \text{for} \qquad |s| < 1. \tag{2.29}$$

Also

$$J_\mu(s) = Cs^{-1/2}\cos(s - \mu\pi/2 - \pi/4) + O\left(e^{2\pi|Im(\mu)|}|s|^{-3/2}\right) \tag{2.30}$$

and

$$|\tilde{J}_\mu(s)| \leq Ce^{2\pi|Im(\mu)|}s^{-Re(\mu)-1/2} \qquad \text{for} \qquad |s| > 1. \tag{2.31}$$

It follows from Lemma 2.4 that $\|\tilde{J}_\mu\|_{L^\infty(\mathbb{R}^+)} = O\left(e^{2\pi|Im(\mu)|}\right)$ for $Re(\mu) \geq -1/2$. The derivative of $\tilde{J}_\nu$ for any complex ν is given by the formulas

$$\tilde{J}'_\nu(s) = \frac{1}{s}(\tilde{J}_{\nu-1}(s) - 2\nu\tilde{J}_\nu(s)) \tag{2.32}$$

$$\tilde{J}'_\nu(s) = -s\tilde{J}_{\nu+1}(s) \tag{2.33}$$

Estimates for $n \geq 4$.

The decay estimate for $n \geq 4$ is obtained by integrating (2.28) by parts multiple times.

Lemma 2.5 *Suppose $\nu > 0$, $\sigma > 3$ and $2\tau + \sigma - \nu - 1/2 < 0$. Then*

$$\int_0^\infty \sin(tr_2)r_2^\tau r^\sigma \tilde{J}_\nu(Rr)dr = \frac{-1}{2t}\int_0^\infty \sin(tr_2 - \pi/2)P_{\nu\sigma\tau}(r)dr \tag{2.34}$$

where

$$\begin{aligned} P_{\nu\sigma\tau}(r) = \frac{d}{dr}[r_2^{\tau+1}r^{\sigma-3}\tilde{J}_\nu(Rr)] &= r_2^{\tau+1}r^{\sigma-4}[(\sigma - 2\nu - 3)\tilde{J}_\nu(Rr) + \tilde{J}_{\nu-1}(Rr)] \\ &\quad + 2(\tau+1)r_2^{\tau-1}r^\sigma\tilde{J}_\nu(Rr). \end{aligned} \tag{2.35}$$

Proof. Since

$$\sin(tr_2) = \frac{r_2}{2tr^3}\frac{d}{dr}\sin(tr_2 - \pi/2), \tag{2.36}$$

integration by parts gives

$$\begin{aligned} \int_0^\infty \sin(tr_2)r_2^\tau r^\sigma\tilde{J}_\nu(Rr)dr &= \frac{1}{2t}\sin(tr_2 - \pi/2)r_2^{\tau+1}r^{\sigma-3}\tilde{J}_\nu(Rr)\Big|_0^\infty \\ &\quad - \frac{1}{2t}\int_0^\infty \sin(tr_2 - \pi/2)\frac{d}{dr}[r_2^{\tau+1}r^{\sigma-3}\tilde{J}_\nu(Rr)]dr. \end{aligned} \tag{2.37}$$

By (2.29) and the hypothesis $\sigma > 3$ the first term vanishes at $r = 0$. It vanishes at infinity as well since $r_2 = O\left(r^2\right)$ as $r \to \infty$ and $2(\tau+1) + \sigma - 3 - \nu - \frac{1}{2} = 2\tau + \sigma - \nu - \frac{3}{2} < 0$ by assumption. The expansion of the derivative follows from (2.32). ■

In order to prove Lemma 2.3 we fix $\alpha = n/4$. The integral in (2.28) is the same as that in Lemma 2.5 with $\tau = -n/4$, $\sigma = n-1$ and $\nu = n/2 - 1$. Thus we may apply Lemma 2.5 to (2.28) N times to deduce that

$$K^s_{n/4}(t)(R) = (2\pi)^n \left(\frac{-1}{2t}\right)^N \int_0^\infty \sin\left(tr_2 - N\frac{\pi}{2}\right) \sum_{j+k+l=N} C_{jkl} f_{jkl} dr \tag{2.38}$$

where

$$f_{jkl}(r,R) = r_2^{-n/4+j+l-k} r^{n-1-4j-4l} \tilde{J}_{\frac{n}{2}-1-j}(Rr) \tag{2.39}$$

or, using the fact that $j+k+l=N$

$$f_{jk}(r,R) = r_2^{N-n/4-2k} r^{n-1-4N+4k} \tilde{J}_{n/2-1-j}(Rr) \qquad 0 \le j+k \le N. \tag{2.40}$$

Thus we need to estimate terms of the form

$$I_{jk}(R) = \int_0^\infty \sin\left(tr_2 - N\frac{\pi}{2}\right) f_{jk}(r,R) dr \tag{2.41}$$

Since $\tilde{J}_{n/2-1-j}(s)$ is bounded, $f_{j0}(r) = O\left(r^{n-1-4N}\right)$ for r near zero. Hence if g_{j0} is to be integrable at $r=0$ we require $N < n/4$. So let $N = \lfloor \frac{n-1}{4} \rfloor$. In view of (2.38) and (2.41), the first part of Lemma 2.3 is equivalent to the following.

Lemma 2.6 *For $n \ge 4$ and $\alpha = n/4$ we have $\|I_{jk}\|_\infty = O\left(t^{N-n/4}\right)$ for $t > 0$.*

Proof. For ease of notation we now fix $\kappa = n/2 - 1 - j$. We first consider the case $t < 1$. We shall estimate $I_{jk}(R)$ over the intervals $[0,1]$, $[1, t^{-1/2}]$ and $[t^{-1/2}, \infty]$ separately. Since $\tilde{J}_\kappa(Rr)$ is bounded, the integral over $[0,1]$ is $O(1)$ in t for all R. Over the second interval we get

$$\begin{aligned} \int_1^{t^{-1/2}} \sin\left(tr_2 - N\frac{\pi}{2}\right) f_{jk}(r,R) dr &\le C \int_1^{t^{-1/2}} r^{n/2-1-2N} dr \\ &= r^{n/2-2N}\Big|_1^{t^{-1/2}} = O(1) + O\left(t^{N-n/4}\right). \end{aligned} \tag{2.42}$$

Finally over the last interval, we integrate by parts to get

$$\begin{aligned} \int_{t^{-\frac{1}{2}}}^\infty \sin(tr_2) f_{jk}(r,R) dr = &-\frac{1}{2t} \cos(tr_2) r_2^{N-n/4+1-2k} r^{n-4-4N+4k} \tilde{J}_\kappa(Rr)\Big|_{t^{-1/2}}^\infty \\ &+ \frac{1}{2t} \int_{t^{-\frac{1}{2}}}^\infty \cos(tr_2) P_j(r) dr \end{aligned} \tag{2.43}$$

where

$$P_{jk}(r,R) = P_{\nu\sigma\tau}(r,R) \tag{2.44}$$

is defined by (2.35) with $\nu = \kappa$, $\sigma = n - 1 - 4N + 4k$ and $\tau = N - n/4 - 2k$. Since $\nu \geq 1/2$ implies $P_{\nu\sigma\tau}(r) = O\left(r^{2\tau+\sigma-2}\right)$ for $r > 1$, it follows that $P_{jk}(r) = O\left(r^{n/2-3-2N}\right)$ for $r > 1$, and therefore the last expression is of order

$$O\left(t^{N-n/4}\right) + O\left(t^{-1}\int_{t^{-1/2}}^{\infty} r^{n/2-3-2N}dr\right) = O\left(t^{N-n/4}\right). \tag{2.45}$$

For $t > 1$ the estimates on I_{jk} are established according to the congruence of the dimension n mod 4. There are two cases.

Case 1: $n \equiv 0(4)$. When $n \equiv 0(4)$, $N = \frac{n-4}{4}$ and we need to show $\|I_{jk}\|_{L^\infty(\mathbb{R}^n)} = O\left(t^{-1}\right)$. For $k = 0$ integrating by parts gand using (2.33) gives

$$\begin{aligned} I_{jk}(R) &= \int_0^\infty \sin\left(tr_2 - N\frac{\pi}{2}\right) r^3 r_2^{-1}\tilde{J}_\kappa(rR)dr = \frac{-1}{2t}\int_0^\infty \frac{d}{dr}\cos\left(tr_2 - N\frac{\pi}{2}\right)\tilde{J}_\kappa(rR)dr \\ &= -\frac{1}{2t}\cos\left(tr_2 - N\frac{\pi}{2}\right)\tilde{J}_\kappa(rR)\Big|_0^\infty + \frac{1}{2t}\int_0^\infty \cos\left(tr_2 - N\frac{\pi}{2}\right) rR^2\tilde{J}_{\kappa+1}(rR)dr \\ &= O\left(t^{-1}\right) + \frac{1}{2t}\int_0^\infty \cos\left(tr_2 - N\frac{\pi}{2}\right) rR^2\tilde{J}_{\kappa+1}(rR)dr. \end{aligned} \tag{2.46}$$

By making the change of variable $s = rR$ the remaining integral may be written as

$$C\int_0^\infty \cos\left(t\left(1 + s^4/R^4\right)^{1/2} - N\frac{\pi}{2}\right) s\tilde{J}_{\kappa+1}(s)ds \tag{2.47}$$

which is bounded independently of R since $\kappa \geq n/2 - 1 - N = n/4 \geq 1$.

For $k \geq 1$ we have $\kappa \geq 2$ and we may apply Lemma 2.5 with $\nu = \kappa$, $\sigma = 3 + 4k$ and $\tau = -1 - 2k$ to obtain

$$I_{jk}(R) = \int_0^\infty \sin\left(tr_2 - N\frac{\pi}{2}\right) r^{3+4k} r_2^{-1-2k}\tilde{J}_\kappa(rR) = \frac{-1}{2t}\int_0^\infty \cos\left(tr_2 - N\frac{\pi}{2}\right) P_{jk}(rR)dr \tag{2.48}$$

where

$$P_{jk}(r,R) = r_2^{-2k} r^{4k-1}[C\tilde{J}_\kappa(Rr) + \tilde{J}_{\kappa-1}(Rr)] + Cr_2^{-2-2k} r^{3+4k}\tilde{J}_\kappa(Rr) \tag{2.49}$$

On the interval $[0,1]$ the function $P_{jk}(r,R)$ is bounded and thus we only need to estimate the integral over the interval $[1,\infty)$. We integrate by parts once again to find

$$\begin{aligned} \int_1^\infty \cos\left(tr_2 - N\frac{\pi}{2}\right) P_{jk}(r,R)dr &= Ct^{-1}\sin\left(tr_2 - N\frac{\pi}{2}\right)\frac{P_{jk}(r,R)r_2}{r^3}\Big|_1^\infty \\ &\quad + Ct^{-1}\int_1^\infty \sin\left(tr_2 - N\frac{\pi}{2}\right)\frac{d}{dr}\left(\frac{P_{jk}(r,R)r_2}{r^3}\right)dr \end{aligned} \tag{2.50}$$

where

$$P_{jk}(r,R) = r_2^{-2k} r^{4k-1}(C\tilde{J}_\kappa(Rr) + \tilde{J}_{\kappa-1}(Rr)) + r_2^{-2k-2} r^{4k+3}\tilde{J}_\kappa(Rr). \tag{2.51}$$

Since P_{jk} is bounded and vanishes as $r \to \infty$ the first term is of order $O\left(t^{-1}\right)$. A simple computation, using the fact that $\kappa \geq 2$ shows that the derivative in the integrand is of order

$O\left(r^{-3}\right)$ for $r > 1$ and therefore the last integral is bounded. This shows that, for $k \geq 1$, we have $I_{jk}(R) \leq Ct^{-1}$. Hence the lemma is proved for $n \equiv 0(4)$.

Case 2: $n \not\equiv 0(4)$. In this case we will estimate I_{jk} over $[0, t^{-1/4}]$ and $[t^{-1/4}, \infty]$ separately. On the interval $[0, t^{-1/4}]$, $f_{jk}(r, R)$ is of order $O\left(r^{n-1-4N+4k}\right)$ so that

$$\begin{aligned} \int_0^{t^{-1/4}} \sin\left(tr_2 - N\frac{\pi}{2}\right) f_{jk}(r, R)dr \leq C \int_0^{t^{-1/4}} r^{n-1-4N+4k}dr \\ = Ct^{N-n/4-l} \leq Ct^{N-n/4}. \end{aligned} \tag{2.52}$$

On the interval $[t^{-1/4}, \infty]$ we may integrate by parts using (2.36) to get

$$\begin{aligned} \int_{t^{-1/4}}^{\infty} \sin\left(tr_2 - N\frac{\pi}{2}\right) f_{jk}(r, R)dr = \frac{1}{2t} \int_{t^{-1/4}}^{\infty} \cos\left(tr_2 - N\frac{\pi}{2}\right) P_{jk}(r, R)dr \\ = -\frac{1}{2t} \cos\left(tr_2 - N\frac{\pi}{2}\right) r_2^{N+1-n/4} r^{n-4-4N} \tilde{J}_\kappa(Rr)\Big|_{t^{-1/4}}^{\infty} \end{aligned} \tag{2.53}$$

where $P_{jk}(r, R)$ is defined as in (2.44). By (2.31), the second term vanishes at the upper limit $r = +\infty$, while at the lower limit (2.29) implies that this term is of order $O\left(t^{N-n/4-l}\right)$. As above it follows that $P_{jk}(r) = O\left(r^{n/2-3-2N}\right)$ for $r > 1$. For $r < 1$, $P_{\nu\sigma\tau}(r) = O\left(r^{\sigma-4}\right)$ and therefore $P_{jk}(r, R) = O\left(r^{n-5-4N+4k}\right)$. Since $2N \geq \frac{n-3}{2}$ for $n \not\equiv 0(4)$ we have $n/2 - 3 - 2N \leq -3/2$ and therefore the remaining integral is of order

$$O\left(t^{-1} \int_{t^{-1/4}}^{1} r^{n-5-4N+4k}dr + t^{-1} \int_1^{\infty} r^{-3/2}dr\right) = O\left(t^{N-n/4}\right). \tag{2.54}$$

Hence

$$I_{jk}(x) = O\left(t^{N-\frac{n}{4}}\right) \tag{2.55}$$

for $n \not\equiv 0(4)$. This completes the proof of Lemma 2.6. ■

Estimates for $n < 4$.

The second part of Lemma 2.3 is proven in the following Lemma.

Lemma 2.7 *For $1 \leq n \leq 3$ we have*

$$\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-n/4} \qquad \textit{for} \qquad t > 1 \tag{2.56}$$

$$\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{1-n/2} \qquad \textit{for} \qquad t < 1. \tag{2.57}$$

Proof. We will first consider the case $t < 1$, and prove (2.57) for dimensions $n = 1$, 2 and 3 separately.

For $n = 1$, $\tilde{J}_{-1/2}(s) = \cos(s)$ and therefore

$$K^s(t)(R) = 2\pi \int_0^{\infty} \sin(tr_2) r_2^{-1} \cos(Rr)dr. \tag{2.58}$$

We need to show $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{1/2}$. Over the interval $[0, t^{-1/2}]$ we use the estimate $\sin(s)s^{-1} \le C$ to get

$$\int_0^{t^{-1/2}} \sin(tr_2)r_2^{-1}\cos(Rr)dr \le C\int_0^{t^{-1/2}} tdr = Ct^{1/2}. \tag{2.59}$$

On the remaining interval

$$\int_{t^{-1/2}}^\infty \sin(tr_2)r_2^{-1}\cos(Rr)dr \le C\int_{t^{-1/2}}^\infty r^{-2}dr = Ct^{1/2}. \tag{2.60}$$

For $n = 2$,

$$K^s(t)(R) = (2\pi)^2\int_0^\infty \sin(tr_2)r_2^{-1}rJ_0(Rr)dr \tag{2.61}$$

and we need to show $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} = O(1)$ in t. On $[0, t^{-1/2}]$ we have

$$\int_0^{t^{-1/2}} \sin(tr_2)r_2^{-1}rJ_0(Rr)dr \le C\int_0^{t^{-1/2}} trdr = C. \tag{2.62}$$

On $[t^{-1/2}, \infty]$ we first consider $R < t^{1/2}$ and integrate by parts to obtain

$$\begin{aligned}\int_{t^{-1/2}}^\infty \sin(tr_2)r_2^{-1}rJ_0(Rr)dr &= \frac{-1}{2t}\cos(tr_2)r^{-2}J_0(Rr)\Big|_{t^{-1/2}}^\infty \\ &- \frac{1}{2t}\int_{t^{-1/2}}^\infty \cos(tr_2)[2r^{-3}J_0(Rr) + r^{-2}RJ_1(Rr)]dr \\ &\le O(1) + Ct^{-1}\int_{t^{-1/2}}^\infty r^{-2}(r^{-1} + R)dr \\ &= O(1) + O\left(t^{-1/2}R\right) = O(1).\end{aligned} \tag{2.63}$$

Finally if $R > t^{1/2}$ we use the estimate $J_0(s) = O\left(s^{-1/2}\right)$ for $s > 1$ to obtain

$$\int_{t^{-1/2}}^\infty \sin(tr_2)r_2^{-1}rJ_0(Rr)dr \le CR^{-1/2}\int_{t^{-1/2}}^\infty r^{-3/2}dr = O\left(R^{-1/2}t^{1/4}\right) = O(1). \tag{2.64}$$

If $n = 3$

$$K^s(t)(R) = (2\pi)^3\int_0^\infty \sin(tr_2)r_2^{-1}r^2\tilde{J}_{1/2}(Rr)dr \tag{2.65}$$

and we need to show $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-1/2}$. On the interval $[0, t^{-1/2}]$ we estimate

$$\int_0^{t^{-1/2}} \sin(tr_2)r_2^{-1}r^2\tilde{J}_{1/2}(Rr)dr \le C\int_0^{t^{-1/2}} tr^2dr = O\left(t^{-1/2}\right). \tag{2.66}$$

On $[t^{-1/2}, \infty]$ we integrate by parts once again to find

$$\begin{aligned}\int_{t^{-1/2}}^\infty \sin(tr_2)r_2^{-1}r^2\tilde{J}_{1/2}(Rr)dr &= \frac{-1}{2t}\cos(tr_2)r^{-1}\tilde{J}_{1/2}(Rr)\Big|_{t^{-1/2}}^\infty \\ &+ \frac{1}{2t}\int_{t^{-1/2}}^\infty \cos(tr_2)r^{-2}\left(\cos(Rr) - 2\frac{\sin(Rr)}{Rr}\right)dr \\ &= O\left(t^{-1/2}\right) + O\left(t^{-1}\int_{t^{-1/2}}^\infty r^{-2}\right) = O\left(t^{-1/2}\right).\end{aligned} \tag{2.67}$$

We remark here that the estimate (2.19) on $K^c(t)$ for $t < 1$ follows in exactly the same way as just shown. The only difference is that an estimate on $\sin(tr_2)$ was used on the interval $[0, t^{-1/2}]$. Using the fact that $n - 3 < n/2 - 1$ for $n < 4$ we see that

$$K^c(t)(R) = O\left(\int_0^{t^{-1/2}} r_2^{-1} r^{n-1} dr\right) = O(1) + O\left(\int_1^{t^{-1/2}} r^{n/2-1}\right) = O\left(t^{-n/4}\right) \tag{2.68}$$

Next, for $t > 1$ we again use a different argument in each dimension. For $n = 1$, we need to show that $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} = O\left(t^{-1/4}\right)$. Define $H'(r) = \sin(tr_2)\cos(Rr)$ with $H(0) = 0$. Then by Corollary 2.9 below,

$$|H(r)| \le Ct^{-1/4} \tag{2.78}$$

and (2.65) implies

$$\begin{aligned} K^s(t)(x) &= 2\pi H(r) r_2^{-1}\Big|_0^\infty + 4\pi \int_0^\infty H(r)\left(\frac{r}{r_2}\right)^3 dr \\ &= O\left(t^{-1/4}\int_0^\infty \left(\frac{r}{r_2}\right)^3 dr\right) = O\left(t^{-1/4}\right). \end{aligned} \tag{2.69}$$

For $n = 2$ we need to show $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-1/2}$. On the interval $[0, t^{-1/4}]$ we write

$$\int_0^{t^{-1/4}} \sin(tr_2) r_2^{-1} r J_0(Rr) dr \le C \int_0^{t^{-1/4}} r dr = Ct^{-1/2} \tag{2.70}$$

On the remaining interval, we first consider the case $R < t^{1/4}$ to obtain

$$\begin{aligned} \int_{t^{-1/4}}^\infty \sin(tr_2) r_2^{-1} r J_0(Rr) dr &= -\frac{\cos(tr_2)}{2tr^2} J_0(Rr)\Big|_{t^{-1/4}}^\infty \\ &\quad + \frac{1}{2t}\int_{t^{-1/4}}^\infty \cos(tr_2)[r^{-3}J_0(Rr) + r^{-2}RJ_1(Rr)]dr \\ &= O\left(t^{-1/2}\right) + O\left(t^{-1}\int_{t^{-1/4}}^\infty r^{-3} + Rr^{-2} dr\right) = O\left(t^{-1/2}\right). \end{aligned} \tag{2.71}$$

Next, if $t > R > t^{1/4}$ we first consider the interval $[t^{-1/4}, (R/t)^{1/3}]$ and use the estimate (2.19) on $J_0(s)$ for large s to get

$$\begin{aligned} \int_{t^{-1/4}}^{(R/t)^{1/3}} \sin(tr_2) r_2^{-1} r J_0(Rr) dr &= O\left(\int_{t^{-1/4}}^{(R/t)^{1/3}} r_2^{-1} r^{1/2} R^{-1/2} dr\right) \\ &= O\left(R^{-1/2}(R/t)^{1/2}\right) = O\left(t^{-1/2}\right). \end{aligned} \tag{2.72}$$

On the interval $[(R/t)^{1/3}, \infty)$ we integrate by parts and use the estimate (2.31) once again to obtain

$$\begin{aligned}
\int_{(R/t)^{1/3}}^{\infty} \sin(tr_2) r_2^{-1} r J_0(Rr) dr &= -\frac{\cos(tr_2)}{2tr^2} J_0(Rr)\Big|_{(R/t)^{1/3}}^{\infty} \\
&+ \frac{1}{2t} \int_{(R/t)^{1/3}}^{\infty} \cos(tr_2)[r^{-3} J_0(Rr) + r^{-2} R J_1(Rr)] dr \\
&= O\left(t^{-1}(R/t)^{-2/3}\right) + O\left(\frac{1}{2t} \int_{(R/t)^{1/3}}^{\infty} r^{-5/2} R^{1/2} dr\right) \\
&= O\left(t^{-1}(R/t)^{-2/3}\right) + O\left(t^{-1}(R/t)^{-1/2} R^{1/2}\right) = O\left(t^{-1/2}\right)
\end{aligned} \tag{2.73}$$

for $t^{1/4} < R < t$. Finally, for $R > t$ we use the asymptotic expansion (2.30) of J_0 again to conclude

$$\int_{t^{-1/4}}^{\infty} \sin(tr_2) r_2^{-1} r J_0(Rr) dr = O\left(R^{-1/2} \int_{t^{-1/4}}^{\infty} r_2^{-1} r^{1/2} dr\right) = O\left(t^{-1/2}\right). \tag{2.74}$$

For $n = 3$ we must show $\|K^s(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-3/4}$. On the interval $[0, t^{-1/4}]$ we have

$$\int_0^{t^{-1/4}} \sin(tr_2) r_2^{-1} r^2 \tilde{J}_{1/2}(Rr) dr \leq C \int_0^{t^{-1/4}} r^2 dr = Ct^{-3/4}. \tag{2.75}$$

On the remaining interval we can use (2.34) to compute

$$\begin{aligned}
\int_{t^{-1/4}}^{\infty} \sin(tr_2) r_2^{-1} r^2 \tilde{J}_{1/2}(Rr) dr &= \frac{-1}{2t} \cos(tr_2) r^{-1} \tilde{J}_{1/2}(Rr)\Big|_{t^{-1/4}}^{\infty} \\
&+ \frac{1}{2t} \int_{t^{-1/4}}^{\infty} \cos(tr_2) r^{-2} \left(\cos(Rr) - 2\frac{\sin(Rr)}{Rr}\right) dr \\
&= O\left(t^{-3/4}\right) + O\left(t^{-1} \int_{t^{-1/4}}^{\infty} r^{-2}\right) = O\left(t^{-3/4}\right).
\end{aligned} \tag{2.76}$$

This completes the proof of Lemma 2.7. ■

Lemma 2.8 *Let $f \in C^k(a, b)$, $k \geq 2$, and define*

$$G = \int_a^b e^{if(r)} dr. \tag{2.77}$$

If $|f'(x)| \geq \lambda$ and $f''(x) \neq 0$ on $[a, b]$ then $|G| \leq C\lambda^{-1}$. If $|f^{(k)}(x)| \geq \lambda$ on $[a, b]$ then $|G| \leq C\lambda^{-1/k}$, where the constant C depends only on k.

Corollary 2.9 *Define $H(r)$ by $H'(r) = \sin(tr_2)\cos(Rr)$ and $H(0) = 0$. Then*

$$|H(r)| \leq Ct^{-1/4} \tag{2.78}$$

Proof. We apply Lemma 2.8 to the function $G_\pm(r)$ defined by $G'_\pm(r) = e^{i(tr_2 \pm Rr)}$, $G_\pm(0) = 0$. The derivatives of $f(r) = tr_2 \pm Rr$ up to fourth order are

$$\begin{aligned} f'(r) &= \frac{2tr^3}{r_2} \pm R & f''(r) &= \frac{2tr^2(3+r^4)}{r_2^3} \\ f'''(r) &= \frac{12tr(1-r^4)}{r_2^5} & f''''(r) &= \frac{12t(1-14r^4+5r^8)}{r_2^7} \end{aligned} \tag{2.79}$$

Since $f''(r) > t$ for $r \geq 1/2$ and $f''''(r) > t$ for $r \leq 1/2$ we have

$$|G_\pm(r)| \leq C_1 t^{-\frac{1}{2}} + C_2 t^{-\frac{1}{4}} \tag{2.80}$$

Since $\sin(tr_2)\cos(Rr)$ is a linear combination of terms of the form $e^{\pm i(tr_2 \pm Rr)}$, this proves (2.78). ■

Proof of Lemma 2.8. The first statement follows by writing $G = \int_a^b \frac{d}{dr}[e^{if(r)}]\frac{-i}{f'(r)}dr$ and integrating by parts to get

$$G = \left.\frac{e^{if(r)}}{f'(r)}\right|_a^b + i\int_a^b e^{if(r)}\frac{f''(r)}{[f'(r)]^2}dr \tag{2.81}$$

The modulus of the first term is bounded by $2/\lambda$, while the second term is bounded by

$$\left|\int_a^b \frac{f''(r)}{[f'(r)]^2}dr\right| = \left|\frac{1}{f'(b)} - \frac{1}{f'(a)}\right| \leq \frac{2}{\lambda} \tag{2.82}$$

since $f''(x)$ does not change sign on $[a, b]$. Proceeding inductively, suppose that $f^{(k+1)}(x) \geq \lambda > 0$ on $[a, b]$. We may also suppose that $f^{(k)}(x) \leq 0$ on $[a, c]$ and $f^{(k)}(x) \geq 0$ on $[c, b]$ for some $c \in [a, b]$. If $c < b$, then for any $\gamma \in (c, b)$ we have $f^{(k)}(x) \geq (\gamma - c)\lambda$ on $[\gamma, b]$, so that the induction hypothesis implies

$$G_1 = \left|\int_c^b e^{if(r)}dr\right| \leq \int_c^\gamma 1dr + \left|\int_\gamma^b e^{if(r)}dr\right| \leq (\gamma - c) + \frac{C}{(\gamma-c)^{1/k}\lambda^{1/k}} \tag{2.83}$$

Choosing $\gamma - c = C'\lambda^{1/(k+1)}$ minimizes the last expression. Hence $G_1 \leq C''\lambda^{-1/(k+1)}$. Similarly $G_2 = \left|\int_a^c e^{if(r)}dr\right| \leq C\lambda^{-1/(k+1)}$ and the lemma is proved. ■

Optimality

We next obtain lower bounds for the rate of decay which, along the line of duality, agree with the decay rates in Theorem 2.1.

Lemma 2.10 *There exists a constant C such that for any $0 < t < 1$ we have*

$$\|T^s(t)\|_{p,q} \geq Ct^{1-\frac{n}{2p}+\frac{n}{2q}} \tag{2.84}$$

and there is a sequence $t_k \to +\infty$ such that

$$\|T^s(t_k)\|_{p,q} \geq Ct_k^{\frac{n}{4q}-\frac{n}{4p}}. \tag{2.85}$$

We remark that Lemma 2.10 shows that the estimates of Theorem 2.1 are sharp since

$$\frac{n}{4q}-\frac{n}{4p}=\frac{n}{2q}-\frac{n}{4} \qquad \text{and} \qquad 1-\frac{n}{2p}+\frac{n}{2q}=1+\frac{n}{q}-\frac{n}{2} \tag{2.86}$$

on the line of duality.

Proof. We first consider the case $t>1$. Let $\psi(x)$ be a function with Fourier transform $\hat{\psi}(\xi)=\phi(|\xi|)$, where $\phi(r)$ is C^∞ with support on $[0,1]$ and satisfies $\phi\geq 0$ and $\phi(r)=1$ for $r\leq 1/2$. Then define $\psi_t(x)=t^{-\frac{n}{4p}}\psi(x/t^{1/4})$. It follows that $\hat{\psi}_t(\xi)=t^{\frac{n}{4p'}}\hat{\psi}(\xi t^{1/4})$ and we have the formula

$$T^s(t)\psi_t(R)=t^{\frac{n}{4p'}}\int_0^\infty \sin(tr_2)r_2^{-1}\tilde{J}_{n/2-1}(Rr)\phi(t^{1/4}r)r^{n-1}dr. \tag{2.87}$$

Since the integral takes place only over the interval $[0,t^{-1/4}]$ we may use the expansion

$$tr_2=t(1+r^4)^{1/2}=t+\frac{1}{2}tr^4+O\left(tr^8\right) \tag{2.88}$$

to get

$$\sin(tr_2)=\sin\left(t+\frac{1}{2}tr^4\right)+O\left(tr^8\right). \tag{2.89}$$

Thus we may break up the integral in (2.87) as $T^s(t)\psi_t(R)=I_1(R)+I_2(R)$ where the main term is

$$I_1(R)=t^{\frac{n}{4p'}}\int_0^{t^{-1/4}}\sin\left(t+\frac{1}{2}tr^4\right)r_2^{-1}\tilde{J}_{n/2-1}(Rr)\phi(t^{1/4}r)r^{n-1}dr. \tag{2.90}$$

Making the change of variable $s=t^{1/4}r$ gives

$$I_1(R)=t^{\frac{n}{4p'}-\frac{n}{4}}\int_0^1\frac{\sin(t+s^4/2)}{(1+s^4/t)^{-1/2}}\tilde{J}_{n/2-1}(Rst^{-1/4})\phi(s)s^{n-1}ds. \tag{2.91}$$

Since the Bessel function $\tilde{J}_{n/2-1}$ is positive on some interval $[0,a]$, we can choose constants c_1, c_2 so that if $c_1t^{1/4}\leq R\leq c_2t^{1/4}$ then $\tilde{J}_{n/2-1}(Rst^{-1/4})>0$ for $s\in(0,1)$. Also, for any $t>1$ we have $(1+s^4/t)^{-1/2}>1/2$. Finally, for $t_k=(4k+1)\frac{\pi}{2}$ we have $\sin(t_k+s^4/2)>1/2$ for $s\in(0,1)$. So for R and t_k chosen in this way we have

$$I_1(R)\geq Ct_k^{\frac{n}{4p'}-\frac{n}{4}}=Ct_k^{-\frac{n}{4p}}. \tag{2.92}$$

Next we must show that the term

$$I_2(R)=O\left(t^{1+\frac{n}{4p'}}\int_0^{t^{-1/4}}r_2^{-1}\tilde{J}_{n/2-1}(Rr)\phi(t^{1/4}r)r^{n+7}dr\right) \tag{2.93}$$

is really an error term. Making the same change of variable as above, we get

$$I_2(R)=O\left(t^{\frac{n}{4p'}-\frac{n}{4}-1}\int_0^1\left(1+\frac{s^4}{t}\right)^{-1/2}\tilde{J}_{n/2-1}(Rst^{-1/4})s^{n+7}ds\right). \tag{2.94}$$

Since the integral is bounded independently of t and R we have $I_2(R) = O\left(t^{\frac{n}{4p'}-\frac{n}{4}-1}\right)$ and therefore

$$T^s(t_k)\psi_{t_k}(R) \geq Ct_k^{\frac{n}{4p'}-\frac{n}{4}} = Ct_k^{-\frac{n}{4p}}. \tag{2.95}$$

Since this holds for R in a set of measure $t_k^{n/4}$ we have

$$\|T^s(t_k)\psi_{t_k}\|_{L^q(\mathbb{R}^n)} \geq Ct_k^{-\frac{n}{4p}} t^{\frac{n}{4q}} = Ct_k^{\frac{n}{4q}-\frac{n}{4p}} \qquad \text{for} \qquad t_k = \frac{(4k+1)\pi}{2}. \tag{2.96}$$

This proves (2.85) since $\|\psi_{t_k}\|_{L^p(\mathbb{R}^n)} = \|\psi\|_{L^p(\mathbb{R}^n)}$ for all k.

Next, for $t < 1$ we define $\psi_t(x) = t^{-\frac{n}{2p}}\psi(x/t^{\frac{1}{2}})$ where ψ is defined as above. We then have $\hat{\psi}_t(\xi) = t^{\frac{n}{2p'}}\hat{\psi}(\xi t^{\frac{1}{2}})$ and therefore

$$T^s(t)\psi_t(R) = t^{\frac{n}{2p'}} \int_0^\infty \sin(tr_2) r_2^{-1} \tilde{J}_{n/2-1}(Rr)\phi(t^{\frac{1}{2}}r) r^{n-1} dr. \tag{2.97}$$

Since ϕ is supported on $[0,1]$ we can make the change of variable $s = t^{1/2}r$ to get

$$T^s(t)\psi_t(R) = t^{\frac{n}{2p'}-\frac{n}{2}} \int_0^1 \frac{\sin(\sqrt{t^2+s^4})}{(1+s^4/t^2)^{1/2}} \tilde{J}_{n/2-1}(Rst^{-1/2})\phi(s)s^{n-1}ds \tag{2.98}$$

Now, for $s \leq 1$ and $t < 1$ we have $\sqrt{t^2+s^4} < \sqrt{2}$ and therefore

$$\sin(\sqrt{t^2+s^4})(1+s^4/t^2)^{-1/2} \geq t/2. \tag{2.99}$$

Also, there is some c_1 such that for $R < c_1 t^{1/2}$ we have $\tilde{J}_{n/2-1}(Rst^{-1/2}) > 0$ for $s \in (0,1)$. Hence

$$T^s(t)\psi_t(R) \geq Ct^{1+\frac{n}{2p'}-\frac{n}{2}} = Ct^{1-\frac{n}{2p}}. \tag{2.100}$$

Since this holds for R in a set of measure $c_2 t^{n/4}$ we have

$$\|T^s(t)\psi_t\|_q \geq Ct^{1-\frac{n}{2p}+\frac{n}{2q}}. \tag{2.101}$$

This proves (2.84) since $\|\psi_t\|_{L^p(\mathbb{R}^n)} = \|\psi\|_{L^p(\mathbb{R}^n)}$ for all t. ∎

3 Space-Time Integrability

In this section we again consider solutions of the linear equation

$$u_{tt} + \Delta^2 u + u = 0 \tag{2.1}$$

with arbitrary initial data in X and show that they are integrable in both space and time. The main result is the following.

Theorem 3.1 *For any n let q satisfy*

$$2+\frac{8}{n}\leq q<q_n=\begin{cases}\frac{2(n+2)}{n-4} & \textit{for} \quad n\geq 5\\ \infty & \textit{for} \quad 1\leq n\leq 4.\end{cases} \tag{3.1}$$

Then the solution of (2.1) *with initial data* $(u_0,v_0)\in X\equiv H^2(\mathbb{R}^n)\oplus L^2(\mathbb{R}^n)$ *satisfies*

$$\|u\|_{L^q(\mathbb{R}^{n+1})}\leq C\|(u_0,v_0)\|_X \tag{3.2}$$

and

$$\|u_t\|_{W^{-2,q}(\mathbb{R}^{n+1})}\leq C\|(u_0,v_0)\|_X. \tag{3.3}$$

Theorem 3.1 is a consequence of the following theorem.

Theorem 3.2 *For any n choose q so that*

$$2+\frac{8}{n}\leq q<\begin{cases}\frac{2(n+2)}{n-2} & \textit{for} \quad n\geq 3\\ \infty & \textit{for} \quad n=1,2\end{cases} \tag{3.4}$$

Let $u(t)$ be the solution of (2.1) *with initial data $u(0)=u_0$ and $u_t(0)=v_0$, where $(u_0,v_0)\in Y\equiv H^1(\mathbb{R}^n)\oplus H^{-1}(\mathbb{R}^n)$. Then*

$$\|u\|_{L^q(\mathbb{R}^{n+1})}\leq C\|(u_0,v_0)\|_Y. \tag{3.5}$$

We prove Theorems 3.1 and 3.2 by using the method of Strichartz [20] to reduce the estimate to an estimate on the Fourier transform of a certain quartic function (3.17). The theorems then follow from the main estimate in Lemma 3.4. For completeness we first repeat here the reduction to the integral estimate.

By taking the Fourier transform of (2.1) in the spatial variable, we obtain the solution

$$u(x,t)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\left(e^{ir_2t}(r_2\hat{u}_0-i\hat{v}_0)+e^{-ir_2t}(r_2\hat{u}_0+i\hat{v}_0)\right)\frac{d\xi}{2\sqrt{1+|\xi|^4}}. \tag{3.6}$$

This formula may then be considered as the Fourier transform of a function on a surface in $\mathbb{R}^{n+1}$. More precisely, we define

$$S=\{(\xi,\tau):\tau^2-|\xi|^4-1=0\} \tag{3.7}$$

and define a measure on S by

$$d\mu=\frac{\delta_0(\tau^2-|\xi|^4-1)d\xi d\tau}{2\sqrt{1+|\xi|^4}} \tag{3.8}$$

where δ_0 denotes the delta function centered at 0. If we then let

$$g(\xi)=\begin{cases}(r_2\hat{u}_0(\xi)-i\hat{v}_0(\xi)) & \text{on } S_-\\ (r_2\hat{u}_0(\xi)+i\hat{v}_0(\xi)) & \text{on } S_+\end{cases} \tag{3.9}$$

where $S_-=S\cap\{\tau<0\}$ and $S_+=S\cap\{\tau>0\}$, we have

$$u(x,t)=\mathcal{F}^{-1}(gd\mu) \tag{3.10}$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^{n+1}$ and $\mathcal{F}^{-1}$ denotes its inverse. We will prove the following result regarding the measure $d\mu$.

Lemma 3.3 *Let q be chosen as is Theorem 3.1 and let $1/p+1/q=1$. If $g \in L^p(\mathbb{R}^{n+1})$, then $\mathcal{F}g\big|_S \in L^2(S,d\mu)$ and $\|\mathcal{F}g\|_{L^2(S,d\mu)} \leq C\|g\|_{L^p(\mathbb{R}^{n+1})}$. Also if $g \in L^2(S,d\mu)$, then $\mathcal{F}(gd\mu) \in L^q(\mathbb{R}^{n+1})$ and $\|\mathcal{F}(gd\mu)\|_{L^q(\mathbb{R}^{n+1})} \leq C\|g\|_{L^2(S,d\mu)}$.*

Proof of Theorem 3.2 By (3.10) and the second part of Lemma 3.3

$$\begin{aligned} \|u\|^2_{L^q(\mathbb{R}^n)} \leq C\|g\|^2_{L^2(S,d\mu)} &= \int_{\mathbb{R}^n} r_2|\hat{u}_0(\xi)|^2 + r_2^{-1}|\hat{u}_1(\xi)|^2 d\xi \\ &= \|(u_0,u_1)\|^2_{H^1(\mathbb{R}^n)\oplus H^{-1}(\mathbb{R}^n)} \end{aligned} \tag{3.11}$$

where $r_2 = (1+|\xi|^4)^{1/2}$ as in Chapter 1. ∎

Proof of Theorem 3.1 By (3.6) it is easily seen that $\nabla_x u = \mathcal{F}^{-1}(\xi g d\mu)$. We also have $|\partial_t|^{1/2}u = \mathcal{F}^{-1}(r_2^{1/2} g d\mu)$. As in the proof of Theorem 3.2, it follows from Lemma 3.3 that

$$\begin{aligned} \|\nabla_x u\|_{L^q(\mathbb{R}^{n+1})} &\leq C\|(u_0,v_0)\|_X \\ \||\partial_t|^{1/2}u\|_{L^q(\mathbb{R}^{n+1})} &\leq C\|(u_0,v_0)\|_X \end{aligned} \tag{3.12}$$

for q chosen as in Theorem 3.2. Hence

$$u \in L^q([0,T],W^{1,q}(\mathbb{R}^n)) \subset L^q([0,T],L^{\frac{nq}{n-q}}(\mathbb{R}^n)) \tag{3.13}$$

and

$$u \in W^{1/2,q}([0,T],L^q(\mathbb{R}^n)) \subset L^{\frac{2q}{2-q}}([0,T],L^q(\mathbb{R}^n)). \tag{3.14}$$

Interpolating between (3.13) and (3.14), with $s = \frac{n}{n+2}$ shows that $u \in L^{\bar{q}}(\mathbb{R}^{n+1})$ where $\bar{q} = \frac{(n+2)q}{n+2-q}$. This proves the first statement since by the choice of q in Theorem 3.2 we have $\bar{q} < \frac{2(n+2)}{n-4}$. The second statement follows since $(1+\Delta^2)^{-1/2}u_t = \mathcal{F}^{-1}(\tilde{g}d\mu)$, where $\tilde{g} = ig$ on S_- and $\tilde{g} = -ig$ on S_+. ∎

Proof of Lemma 3.3 We first note that by duality the two statements are identical. Also the first statement is implied by

$$\|\mathcal{F}(d\mu) * g\|_{L^q(\mathbb{R}^{n+1})} \leq C\|g\|_{L^p(\mathbb{R}^{n+1})} \tag{3.15}$$

where $1/p+1/q=1$. For if we assume (3.15) holds, then

$$\begin{aligned} \int_S |\mathcal{F}g|^2 d\mu = \int_S \overline{\mathcal{F}g}\mathcal{F}g d\mu &= \int_{\mathbb{R}^n} \bar{g}\mathcal{F}^{-1}(\mathcal{F}g d\mu) \\ &= \int_{\mathbb{R}^n} \bar{g}(\mathcal{F}(d\mu) * g) \leq C\|g\|_{L^p(\mathbb{R}^n)}\|\mathcal{F}(d\mu) * g\|_{L^q(\mathbb{R}^n)} \\ &\leq C\|g\|^2_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{3.16}$$

Hence it suffices to show that the operator $T = \mathcal{F}(d\mu)*$ is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^q(\mathbb{R}^{n+1})$ for p and q as above.

Let

$$G(z) = \Gamma(z+1)^{-1}(1-\tau^2+|\xi|^4)^z_+. \tag{3.17}$$

If we define $S_\sigma = \{(\xi, \tau) : \tau^2 - |\xi|^4 - \sigma = 0\}$ with the corresponding measure

$$d\mu_\sigma = \frac{\delta_0(\tau^2 - |\xi|^4 - \sigma)d\xi d\tau}{2\sqrt{\sigma + |\xi|^4}} \tag{3.18}$$

then for any test function $\varphi(\xi, \tau)$ we may make the change of variable $\sigma = \tau^2 + |\xi|^4$ to obtain

$$\int_{\mathbb{R}^{n+1}} G(z)\varphi(\xi, \tau)d\xi d\tau = \Gamma(z+1)^{-1} \int_{\mathbb{R}} \left(\int_{S_\sigma} \varphi d\mu_\sigma \right) (1-\sigma)_+^z d\sigma. \tag{3.19}$$

Since

$$\left. \frac{(1-\sigma)_+^z}{\Gamma(z+1)} \right|_{z=-1} = \delta_0(1-\sigma) \tag{3.20}$$

(see [3] p57) it follows that

$$\lim_{z \to -1} \int_{\mathbb{R}^{n+1}} G(z)\varphi(\xi, \tau)d\xi d\tau = C \int_S \varphi(\xi, \tau)d\mu. \tag{3.21}$$

So if we define the family of linear operators

$$T_z(g) = \mathcal{F}^{-1}G(z) * g \tag{3.22}$$

then Lemma 3.3 is equivalent to the statement that T_{-1} is a bounded operator from $L^p(\mathbb{R}^{n+1})$ to $L^q(\mathbb{R}^{n+1})$.

Now, for $Re(z) = 0$, $G(z)$ is bounded by $Ce^{|Im(z)|}$ and therefore $\|T_z\|_{2,2} \le Ce^{|Im(z)|}$. According to Lemma 3.4 below we also have $\|T_z\|_{1,\infty} \le Ce^{C|Im(z)|}$. Therefore Lemma 3.3 follows from Theorem 2.2. ■

Lemma 3.4 *Let $G(z)$ be defined as is* (3.17). *Then $\mathcal{F}(G(z)) \in L^\infty(\mathbb{R}^{n+1})$ and for $|Im(z)|$ large we have*

$$\|\mathcal{F}(G(z))\|_{L^\infty(\mathbb{R}^n)} \le Ce^{2\pi|Im(z)|} \tag{3.23}$$

where z satisfies

$$-n/4 - 1 \le Re(z) < \begin{cases} -n/4 - 1/2 & for \quad n \ge 3 \\ -1 & for \quad n = 1, 2. \end{cases} \tag{3.24}$$

The rest of this section is spent proving Lemma 3.4. We first observe that, for fixed ξ, the quantity in (3.17) is quadratic in τ and therefore its Fourier transform in τ may be explicitly computed using Lemma 3.5 below. Since the result is a radial function of ξ we may then use the formula from Section 2 (2.26) for the Fourier transform of a radial function. The problem is then reduced to bounding an integral (2.23) in the same way as in Section 2. We let $\mathcal{F}_\tau$ denote the Fourier transform with respect to τ.

Lemma 3.5 *Let $f(\tau) = (a^2 - \tau^2)_+^z$. Then*

$$\mathcal{F}_\tau f(t) = \sqrt{\pi}\Gamma(z+1)a^{2z+1} \left(\frac{at}{2} \right)^{-z-1/2} J_{z+1/2}(at). \tag{3.25}$$

Proof. This follows by making the dilation $\tau \to \tau/a$ in the formula

$$\mathcal{F}_\tau(1-\tau^2)_+^z = \sqrt{\pi}\Gamma(z+1)\left(\frac{t}{2}\right)^{-z-1/2} J_{z+1/2}(t), \tag{3.26}$$

which can be found in [3], p185. ■

Using Lemma 3.5 with $a^2 = 1+|x|^4$ gives

$$\mathcal{F}_\tau(G(z)) = \sqrt{\pi}(1+|\xi|^4)^{z/2+1/4}\left(\frac{t}{2}\right)^{-z-1/2} J_{z+1/2}(t(1+|\xi|^4)^{1/2}) \tag{3.27}$$

where J_ν once again denotes the Bessel function of (complex) order ν. Since the result is a radial function of the variable ξ, we can use (2.26) to obtain

$$\mathcal{F}(G(z)) = CR^{1-n/2}\left(\frac{2}{t}\right)^{z+1/2}\int_0^\infty r_2^{z+1/2} J_{z+1/2}(tr_2)J_{n/2-1}(Rr)r^{n/2}dr. \tag{3.28}$$

With $\tilde{J}_\nu(s) = s^{-\nu}J_\nu(s)$ defined as in Chapter 1 we may write

$$\mathcal{F}(G(z)) = C2^{z+1/2}\int_0^\infty r_2^{2z+1}\tilde{J}_{z+1/2}(tr_2)\tilde{J}_{n/2-1}(Rr)r^{n-1}dr. \tag{3.29}$$

We next prove the analog of Lemma 2.5.

Lemma 3.6 *Suppose $-2Re(\mu)+2Re(\tau)+\sigma-\nu-5/2<0$ and $\sigma>3$. Then*

$$ir\tilde{J}_\mu(tr_2)r_2^\tau r^\sigma \tilde{J}_\nu(rR)dr = -\frac{1}{2}\int_0^\infty \tilde{J}_{\mu+1}(tr_2)P_{\nu\sigma\tau\mu}(r,R)dr \tag{3.30}$$

where

$$\begin{aligned} P_{\nu\sigma\tau\mu}(r,R) &= \frac{d}{dr}\left(r_2^{\tau+2}r^{\sigma-3}\tilde{J}_\nu(Rr)\right) - 4(\mu+1)r_2^\tau r^\sigma \tilde{J}_\nu(Rr) \\ &= r_2^{\tau+2}r^{\sigma-4}[(\sigma-2\nu-3)\tilde{J}_\nu(Rr)+\tilde{J}_{\nu-1}(Rr)] \\ &\quad + 2(\tau-2\mu-2)r_2^\tau r^\sigma \tilde{J}_\nu(Rr). \end{aligned} \tag{3.31}$$

Proof. By Lemma 2.4 (2.32) we have

$$\frac{d}{dr}\left(\tilde{J}_\nu(tr_2)\right) = \frac{2r^3}{r_2^2}(\tilde{J}_{\nu-1}(tr_2) - 2\nu\tilde{J}_\nu(tr_2)) \tag{3.32}$$

and consequently

$$\tilde{J}_\nu(tr_2) = 2(\nu+1)\tilde{J}_{\nu+1}(tr_2) + \frac{r_2^2}{2r^3}\frac{d}{dr}\left(\tilde{J}_{\nu+1}(tr_2)\right) \tag{3.33}$$

Using (3.33) we have

$$\int_0^\infty \tilde{J}_\mu(tr_2)r_2^\tau r^\sigma \tilde{J}_\nu(Rr)dr = \int_0^\infty 2(\mu+1)\tilde{J}_{\mu+1}(tr_2)r_2^\tau r^\sigma \tilde{J}_\nu(Rr)dr + \frac{1}{2}\tilde{J}_{\mu+1}(tr_2)r_2^{\tau+2}r^{\sigma-3}\tilde{J}_\nu(Rr)\Big|_0^\infty - \frac{1}{2}\int_0^\infty \tilde{J}_{\mu+1}(tr_2)\frac{d}{dr}\left(r_2^{\tau+2}r^{\sigma-3}\tilde{J}_\nu(Rr)\right)dr. \tag{3.34}$$

By Lemma 2.4 the second term in (3.34) is of order $O\left(r^{-2Re(\mu)+2Re(\tau)+\sigma-\nu-5/2}\right)$ for large r and therefore vanishes as $r\to\infty$. It vanishes at $r=0$ as well since $\sigma>3$. The result follows by expanding the derivative inside the integrand, using (2.32). ■

Estimates for $n \le 2$.

Define $g_n(r,R,t,z) = r_2^{2z+1}\tilde{J}_{z+1/2}(tr_2)\tilde{J}_{n/2-1}(Rr)r^{n-1}$ for $n=1,2$. Then (3.29) implies

$$\mathcal{F}(G(z)) = C_z\int_0^\infty g_n(r,R,t,z)dr. \tag{3.35}$$

For $n=1$

$$g_1(r,R,t,z) = r_2^{2z+1}\tilde{J}_{z+1/2}(tr_2)\cos(Rr). \tag{3.36}$$

If $t<1$, g_1 is bounded by $Ce^{C|Im(z)|}$ for $r\in[0,1]$ independently of R and t and thus

$$\int_0^1 g_1(r,R,t,z)dr = O\left(e^{C|Im(z)|}\right). \tag{3.37}$$

On the interval $[1,t^{-1/2}]$ we have $tr_2\le 2$, and for such arguments the Bessel function bounded by $Ce^{C|Im(z)|}$. Thus

$$\int_1^{t^{-1/2}} g_1(r,R,t,z)dr = O\left(e^{C|Im(z)|}\int_1^{t^{-1/2}} r^{4Re(z)+2}dr\right) = O\left(e^{C|Im(z)|}\right) \tag{3.38}$$

since $Re(z)\le -1$. Finally, on $[t^{-1/2},\infty]$ the estimate (3.33) implies that

$$\int_{t^{-1/2}}^\infty g_1(r,R,t,z)dr = O\left(e^{C|Im(z)|}\int_{t^{-1/2}}^\infty t^{-Re(z)-1}r^{2Re(z)}dr\right) = O\left(e^{C|Im(z)|}t^{-2Re(z)-3/2}\right) = O\left(e^{C|Im(z)|}\right) \tag{3.39}$$

since $Re(z)\le -1$.

If $t>1$ we use (2.30) to obtain

$$\mathcal{F}(G(z)) = C_z\int_0^\infty r_2^z t^{-z-1}\cos(tr_2+C_z)\cos(Rr)dr + O\left(e^{C|Im(z)|}\int_0^\infty r_2^{Re(z)-1}t^{-Re(z)-2}dr\right). \tag{3.40}$$

Since $-5/4 \leq Re(z) \leq -1$ the last integral is bounded. To estimate the first integral we define $H(r)$ by $H'(r) = \cos(tr_2 + C_z)\cos(Rr)$ with $H(0) = 0$. Lemma 2.8 then implies that

$$|H(r)| \leq Ct^{-1/4}e^{C|Im(z)|} \tag{3.41}$$

and therefore integration by parts yields

$$\begin{aligned}\mathcal{F}(G(z)) &= C_z H(r) r_2^z t^{-z-1}\Big|_0^\infty + O\left(e^{C|Im(z)|}\right) \\ &- 2zt^{-z-1}\int_0^\infty H(r) r_2^{z-2} r^3 dr\end{aligned} \tag{3.42}$$

The first term vanishes since $Re(z) < 0$ and $H(0) = 0$. Finally, the estimate (3.24) shows that the integral is of order

$$O\left(e^{C|Im(z)|}t^{-Re(z)-5/4}\int_0^\infty r_2^{2Re(z)-2} r^3 dr\right). \tag{3.43}$$

Since $Re(z) < 0$ the integral is bounded, and since $Re(z) \geq -5/4$ this term is of order $O\left(e^{C|Im(z)|}\right)$. This proves the estimate in dimension one.

For $n = 2$, $g_2(r, R, t, z) = r_2^{2z+1}\tilde{J}_{z+1/2}(tr_2) r J_0(Rr)$. When $t < 1$ (2.29) implies

$$\int_0^1 g_2(r, R, t, z) dr = O\left(e^{C|Im(z)|}\right) \tag{3.44}$$

and

$$\begin{aligned}\int_1^{t^{-1/2}} g_2(r, R, t, z) dr &= O\left(e^{C|Im(z)|}\int_1^{t^{-1/2}} r^{4Re(z)+3} dr\right) \\ &= O\left(e^{C|Im(z)|}t^{-2Re(z)-2}\right) = O\left(e^{C|Im(z)|}\right)\end{aligned} \tag{3.45}$$

if $Re(z) < -1$. Over $[t^{-1/2}, \infty]$ it follows from (2.31) that

$$\begin{aligned}\int_{t^{-1/2}}^\infty g_2(r, R, t, z) dr &= O\left(e^{C|Im(z)|}\int_{t^{-1/2}}^\infty t^{-Re(z)-1} r^{2Re(z)+1} dr\right) \\ &= O\left(e^{C|Im(z)|}t^{-2Re(z)-2}\right) = O\left(e^{C|Im(z)|}\right)\end{aligned} \tag{3.46}$$

since $Re(z) < -1$. Next, if $t > 1$ we can use (2.30) to obtain

$$\begin{aligned}\widehat{G(z)} &= C\int_0^\infty t^{-z-1} r_2^z \cos(tr_2 + C_z) r J_0(Rr) dr \\ &+ O\left(e^{C|Im(z)|}t^{-Re(z)-2}\int_0^\infty r_2^{Re(z)-1} r dr\right).\end{aligned} \tag{3.47}$$

For $Re(z) < -1/2$ the second integral is bounded. Therefore, since $Re(z) \geq -3/2$ this term is of order $O\left(e^{C|Im(z)|}\right)$. We bound the remaining integral first on $[0, t^\beta]$ where $\beta =$

$Re(z)/2+1/2$. Using (2.31) we obtain

$$\begin{aligned}\int_0^{t^\beta} t^{-z-1} r_2^z \cos(tr_2+C_z) r J_0(Rr)dr &= O\left(e^{C|Im(z)|} t^{-2\beta}\int_0^{t^\beta} rdr\right)\\ &= O\left(e^{C|Im(z)|}\right).\end{aligned} \tag{3.48}$$

On $[t^\beta,\infty]$ first consider $R<t^{3\beta+1}$. Integration by parts yields

$$\begin{aligned}\int_{t^\beta}^\infty t^{-z-1} r_2^z \cos(tr_2+C_z) r J_0(Rr)dr &= t^{-z-2}\frac{\sin(tr_2)}{2r^2} r_2^{z+1} J_0(Rr)\Big|_{t^\beta}^\infty\\ &- t^{-z-2}\int_{t^\beta}^\infty \sin(tr_2)J_0(Rr)[Cr_2^{z-1}r+Cr_2^{z+1}r^{-3}]dr\\ &- t^{-z-2}\int_{t^\beta}^\infty \sin(tr_2)J_0(Rr)r_2^{z+1}r^{-2}RJ_1(Rr)dr.\end{aligned} \tag{3.49}$$

The first term vanishes at infinity, while at $r=t^\beta$ it is of order $O\left(e^{C|Im(z)|}t^{-2Re(z)-3}\right) = O\left(e^{C|Im(z)|}\right)$ since $Re(z)\geq -3/2$. The second term is also of order $O\left(e^{C|Im(z)|}t^{-2Re(z)-3}\right)$ while the last term is order

$$O\left(e^{C|Im(z)|}t^{-3\beta-1}R\right) = O\left(e^{C|Im(z)|}\right) \tag{3.50}$$

for $R<t^{3\beta+1}$.

For $t^{3\beta+1}<R<t$ we first consider the interval $[t^\beta,(R/t)^{1/3}]$. Since $rR>1$ on this interval,we may use (2.31) to see that

$$\begin{aligned}\int_{t^\beta}^{(R/t)^{1/3}} t^{-z-1} r_2^z \cos(tr_2) r J_0(Rr)dr &= O\left(e^{C|Im(z)|}t^{-2\beta}\int_{t^\beta}^{(R/t)^{1/3}} r^{1/2}R^{-1/2}dr\right)\\ &= O\left(e^{C|Im(z)|}t^{-2\beta}(R/t)^{1/2}R^{-1/2}\right)\\ &= O\left(e^{C|Im(z)|}t^{-Re(z)-3/2}\right) = O\left(e^{C|Im(z)|}\right)\end{aligned} \tag{3.51}$$

for $Re(z)\geq -3/2$. On $[(R/t)^{1/3},\infty]$ integration by parts gives

$$\begin{aligned}\int_{(R/t)^{1/3}}^\infty t^{-z-1} r_2^z \cos(tr_2) r J_0(Rr)dr &= t^{-z-2}\frac{\sin(tr_2)}{2r^2} r_2^{z+1} J_0(Rr)\Big|_{(R/t)^{1/3}}^\infty\\ &- t^{-z-2}\int_{(R/t)^{1/3}}^\infty \sin(tr_2)J_0(Rr)[Cr_2^{z-1}r+Cr_2^{z+1}r^{-3}]dr\\ &- t^{-z-2}\int_{(R/t)^{1/3}}^\infty \sin(tr_2)J_0(Rr)r_2^{z+1}r^{-2}RJ_1(Rr)dr.\end{aligned} \tag{3.52}$$

Using the estimate (2.31) again, this reduces to

$$\begin{aligned}\int_{(R/t)^{1/3}}^\infty t^{-z-1} r_2^z \cos(tr_2) r J_0(Rr)dr &= O\left(e^{C|Im(z)|}t^{-Re(z)-2}(R/t)^{-2/3}\right)\\ &+ O\left(e^{C|Im(z)|}t^{-Re(z)-2}\int_{(R/t)^{1/3}}^\infty r^{-5/2}R^{1/2}dr\right)\\ &= O\left(e^{C|Im(z)|}t^{-Re(z)-4/3}R^{-2/3}\right) + O\left(e^{C|Im(z)|}t^{-Re(z)-3/2}\right)\\ &= O\left(e^{C|Im(z)|}\right).\end{aligned} \tag{3.53}$$

Finally, for $R > t$ we use (2.31) again to obtain

$$\begin{aligned}\int_{t^{-1/4}}^{\infty} \cos(tr_2)t^{-z-1}r_2^z r J_0(Rr)dr &= O\left(\frac{e^{C|Im(z)|}}{R^{1/2}t^{Re(z)+1}} \int_{t^{-1/4}}^{\infty} r^{1/2} r_2^{Re(z)} dr\right) \\ &= O\left(e^{C|Im(z)|}t^{-Re(z)-1}R^{-1/2}\right) \\ &= O\left(e^{C|Im(z)|}t^{-Re(z)-3/2}\right) = O\left(e^{C|Im(z)|}\right)\end{aligned} \tag{3.54}$$

for $Re(z) \geq -3/2$. This proves Lemma 3.4 in dimensions $n \leq 2$.

Estimates for $n \geq 3$.

In dimension $n \geq 3$ we apply Lemma 3.6 multiple times. After N iterations we arrive at

$$\mathcal{F}(G(z)) = C \int_0^{\infty} \tilde{J}_{z+1/2+N}(tr_2) \sum_{j+k+l=N} h_{jkl}(r,R)dr \tag{3.55}$$

where

$$h_{jkl}(r,R) = C_{jkl} r_2^{2z+1+2j+2l} r^{n-1-4j-4l} \tilde{J}_{n/2-1-j}(Rr) \tag{3.56}$$

Once again, since $j+k+l = N$ we must consider terms of the form

$$I_{jk}(R,t) = \int_0^{\infty} h_{jk}(r,R,t)dr \qquad 0 \leq j+k \leq N \tag{3.57}$$

where

$$h_{jk}(r,R,t) = \tilde{J}_{z+1/2+N}(tr_2) r_2^{2z+1+2N-2k} r^{n-1-4N+4k} \tilde{J}_{n/2-1-j}(Rr). \tag{3.58}$$

In order for Lemma 3.6 to apply in the Nth stage of iteration it is necessary that $\sigma = n+3-4N > 3$, or $N < n/4$. We define $N = \lfloor \frac{n-1}{4} \rfloor$ as in Section 2 and let $\kappa = n/2-1-j$. For N chosen in this way it is shown below in Lemma 3.7 that $\|I_{jk}\|_{L^\infty(\mathbb{R}^n)} \leq Ce^{C|Im(z)|}$. Since $N < n/4$ this proves Lemma 3.4.

Lemma 3.7 *Suppose $n \geq 3$ and $-n/4-1 \leq Re(z) < -n/4-1/2$. Then*

$$\|I_{jk}\|_{L^\infty(\mathbb{R}^n)} \leq Ce^{2\pi|Im(z)|}. \tag{3.59}$$

Proof. We first prove the estimate for $n \equiv 0(4)$. In this case

$$h_{jk}(r,R,t) = \tilde{J}_{z+n/4-1/2}(tr_2) r_2^{2z+n/2-1-2k} r^{3+4k} \tilde{J}_\kappa(Rr) \tag{3.60}$$

When $k = 0$ we may integrate by parts once again and use (2.29) to get

$$\begin{aligned}I_{jk}(R,t) &= \frac{1}{2}\tilde{J}_{z+n/4+1/2}(tr_2) r_2^{2z+n/2+1} \tilde{J}_\kappa(Rr)\Big|_0^{\infty} \\ &+ \frac{1}{2}\int_0^{\infty} \tilde{J}_{z+n/4+1/2}(tr_2) r_2^{2z+n/2+1} R^2 r \tilde{J}_{\kappa+1}(Rr)dr.\end{aligned} \tag{3.61}$$

Since $\kappa \geq 1$ and $Re(z) + n/4 + 1/2 \geq -1/2$, Lemma 2.4 shows that the first term on the right is bounded at $r = 0$ by $Ce^{\pi|Im(z)|}$. Since $2Re(z) + n/2 + 1 < 0$ this term vanishes at $r = \infty$. In the remaining integral we make the change of variable $u = Rr$ to obtain

$$\int_0^\infty \tilde{J}_{z+n/4+1/2}(tu_2)u_2^{2z+n/2+1}u\tilde{J}_{\kappa+1}(u)du \tag{3.62}$$

where $u_2^2 = 1 + u^4R^{-4}$. Since Lemma 2.4 implies that $\tilde{J}_{z+n/4+1/2}(tu_2)$ is bounded, and since $\kappa \geq 1$ and $2Re(z) + n/2 + 1 < 0$, this integral is bounded independently of both R and t by $Ce^{\pi|Im(z)|}$.

For $k \geq 1$ we have $\kappa \geq 2$ and we may apply Lemma 3.6 with $\nu = \kappa$, $\sigma = 3 + 4k$ and $\tau = 2z + n/2 - 1 - 2k$ and $\mu = z + n/4 - 1/2$ to obtain

$$I_{jk}(R,t) = \int_0^\infty \tilde{J}_{z+n/4+1/2}(tr_2)\tilde{H}_{jk}(r,R)dr \tag{3.63}$$

where

$$\tilde{H}_{jk}(r,R) = -\frac{1}{2}P_{\nu\sigma\tau\mu}(r,R). \tag{3.64}$$

Since $Re(z) + n/4 + 1/2 \geq -1/2$ it follows from Lemma 2.4 that

$$|\tilde{J}_{z+n/4+1/2}(tr_2)| \leq Ce^{C|Im(z)|} \tag{3.65}$$

for all t and r. Also, since $\sigma - 4 = 4k - 1 \geq 3$, $2Re(\tau) + \sigma = 4Re(z) + n + 1 < -1$ and

$$|\tilde{H}_{jk}(r,R)| \leq C(r^{\sigma-4} + r^\sigma r_2^\tau) \tag{3.66}$$

it follows that $\tilde{H}_{jk}$ is integrable. Hence Lemma 3.7 holds for $n \equiv 0(4)$.

For $n \not\equiv 0(4)$ we first consider the interval $[1,\infty]$. Using (2.33) we obtain

$$\begin{aligned}\int_1^\infty h_{jk}(r,R,t)dr &= \int_1^\infty \tilde{J}_{z+1/2+N}(tr_2)r_2^{2z+1+2N-2k}r^{n-1-4N+4k}\tilde{J}_\kappa(Rr)dr \\ &= (2z+3+2N)\int_1^\infty \tilde{J}_{z+3/2+N}(tr_2)r_2^{2z+1+2N-2k}r^{n-1-4N+4k}\tilde{J}_\kappa(Rr)dr \\ &+ \frac{1}{2}\int_1^\infty \frac{d}{dr}\left(\tilde{J}_{z+3/2+N}(tr_2)\right)r_2^{2z+3+2N-2k}r^{n-4-4N+4k}\tilde{J}_\kappa(Rr)dr.\end{aligned} \tag{3.67}$$

Integrating by parts gives

$$\begin{aligned}\int_1^\infty h_{jk}(r,R,t)dr = \tilde{J}_{z+3/2+N}(tr_2)r_2^{2z+3+2N-2k}r^{n-4-4N+4k}\tilde{J}_\kappa(Rr)\Big|_1^\infty \\ - \frac{1}{2}\int_1^\infty \tilde{J}_{z+3/2+N}(tr_2)\tilde{h}_{jk}(r,R)dr\end{aligned} \tag{3.68}$$

where

$$\tilde{h}_{jk}(r,R) = r_2^{2z+3+2N-2k}r^{n-5-4N+4k}[C\tilde{J}_\kappa(Rr) + \tilde{J}_{\kappa-1}(Rr)]. \tag{3.69}$$

By Lemma 2.4 the first term is of order $O\left(r^{2Re(z)+n/2-3/2-2N+j}\right)$ for large r. Since $2Re(z)+n/2-3/2-2N+j \leq -5/2-N$ it follows that this term vanishes at $r=\infty$. At $r=1$, Lemma 2.4 implies that its norm in $L^\infty(\mathbb{R}^n)$ is bounded exponentially in $Im(z)$. Next we observe that since $\kappa \geq 1/2$ each term in $\tilde{h}_{jk}(r,R)$ is of order $O\left(r^{4Re(z)+n+1}\right)$ on $[1,\infty]$ for every R. Since $Re(z)+3/2+N \geq -1/2$ Lemma 2.4 implies that the remaining integral is bounded by

$$Ce^{2\pi|Im(z)|}\int_1^\infty r^{4Re(z)+n+1}dr = Ce^{2\pi|Im(z)|} \tag{3.70}$$

for $4Re(z)+n+1<-1$.

On $[0,1]$ we consider the cases $t<1$ and $t>1$ separately. When $t<1$, Lemma 2.4 implies that $h_{jk}(r,R,t) \leq Ce^{\pi|Im(z)|}$ for all R and for all $r \in [0,1]$. Hence

$$\int_0^1 h_{jk}(r,R,t)dr \leq Ce^{\pi|Im(z)|}. \tag{3.71}$$

When $t>1$ we divide the interval $[0,1]$ into $[0,t^{-1/4}]$ and $[t^{-1/4},1]$ and estimate, using Lemma 2.4, as follows:

$$\begin{aligned}\int_0^{t^{-1/4}} h_{jk}(r,R,t)dr &\leq Ce^{\pi|Im(z)|}t^{-Re(z)-1-N}\int_0^{t^{-1/4}} r^{n-1-4N+4k}dr \\ &= Ce^{\pi|Im(z)|}t^{-Re(z)-1-N}r^{n-4N+4k}\Big|_0^{t^{-1/4}} \\ &\leq Ce^{\pi|Im(z)|}\end{aligned} \tag{3.72}$$

as long as $Re(z) \geq -n/4-1$. On the remaining interval we integrate by parts once again to obtain

$$\begin{aligned}\int_{t^{-1/4}}^1 h_{jk}(r,R,t)dr = {} & \tilde{J}_{z+3/2+N}(tr_2)r_2^{2z+3+2N-2k}r^{n-4-4N+4k}\tilde{J}_\kappa(Rr)\Big|_{t^{-1/4}}^1 \\ & -\frac{1}{2}\int_{t^{-1/4}}^1 \tilde{J}_{z+3/2+N}(tr_2)\tilde{h}_{jk}(r,R)dr.\end{aligned} \tag{3.73}$$

As above, the first term has norm in $L^\infty(\mathbb{R}^{n+1})$ bounded exponentially in $|Im(z)|$ at $r=1$, while at $r=t^{-1/4}$ we have $tr_2=O(t)$ and Lemma 2.4 shows that this term is of order $O\left(t^{-Re(z)-1-n/4-k}e^{\pi|Im(z)|}\right)=O\left(e^{\pi|Im(z)|}\right)$ if $Re(z) \geq -n/4-1$. Finally, for $r \in [t^{-1/4},1]$, $\tilde{h}_{jk}(r,R)$ is of order $O\left(r^{n-5-4N+4k}\right)$ and therefore of order $O\left(r^{n-5-4N}\right)$. Since $n-5-4N \neq -1$ for $n \not\equiv 0(4)$, we have

$$\begin{aligned}\int_{t^{-1/4}}^1 h_{jk}(r,R,t)dr &= O\left(e^{\pi|Im(z)|}\right)+O\left(e^{\pi|Im(z)|}\int_{t^{-1/4}}^1 t^{-Re(z)-2-N}r^{n-5-4N}dr\right) \\ &= O\left(e^{\pi|Im(z)|}\left(1+t^{-Re(z)-2-N}r^{n-4-4N}\Big|_{t^{-1/4}}^1\right)\right) \\ &= O\left(e^{\pi|Im(z)|}\right)\end{aligned} \tag{3.74}$$

since $Re(z) \geq -\frac{n+4}{4}$. This completes the proof of Lemma 3.7. ■

4 Local Existence

In this section we find precise conditions on f so that for arbitrary initial data in the energy space $X = H^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ solutions of the nonlinear equation

$$u_{tt} + \Delta^2 u + u = f(u) \tag{1.1}$$

exist locally in time and are strongly continuous with values in X. We assume the nonlinearity satisfies the following hypotheses.

$$\begin{aligned} &(i) \qquad f(0) = 0 \\ &(ii) \qquad f \in C^1(\mathbb{R}) \qquad \text{and} \qquad |f'(s)| \leq C|s|^{p-1}. \end{aligned} \tag{4.1}$$

We first write (1.1) in the form of a system as

$$\begin{aligned} u_t &= v \\ v_t &= -\Delta^2 u - u + f(u) \end{aligned} \tag{4.2}$$

If we denote $w = (u, v)$ then the functionals

$$\begin{aligned} E(w) &= \int_{\mathbb{R}^n} \frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|u|^2 - F(u)dx \\ Q(w) &= \int_{\mathbb{R}^n} v \nabla u dx \end{aligned} \tag{4.3}$$

where $F'(s) = f(s)$ and $F(0) = 0$, are formally invariants of (4.2). We now state the main result of this chapter.

Theorem 4.1 *For any n let p satisfy $1 < p < 2^{**} - 1$. Given initial data $g \in X$, there exists $T > 0$ which depends only on $\|g\|_X$ and a unique solution $w = (u, v)$ of (4.2) in $C([0, T), X)$ such that $w(0) = g$ and $E(w(t)) = E(g)$ for all $t \in [0, T]$.*

We prove the theorem directly in dimension $n \leq 4$ using standard semigroup techniques, while for higher dimensions we proceed by first establishing existence in a weaker space using the decay estimates from Chapter 1 and then approximating the nonlinear term with Lipschitz functions. The system (4.2) may be rewritten as

$$\frac{dw}{dt} = Bw + P(w) \tag{4.4}$$

where

$$B = \begin{pmatrix} 0 & I \\ -\Delta^2 - I & 0 \end{pmatrix} \qquad P(w) = (0, f(u)). \tag{4.5}$$

The theorem then follows using standard semi-group results (see [13],[11]) once we show that B is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X and that P is locally Lipschitz on X. This is the content of the following two Lemmas.

Lemma 4.2 *The operator B is the infinitesimal generator of a C_0-semigroup of unitary operators on X.*

Proof. Define an inner product on X by

$$((u_1,v_1),(u_2,v_2)) = \int_{\mathbb{R}^n} (\Delta u_1 \Delta u_2 + u_1 u_2 + v_1 v_2) dx. \tag{4.6}$$

Then for $w \in D(B) = H^4(\mathbb{R}^n) \oplus H^2(\mathbb{R}^n)$,

$$\begin{aligned}(Bw,w) &= \big((v, -\Delta^2 u - u),(u,v)\big) \\ &= \int_{\mathbb{R}^n} (\Delta v \Delta u + vu - v\Delta^2 u - vu) dx = 0\end{aligned} \tag{4.7}$$

and therefore B is skew adjoint. The lemma follows from Stone's theorem. ■

Lemma 4.3 *Let $1 \le n \le 4$ and let f satisfy the hypotheses in (4.1) with $1 < p < \infty$. Then the map $P : X \to X$ given by $P(w) = (0, f(u))$ is locally Lipschitz.*

Proof. For $1 \le n \le 4$ the Sobolev inequality implies that $H^2(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $2 \le p < \infty$. So let $w_1, w_2 \in X$ and compute

$$\begin{aligned}\|P(w_1) - P(w_2)\|_X^2 &= \int_{\mathbb{R}^n} \big|f(u_1) - f(u_2)\big|^2 dx \\ &= \int_{\mathbb{R}^n} |f'(\lambda(x)u_1 + (1-\lambda(x))u_2)(u_1 - u_2)|^2 dx \\ &\le C \int_{\mathbb{R}^n} \big||u_1| + |u_2|\big|^{2(p-1)} |u_1 - u_2|^2 dx \\ &\le C \||u_1| + |u_2|\|_{L^{2p}(\mathbb{R}^n)}^{2(p-1)} \|u_1 - u_2\|_{L^{2p}(\mathbb{R}^n)}^2 \\ &\le C(\|w_1\|_X + \|w_2\|_X)^{2(p-1)} \|w_1 - w_2\|_X^2.\end{aligned} \tag{4.8}$$

and therefore P is locally Lipschitz. ■

For $n > 4$ we proceed as follows. Let $U_0(t)$ denote the solution operator for the linear equation (2.1) at time t. Then clearly the map $U_0(t) : X \to X$ is unitary for all t. If we denote by $w = (u, u_t)$ then (1.1) may be written as an integral equation

$$w(t) = U_0(t)g + \int_0^t U_0(t-\tau) P(w(\tau)) d\tau \tag{4.9}$$

where $w(0) = g \in X$ is the initial data. We define the following spaces in which to solve (4.9). Let

$$\begin{aligned}X_1 &= \{0\} \oplus L^{1+1/p}(\mathbb{R}^n) \\ X_3 &= L^{p+1}(\mathbb{R}^n) \oplus W^{-2,p+1}(\mathbb{R}^n) \\ \tilde{Z} &= L^r([0,T], X_3)\end{aligned} \tag{4.10}$$

where

$$r = \frac{p-1}{1-d} \qquad d = \frac{n(p-1)}{4(p+1)}. \tag{4.11}$$

Lemma 4.4 *For any $p > 1$*

$$\|P(w)\|_{X_1} \le C\|w\|_{X_3}^p \tag{4.12}$$

and

$$\|P(w_1) - P(w_2)\|_{X_1} \le C(\|w_1\|_{X_3}^{p-1} + \|w_2\|_{X_3}^{p_1})\|w_1 - w_2\|_{X_3}. \tag{4.13}$$

Proof. This follows immediately from (4.1) and the definitions of X_1 and X_3. ■

Lemma 4.5 *Let $n \ge 5$ and suppose $1 \le p \le 2^{**} - 1$. Then*

$$\|U_0(t)\|_{X_1, X_3} \le Ct^{-d}. \tag{4.14}$$

Proof. Let $w_0 = (0, v_0) \in X_1$. Then

$$U_0(t)w_0 = (u(t), v(t)) \tag{4.15}$$

where $u(t)$ is the solution of (2.1) with initial data $u(0) = 0$, $u_t(0) = v_0$ and $v(t)$ is the solution of (2.1) with initial data $v(0) = v_0$, $v_t(0) = 0$. Since $(1 + \Delta^2)^{-1/2}v(t)$ is the solution of (2.1) with initial data $((1 + \Delta^2)^{-1/2}v_0, 0)$ it follows from Theorem 2.1 with $q = p + 1$ that

$$\|u(t)\|_{L^{p+1}(\mathbb{R}^n)} \le Ct^{-d}\|v_0\|_{L^{1+1/p}(\mathbb{R}^n)} \tag{4.16}$$

and

$$\|(1 + \Delta^2)^{-1/2}v(t)\|_{L^{p+1}(\mathbb{R}^n)} \le Ct^{-d}\|v_0\|_{L^{1+1/p}(\mathbb{R}^n)}. \tag{4.17}$$

This proves the Lemma. ■

We now define the space

$$\tilde{Z}(\delta) = \{v : \|v\|_{\tilde{Z}} < \delta\} \tag{4.18}$$

and prove that for small enough δ solutions to (4.9) exist in $\tilde{Z}(\delta)$.

Lemma 4.6 *For $n \ge 5$ suppose that $1 < p \le 2^{**} - 1$. Then for any $g \in X$ there exists some $T > 0$ depending only on $\|g\|_X$, some $\delta > 0$ and a unique solution $w \in \tilde{Z}(\delta)$ of (4.9).*

Proof. First define

$$N(w)(t) = \int_0^t U_0(t - \tau)P(w(\tau))d\tau \tag{4.19}$$

Lemma 4.4 and Lemma 4.5 imply that

$$\|N(w)(t)\|_{X_3} \le C\int_0^t |t - \tau|^{-d}\|w(\tau)\|_{X_3}^p d\tau. \tag{4.20}$$

Using the singular integral inequality this implies

$$\|N(w)\|_{\tilde{Z}} \le C\|w\|_{\tilde{Z}}^p \tag{4.21}$$

since $1 + 1/r = p/r + d$. Next we choose δ so that $C\delta^{p-1} < 1/2$. It follows that N maps $\tilde{Z}(\delta)$ to itself. Also since

$$\|N(w_1) - N(w_2)\|_{\tilde{Z}} \leq C(\|w_1\|_{\tilde{Z}}^{p-1} + \|w_2\|_{\tilde{Z}}^{p-1})\|w_1 - w_2\|_{\tilde{Z}} \tag{4.22}$$

this implies that N is a contraction mapping on $\tilde{Z}(\delta)$. Since $U_0(\cdot)$ is unitary on X, we have $U_0(\cdot)g \in L^\infty(\mathbb{R}, X) \subset L^\infty(\mathbb{R}, X_3) \subset \tilde{Z}$ and therefore

$$\begin{aligned} \|U_0(\cdot)g\|_{\tilde{Z}}^r &\leq C \int_0^T \|U_0(\tau)g\|_{X_3}^r d\tau \\ &\leq CT\|U_0(\cdot)g\|_{L^\infty(\mathbb{R},X_3)}^r. \end{aligned} \tag{4.23}$$

Hence

$$U_0(\cdot)g \in \tilde{Z}(\delta/2) \tag{4.24}$$

if T is chosen small enough. So $\tilde{N}(w) = U_0(\cdot)g + N(w)$ is also a contraction on $\tilde{Z}(\delta)$ and there exists a unique fixed point w which solves (4.9). ∎

We now show that the solution obtained above is continuous with values in X and satisfies the energy equality. We first prove a Lemma which will be needed later.

Lemma 4.7 *Given an interval I denote $Y = L^r(I, X_3) \cap L^\infty(I, X_3)$. Suppose $w \in Y \cap B(I, X_3)$ is a solution of*

$$w(t) = U_0(t)g + \int_s^t U_0(t-\tau)P(w(\tau))d\tau \tag{4.25}$$

where $s \in I$ and $g \in X$. Then $w \in C^{0,1-d}(I, X_3)$ and $\|w\|_{C^{0,1-d}(I,X_3)} \leq C\|w\|_Y^p$.

Proof. For $t_1 < t_2$ we subtract to obtain

$$\begin{aligned} w(t_2) - w(t_1) &= (U_0(t_2) - U_0(t_1))g + \int_{t_1}^{t_2} U_0(t_2-\tau)P(w(\tau))d\tau \\ &\quad + \int_s^{t_1} (U_0(t_2-\tau) - U_0(t_1-\tau))P(w(\tau))d\tau \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{4.26}$$

We then notice that

$$(U_0(t_2) - U_0(t_1))g = (K^c(t_2) - K^c(t_1)) * (1+\Delta)^{1/2} g_1 + (K^s(t_2) - K^s(t_1)) * g_2 \tag{4.27}$$

where $K^c(t)$ and $K^s(t)$ are defined in (2.7). By (2.7) it follows that

$$\|(U_0(t_2) - U_0(t_1))g\|_X \leq |t_2 - t_1|\|g\|_X \tag{4.28}$$

so that

$$\|A_1\|_{X_3} \leq C|t_2 - t_1|. \tag{4.29}$$

To estimate the second term we use Lemma 4.4 and Lemma 4.5 and the fact that $w \in L^\infty(I, X_3)$ to obtain

$$\|A_2\|_{X_3} \le C \int_{t_1}^{t_2} |t_2 - \tau|^{-d} \|w(\tau)\|_{X_3}^p d\tau \le C\|w\|_Y^p |t_2 - t_1|^{1-d}. \tag{4.30}$$

Finally, we notice that

$$(U_0(t_2 - \tau) - U_0(t_1 - \tau))(0, g_2) = (K_{t_2-\tau} - K_{t_1-\tau}) * g_2. \tag{4.31}$$

So if we define

$$Lv = (K^s(t_2 - \tau) - K^s(t_1 - \tau)) * v \tag{4.32}$$

then L maps $L^2(\mathbb{R}^n)$ to itself with norm

$$\|L\|_{2,2} \le |t_2 - t_1|. \tag{4.33}$$

Also, since

$$L = T^s(t_2 - \tau) - T^s(t_1 - \tau) \tag{4.34}$$

it follows from (2.9) that

$$\|L\|_{q',q} \le C(|t_2 - \tau|^{-1} + |t_1 - \tau|^{-1}) \tag{4.35}$$

where $q = 2^{**} > p + 1$. Interpolating between (4.33) and (4.35) yields

$$\|L\|_{1+1/p,p+1} \le C|t_2 - t_1|^{1-d}(|t_2 - \tau|^{-1} + |t_1 - \tau|^{-1})^d \tag{4.36}$$

Hence

$$\begin{aligned} \|A_3\|_{X_3} &\le C \int_s^{t_1} \|L(P(w(\tau)))\|_{L^{p+1}(\mathbb{R}^n)} d\tau \\ &\le C|t_2 - t_1|^{1-d} \|w\|_Y^p \int_s^{t_1} (|t_2 - \tau|^{-d} + |t_1 - \tau|^{-d})\, d\tau \end{aligned}. \tag{4.37}$$

Since $0 < d < 1$, the last integral is bounded independently of s, t_1 and t_2 and the lemma is proved. ∎

Lemma 4.8 *For $n \ge 5$ and $1 < p \le 2^{**} - 1$ the solution $w(t)$ of (4.9) satisfies $w \in C([0,T], X)$ and $E(w(t)) = E(g)$ for all $t \in [0,T]$.*

Proof. We use arguments similar to those in [16] and [17] together with Lemma 4.7. We first approximate the nonlinearity f by a sequence of Lipschitz functions f_k, chosen so that $f_k(s) \to f(s)$ uniformly on compact subsets of $\mathbb{R}$ and $|f_k(s)| \le |f(s)|$. Since the operators $P_k(u, v) = (0, f_k(u))$ are Lipschitz on X, it follows from the same argument given in dimension $n \le 4$ that there exist solutions $w_k \in C([0,T], X)$ of

$$w_k(t) = U_0(t)g + \int_0^t U_0(t - \tau)P_k(w_k(\tau))d\tau \tag{4.38}$$

with constant energies

$$E_k = \frac{1}{2}\|w_k(t)\|_X^2 - \int_{\mathbb{R}^n} F_k(u_k(t))dx. \tag{4.39}$$

Since $w_k(0) = g$ for all k we have

$$\begin{aligned} E_k &= \frac{1}{2}\|g\|_X^2 - \int_{\mathbb{R}^n} F_k(g_1)dx \\ &\leq \frac{1}{2}\|g\|_X^2 + C\|g_1\|_{L^{p+1}(\mathbb{R}^n)} \leq C. \end{aligned} \tag{4.40}$$

Hence E_k is a bounded sequence. To show that the nonlinear part of the energies is also bounded, we define the space $Y_t = L^\infty([0,t], X_3)$ and set $M_k(t) = \|w_k\|_{Y_t}$. Taking the X_3 norm of both sides of equation (4.38) we obtain

$$M_k(t) \leq \|g\|_X + CT^{1-d}(M_k(t))^p \tag{4.41}$$

for all $t \in [0,T]$, where the constant C is independent of t, T and k. Now, for T chosen small enough, the function $h(M) = M - \|g\|_X - CT^{1-d}M^p$ is positive on some interval (a,b), where $\|g\|_X < a < b < \infty$. Since $M_k(0) = \|g\|_{X_3}$ and $M_k(t)$ is continuous in t ($w_k \in C([0,T], X)$) it follows that $M_k(t) \leq a$ for all $t \in [0,T]$ independently of k. Thus

$$\left| \int_{\mathbb{R}^n} F_k(u_k(t))dx \right| \leq C\|w_k\|_{Y_T} \tag{4.42}$$

is bounded, and therefore (4.39) implies that $\|w_k(t)\|_X$ is bounded independently of t and k. Hence there is some subsequence which converges weak-$*$ in $L^\infty([0,T], X)$ to some $\tilde{w} = (\tilde{u}, \tilde{v}) \in L^\infty([0,T], X)$. Also, since $u_k(t) \in H^2(\mathbb{R}^n)$ and $u_k'(t) = v_k(t) \in L^2(\mathbb{R}^n)$ it follows that $u_k \in H^1(\mathbb{R}^n \times [0,T])$ and therefore for any compact subset K of $\mathbb{R}^n$, u_k converges strongly in $L^2(K \times [0,T])$ and thereby almost everywhere in $\mathbb{R}^n \times [0,T]$ to $\tilde{u}$. Thus $f_k(u_k) \to f(\tilde{u})$ a.e. in $\mathbb{R}^n \times [0,T]$ and since $f_k(u_k)$ is bounded in $L^{1+1/p}(\mathbb{R}^n)$ it follows that $f_k(u_k) \to f(\tilde{u})$ in $L^1_{loc}(\mathbb{R}^n \times [0,T])$ and therefore $\tilde{w}$ is a solution of (4.9). As in the proof of Lemma 4.6 the condition (4.1) along with Lemmas 4.4 and 4.5 imply that $\tilde{w} \in \widetilde{Z}(\delta)$. By uniqueness, $\tilde{w} = w$.

We next show that $w(t)$ is weakly continuous on $[0,T]$ with values in X. First write

$$\langle u_k', \phi \rangle + \langle u_k, \phi' \rangle = (u_k(t_0), \phi(t_0)) - (g_1, \phi(0)) \tag{4.43}$$

where $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ denote the inner products on $L^2(\mathbb{R}^n \times [0, t_0])$ and $L^2(\mathbb{R}^n)$ respectively, and ϕ is any test function. Since $w_k(t_0)$ is bounded in X it has a subsequence which converges weakly to some $(\tilde{u}, \tilde{v}) \in X$. Taking the limit as $k \to \infty$ in (4.43) gives

$$\langle u', \phi \rangle + \langle u, \phi' \rangle = (\tilde{u}, \phi(t_0)) - (g_1, \phi(0)). \tag{4.44}$$

If we choose $\phi(x,t) = \varphi(t)\psi(x)$ in such a way that φ is zero for $t < t_0 - \epsilon$, one at $t = t_0$ and linear between, we obtain

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0 - \epsilon}^{t_0} (u(t), \psi)dt = (\tilde{u}, \psi). \tag{4.45}$$

But since

$$\begin{aligned} u &\in L^\infty([0,T],H^2(\mathbb{R}^n)) \subset L^2([0,T],L^2(\mathbb{R}^n)) \\ u' &\in L^\infty([0,T],L^2(\mathbb{R}^n)) \subset L^2([0,T],L^2(\mathbb{R}^n)) \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} u'' = -\Delta^2 u - u + f(u) &\in L^\infty([0,T],H^{-2}(\mathbb{R}^n)) \\ &\subset L^2([0,T],H^{-2}(\mathbb{R}^n)) \end{aligned} \tag{4.47}$$

it follows that $w \in H^1([0,T],L^2(\mathbb{R}^n)\oplus H^{-2}(\mathbb{R}^n)) \subset C([0,T],L^2(\mathbb{R}^n)\oplus H^{-2}(\mathbb{R}^n))$ and therefore the limit in (4.45) is $(u(t_0),\psi)$. Since this holds for all test functions ψ it follows that $\tilde{u} = u(t_0) \in H^2(\mathbb{R}^n)$. Thus, letting t_0 vary proves that $u(t)$ is bounded in $H^2(\mathbb{R}^n)$ for *all* $t \in [0,T]$. It follows similarly from the equation

$$(u_k'(t_0),\phi(t_0)) - (g_1,\phi(0)) - \langle u_k',\phi'\rangle + \langle \Delta u_k, \Delta\phi\rangle = \langle f_k(u_k),\phi\rangle \tag{4.48}$$

that $\tilde{v} = u'(t_0)$, and therefore $u'(t)$ is bounded in $L^2(\mathbb{R}^n)$ for all $t \in [0,T]$. We next consider a sequence $t_j \to t$. Since $w(t_j)$ is bounded, some subsequence converges weakly in X. By the continuity of w in $L^2(\mathbb{R}^n) \oplus H^{-2}(\mathbb{R}^n)$ it follows that the limit is in fact $w(t)$ and hence $w(t)$ is weakly continuous in X.

Now fix $t \in [0,T]$. Since $w_k(t) \to w(t)$ weakly in X we have

$$\|w(t)\|_X \le \liminf_{k\to\infty} \|w_k\|_X. \tag{4.49}$$

Also, since $u_k(t)$ is bounded in $H^2(\mathbb{R}^n)$ it follows that $u_k(t) \to u(t)$ strongly in $L^2(K)$ for any compact subset $K \subset \mathbb{R}^n$. Thus $u_k(x,t) \to u(x,t)$ for a.e. $x \in \mathbb{R}^n$. If we subtract (4.9) from (4.38) and define $\|w\|_B = \sup\{\|w(t)\|_{X_3} : t \in [0,T]\}$ we find that

$$\begin{aligned} \|w_k(t) - w(t)\|_{X_3} &\le C\int_0^t |t-\tau|^{-d}\|P_k(w_k(\tau)) - P_k(w(\tau))\|_{X_1} d\tau \\ &\quad + C\int_0^t |t-\tau|^{-d}\|P_k(w(\tau)) - P(w(\tau))\|_{X_1} d\tau \\ &\le CT^{1-d}\|w_k - w\|_B \\ &\quad + C\int_0^t |t-\tau|^{-d}\|f_k(u(\tau)) - f(u(\tau))\|_{L^{1+1/p}(\mathbb{R}^n)} d\tau \end{aligned} \tag{4.50}$$

since w_k is a bounded sequence in $L^\infty([0,T],X)$. For T chosen so small that $CT^{1-d} < 1/2$ and for $\epsilon > 0$ chosen so that $d(1+\epsilon) < 1$ we have

$$\|w_k - w\|_B \le CT^{\frac{1-(1+\epsilon)d}{1+\epsilon}} \left(\int_0^T \|f_k(u(\tau)) - f(u(\tau))\|_{L^{1+1/p}(\mathbb{R}^n)}^{1+1/\epsilon} d\tau \right)^{\epsilon/(1+\epsilon)} \tag{4.51}$$

We now show that the last term vanishes as $k \to \infty$. Define

$$g_k(\tau) = \|f_k(u(\tau)) - f(u(\tau))\|_{L^{1+1/p}(\mathbb{R}^n)}^{1+1/\epsilon} \tag{4.52}$$

Since $f_k \to f$ we have $f_k(u(x,\tau)) - f(u(x,\tau)) \to 0$ for a.e. $x \in \mathbb{R}^n$, for any fixed $\tau \in [0,T]$. Since $|f_k(s)| \le |f(s)|$ for all s it follows that

$$|f_k(u(x,\tau)) - f(u(x,\tau))|^{1+1/p} \le |2f(u(x,\tau))|^{1+1/p} \in L^1(\mathbb{R}^n). \tag{4.53}$$

Thus the dominated convergence theorem implies that

$$\lim_{k\to\infty} g_k(\tau) = 0 \qquad \text{for all} \qquad \tau \in [0,T]. \tag{4.54}$$

Next, if we choose ϵ so that $p(1+1/\epsilon) > r$ it follows that

$$\begin{aligned} u &\in L^\infty([0,T], L^{p+1}(\mathbb{R}^n)) \cap L^r([0,T], L^{p+1}(\mathbb{R}^n)) \\ &\subset L^{p(1+1/\epsilon)}([0,T], L^{p+1}(\mathbb{R}^n)). \end{aligned} \tag{4.55}$$

Thus since

$$\|f(u(\tau))\|^{1+1/\epsilon}_{L^{1+1/p}(\mathbb{R}^n)} \le C\|u(\tau)\|^{p(1+1/\epsilon)}_{L^{p+1}(\mathbb{R}^n)} \tag{4.56}$$

we have

$$g_k(\tau) \le \|2f(u(\tau))\|^{1+1/\epsilon}_{L^{1+1/p}} \in L^1([0,T]). \tag{4.57}$$

Thus we may conclude using the dominated convergence theorem again that

$$\int_0^T g_k(\tau)^{1+1/\epsilon} d\tau \to 0 \qquad \text{as} \qquad k \to \infty \tag{4.58}$$

By (4.51) this shows that

$$\lim_{k\to\infty} \|w_k - w\|_B = 0 \tag{4.59}$$

In particular $w_k(t) \to w(t)$ in X_3, so that $u_k(t) \to u(t)$ in $L^{p+1}(\mathbb{R}^n)$, and therefore

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} F_k(u_k(t))dx = \int_{\mathbb{R}^n} F(u(t))dx. \tag{4.60}$$

Hence

$$\frac{1}{2}\|w(t)\|^2_X - \int_{\mathbb{R}^n} F(u(t))dx \le \frac{1}{2}\|g\|^2_X - \int_{\mathbb{R}^n} F(g_1)dx \tag{4.61}$$

for all $t \in [0,T]$. To obtain the opposite inequality we consider the solution $y(s)$ of

$$y(s) = U_0(s)(w(t)) + \int_0^s U_0(s-\tau)P(y(\tau))d\tau. \tag{4.62}$$

The solution exists on an interval $[-T,0]$ (since the existence time depends only on $\|w(t)\|_X \le \|g\|_X$) and satisfies $y(0) = w(t)$. By uniqueness, $y(-s) = w(t-s)$ and therefore $y(-t) = g$. The same arguments as above therefore imply that

$$\begin{aligned} \frac{1}{2}\|g\|^2_X - \int_{\mathbb{R}^n} F(g_1)dx &= \frac{1}{2}\|y(-t)\|^2_X - \int_{\mathbb{R}^n} F(y_1(-t))dx \\ &\le \frac{1}{2}\|y(0)\|^2_X - \int_{\mathbb{R}^n} F(y_1(0))dx \\ &= \frac{1}{2}\|w(t)\|^2_X - \int_{\mathbb{R}^n} F(u(t))dx. \end{aligned} \tag{4.63}$$

This proves the equality

$$\frac{1}{2}\|w(t)\|_X^2 - \int_{\mathbb{R}^n} F(u(t))dx = \frac{1}{2}\|g\|_X^2 - \int_{\mathbb{R}^n} F(g_1)dx. \tag{4.64}$$

To show that $w(t)$ is in fact strongly continuous, we apply Lemma 4.7 with $I = [0, T]$ and $s = 0$ to conclude that w is strongly continuous with values in X_3. Hence

$$\int_{\mathbb{R}^n} F(u(t))dx \tag{4.65}$$

is a continuous function of t. Together with (4.64) this shows that the norm $\|w(t)\|_X$ is continuous on $[0, T]$. Since $w(t)$ is weakly continuous from $[0, T]$ to X, this proves the strong continuity. ■

5 Low Energy Scattering

In this section we consider the asymptotic behavior of solutions of

$$u_{tt} + \Delta^2 u + u = f(u) \tag{1.1}$$

with small initial data. As in Section 4 we write this as

$$\frac{dw}{dt} = Bw + P(w) \tag{4.4}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -\Delta^2 - I & 0 \end{pmatrix} \qquad P(u, v) = (0, f(u)). \tag{4.5}$$

This may be written as the integral equation

$$w(t) = U_0(t)g + \int_s^t U_0(t-\tau)P(w(\tau))d\tau \tag{5.1}$$

where the initial data is $w(s) = U_0(s)g$. The main result of this section is the following theorem. It can be interpreted as saying that the scattering operator maps a neighborhood of the energy space X into X.

Theorem 5.1 *For any n choose p so that $1 + 8/n \le p < 2^{**} - 1$. Then there exists a $\delta > 0$ such that for $g_- \in X$ with $\|g_-\|_X \le \delta$ there exists a unique solution w of (5.1) which is continuous in t with values in X and satisfies*

$$\|w(t) - U_0(t)g_-\|_X \to 0 \qquad as \qquad t \to -\infty \tag{5.2}$$

Also there exists a unique $g_+ \in X$ so that

$$\|w(t) - U_0(t)g_+\|_X \to 0 \qquad as \qquad t \to +\infty \tag{5.3}$$

and

$$E(w(t)) = \|g_-\|_X = \|g_+\|_X. \tag{5.4}$$

The space in which we shall solve (5.1) is

$$Z = L^r(\mathbb{R}, X_3) \cap L^\infty(\mathbb{R}, X_3) \tag{5.5}$$

where r and X_3 are defined as in Section 4:

$$X_3 = L^{p+1}(\mathbb{R}^n) \oplus W^{-2,p+1}(\mathbb{R}^n) \tag{5.6}$$

$$r = \frac{p-1}{1-d} \qquad\qquad d = \frac{n(p-1)}{4(p+1)} \tag{5.7}$$

In addition we define

$$X_1 = \{0\} \oplus L^{1+1/p}(\mathbb{R}^n) \tag{5.8}$$

$$X_4 = W^{-k,p+1}(\mathbb{R}^n) \oplus W^{-k-2,p+1}(\mathbb{R}^n) \tag{5.9}$$

Theorem 5.1 is a consequence of the results in [17] and [18] once we verify the following hypotheses.

(I) The space X is a Hilbert space and the solution operator $U_0(t)$ of the linear equation is untary on X.

(II) The operator P (see (4.5)) maps X_3 into X_1 with $P(0) = 0$ and

$$\|P(w_1) - P(w_2)\|_{X_1} \leq C(\|w_1\|_{X_3}^{p-1} + \|w_2\|_{X_3}^{p_1})\|w_1 - w_2\|_{X_3} \tag{5.10}$$

(III) The spaces X, X_1 and X_3 are continuously embedded in X_4. Furthermore X is continuously and densely embedded into X_3.

(IV) For each $g \in X$ the function $U_0(\cdot)g$ is contained in $L^r(\mathbb{R}, X_3)$.

(V) The restriction of $U_0(t)$ to $X \cap X_1$ can be extended to all of X_1 in such a way that it maps X_1 to X_3 with norm

$$\|U_0(t)\|_{X_1,X_3} \leq Ct^{-d}. \tag{5.11}$$

and $0 < 1/p < d < 1$. Furthermore, the restriction of $U_0(t)$ to $X \cap X_3$ extends to a continuous linear map from X_3 to X_4.

(VI) The functional

$$G(u) = \int_{\mathbb{R}^n} F(u)dx \tag{5.12}$$

is continuous on X_3.

(VII) Whenever I is a time interval, $s \in I$, $g \in X$, $w \in Z$ with $\|w\|_Z$ sufficiently small, and w satisfies

$$w(t) = U_0(t)g + \int_s^t U_0(t-\tau)P(w(\tau))d\tau \tag{5.13}$$

for t in the interval I, then $u \in C(I, X)$.

Hypothesis (I) is trivial and hypothesis (II) was proven in Lemma 4.4. The embeddings in hypothesis (III) are obvious. Hypothesis (IV) is proved using the following Lemma.

Lemma 5.2 *For any n choose p such that $1 + 8/n \leq p \leq 2^{**} - 1$. If $g \in X$ then $U_0(\cdot)g \in Z$ with $\|U_0(\cdot)g\|_Z \leq C\|g\|_X$.*

Proof. Since $\|U_0(t)g)\|_{X_3} \leq C\|U_0(t)g\|_X = C\|g\|_X$ we have $U_0(\cdot)g \in L^\infty(\mathbb{R}, X_3)$. If we now define $Z_{q_1,q_2} = L^{q_1}(\mathbb{R}, L^{q_2}(\mathbb{R}^n) \oplus W^{-2,q_2})$, then $L^r(\mathbb{R}, X_3) = Z_{r,p+1}$ and Theorem 2.1 implies that

$$\|U_0(\cdot)g\|_{Z_{q,q}} \leq C\|g\|_X \tag{5.14}$$

where

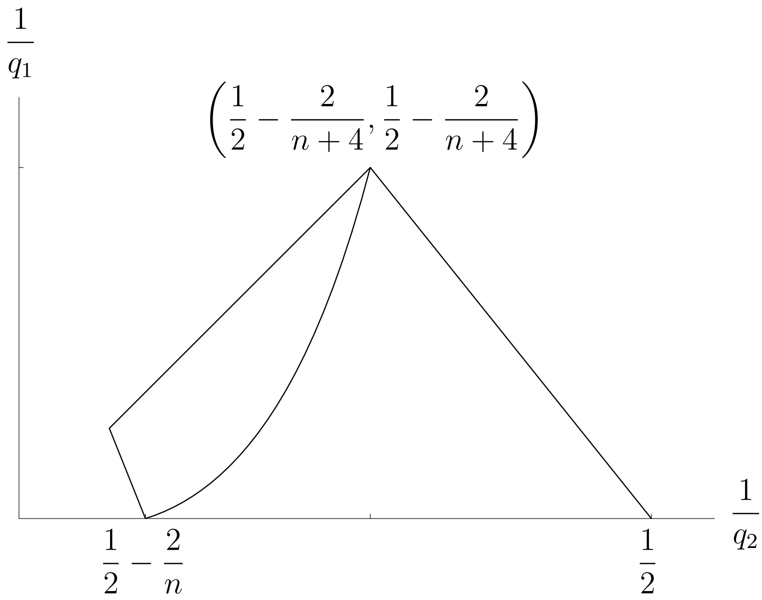
$$2 + \frac{8}{n} \leq q < \frac{2(n+2)}{n-4} \qquad \text{for} \qquad n \geq 5 \tag{5.15}$$

$$2 + \frac{8}{n} \leq q < \infty \qquad \text{for} \qquad 1 \leq n \leq 4. \tag{5.16}$$

Along with energy conservation, this shows that

$$\|U_0(\cdot)g\|_{Z_{q_1,q_2}} \leq C\|g\|_X \tag{5.17}$$

for q_1 and q_2 with $(1/q_2, 1/q_1)$ lying inside the quadrilateral shown below.



Region on which U_0 is bounded (shown for $n = 5$).

The points $(1/(p+1), 1/r)$ which satisfy $r = \frac{p-1}{1-d}$ lie on the curve

$$\frac{1}{q_1} = \frac{1 - \frac{n}{4} + \frac{n}{2q_2}}{q_2 - 2} \tag{5.18}$$

which is contained inside this region. Hence $U_0(\cdot)g \in L^r(\mathbb{R}, X_3)$ and therefore $U_0(\cdot)g \in Z$ with norm bounded by $\|g\|_X$. ■

The first part of hypothesis (V) follows from Lemma 4.5 and the definition of d. For the second part we need the following Lemma, whose proof is left for the reader.

Lemma 5.3 *Let* $r_2 = (1 + |\xi|^4)^{1/2}$. *Then for* k *large enough, the kernels* $r_2^{-k}\sin(tr_2)$ *and* $r_2^{-k}\cos(tr_2)$ *are in* $W^{1,n+1}(\mathbb{R}^n)$.

Hence $(1+\Delta^2)^{-k/2}K^c(t)$ and $(1+\Delta^2)^{-k/2}K^s(t)$ (see (2.7)) are in $L^1(\mathbb{R}^n)$ for large enough k. Consequently (2.6) implies that $U_0(t)(u_0,v_0) \in X_4$ if $(u_0,v_0) \in X_3$. Next, since $F' = f$ is C^1 and satisfies $|f'(s)| \le C|s|^{p-1}$, hypothesis (VI) follows. Finally we prove that hypothesis (VII) holds.

Lemma 5.4 *For any* n *choose* p *so that* $1 + 8/n \le p < 2^{**} - 1$ *and let* $Y = L^r(I, X_3) \cap L^\infty(I, X_3)$ *for some interval* I. *Then there is some* $\delta > 0$ *such that if* $g \in X$, $s \in I$ *and* $w \in Y$ *is the solution of* (5.1) *with* $\|w\|_Y < \delta$, *then* $w \in C(I, X)$ *and* $E(w(t))$ *is constant.*

Proof. We proceed as in the proof of Lemma 4.8.

1. First let f_k be a sequence of Lipschitz functions such that $f_k(s) \to f(s)$ uniformly on compact subsets of $\mathbb{R}$ and $|f_k(s)| \le |f(s)|$. Then, since the functions $P_k(u,v) = (0, f_k(u))$ are Lipschitz from X to itself, it follows using standard semigroup arguments that there exist solutions $w_k \in C(I, X)$ of the equations

$$w_k(t) = U_0(t)g + \int_s^t U_0(t-\tau)P_k(w_k(\tau))d\tau \tag{5.19}$$

with constant energies

$$E_k = \frac{1}{2}\|w_k(t)\|_X^2 - \int_{\mathbb{R}^n} F_k(u_k)dx \tag{5.20}$$

Since at $t = s$ each of these solutions has the initial data $U_0(s)g$ we have

$$\begin{aligned} E_k &= \frac{1}{2}\|U_0(s)g\|_X^2 - \int_{\mathbb{R}^n} F_k((U_0(s)g)_1)dx \\ &\le \frac{1}{2}\|U_0(s)g\|_X^2 + C\|(U_0(s)g)_1\|_{L^{p+1}(\mathbb{R}^n)} \le C \end{aligned} \tag{5.21}$$

Hence E_k is a bounded sequence.

2. We need to show that the nonlinear part of the energies is also a bounded sequence. First define

$$N(w)(t) = \int_s^t U_0(t-\tau)P(w(\tau))d\tau \tag{5.22}$$

By (5.1) we have

$$U_0(\cdot)g = w - N(w). \tag{5.23}$$

Using Lemma 4.4 and Lemma 4.5 it follows that

$$\|N(w)(t)\|_{X_3} \le C\int_s^t |t-\tau|^{-d}\|w(\tau)\|_{X_3}^p d\tau. \tag{5.24}$$

Since $1 + 1/r = p/r + d$ the Hardy-Littlewood-Sobolev inequality implies

$$\|N(w)\|_{L^r(I,X_3)} \le C\|w\|_Y^p. \tag{5.25}$$

Next we let $I_1 = I \cap [t-1, t+1]$ and $I_2 = I \cap I_1^c$ and write

$$\int_{I_1} |t-\tau|^{-d} \|w(\tau)\|_{X_3}^p d\tau \le C\|w\|_Y^p \tag{5.26}$$

and

$$\int_{I_2} |t-\tau|^{-d} \|w(\tau)\|_{X_3}^p d\tau \le \left(\int_{I_2} |t-\tau|^{-\frac{rd}{r-p}} d\tau \right) \|w\|_{L^r(I,X_3)}^p. \tag{5.27}$$

Since $rd > r-p$ the last integral is finite and bounded independently of I, so that, together with (5.24) we have

$$\|N(w)\|_Y \le C\|w\|_Y \tag{5.28}$$

If δ is chosen small enough, we then have

$$\|U_0(\cdot)g\|_Y < 2\delta. \tag{5.29}$$

Now let $t \in I$ and define $Z_t = L^r(I_t, X_3) \cap L^\infty(I_t, X_3)$ where $I_t = [s,t]$ if $s \le t$ and $I_t = [t,s]$ if $s > t$. Set $M_k(t) = \|w_k\|_{Z_t}$. Then the same argument used to prove (5.28) shows that

$$M_k(t) \le 2\delta + C\left(M_k(t)\right)^p \tag{5.30}$$

where C is independent of t and k. Since $w_k \in C(I,X) \subset C(I,X_3)$ the norms $M_k(t)$ are continuous functions of t and since $M_k(s) = \|U_0(s)g\|_{X_3} < \delta$, this implies that if δ is chosen sufficiently small, then $M_k(t)$ is bounded on I by 3δ independent of k. Hence

$$\|w_k\|_Y < 3\delta \qquad \text{for all } k \tag{5.31}$$

and therefore

$$\left| \int_{\mathbb{R}^n} F_k(u_k(t))dx \right| \le C\|w_k\|_Y^2 \tag{5.32}$$

is bounded and therefore $\|w_k(t)\|_X$ is bounded independently of t and k.

3. It follows as in the Proof of Lemma 4.8 that there is a subsequence, renamed w_k which converges weak-$*$ to some w in $L^\infty(I,X)$, and that w is in fact bounded and weakly continuous with values in X. It also follows using the argument in Lemma 4.8 that w is strongly continuous with values in X_3.

4. We claim that $w_k(t)$ converges to $w(t)$ in X_3. We subtract the equations satisfied by w_k and w and use Lemmas 4.4 and 4.5 to obtain

$$\begin{aligned} \|w_k(t) - w(t)\|_{X_3} \le C \int_s^t |t-\tau|^{-d} (\|w(\tau)\|_{X_3}^{p-1} + \|w_k(\tau)\|_{X_3}^{p-1}) \|w_k(\tau) - w(\tau)\|_{X_3} d\tau \\ + C \int_s^t |t-\tau|^{-d} \|P_k(w(\tau)) - P(w(\tau))\|_{X_1} d\tau \end{aligned} \tag{5.33}$$

Using the Hardy-Littlewood-Soblev inequality again, it follows that

$$\begin{aligned} \|w_k - w\|_{L^r(I,X_3)} \le C(\|w\|_{L^r(I,X_3)} + \|w_k\|_{L^r(I,X_3)})^{p-1} \|w_k - w\|_{L^r(I,X_3)} \\ + C\|P_k(w) - P(w)\|_{L^r(I,X_1)}. \end{aligned} \tag{5.34}$$

By (5.31) and the hypothesis $\|w\|_Y < \delta$ in the statement of the Lemma the first term is bounded by $C\delta^{p-1}\|w_k - w\|_{L^r(I,X_3)}$ and therefore if δ is chosen small enough we have

$$\begin{aligned}\|w_k - w\|_{L^r(I,X_3)} &\leq C\|P_k(w) - P(w)\|_{L^r(I,X_1)} \\ &= \left(\int_I \|f_k(u(\tau)) - f(u(\tau))\|^r_{L^{1+1/p}(\mathbb{R}^n)} d\tau\right)^{1/r}.\end{aligned} \tag{5.35}$$

If we now define

$$g_k(\tau) = \|f_k(u(\tau)) - f(u(\tau))\|^r_{L^{1+1/p}(\mathbb{R}^n)} \tag{5.36}$$

it follows as in the the proof of Lemma 4.8 that the dominated convergence implies

$$\int_I g_k(\tau)d\tau \to 0 \qquad \text{as} \qquad k \to \infty. \tag{5.37}$$

Therefore (5.35) implies that $w_k \to w$ in $L^r(I, X_3)$, and there is some subsequence, renamed $w_k(t)$, which converges to $w(t)$ in X_3 for a.e. $t \in I$.

5. We now show that $w_k(t)$ converges to $w(t)$ in X_3 for every t in I. By (5.31) we may apply Lemma 4.7 to conlude that the w_k are Holder continuous from I to X_3 with norms bounded independently of k. For any $t_0 \in I$ let $\epsilon > 0$ be given and choose any t such that $C|t_0 - t|^{1-d} < \epsilon/3$ and $w_k(t) \to w(t)$ in X_3. This is possible by the convergence of w_k to w for a.e. t in I. If we now choose k so large that $\|w_k(t) - w(t)\|_{X_3} \leq \epsilon/3$ the claim above implies that

$$\begin{aligned}\|w_k(t_0) - w(t_0)\|_{X_3} &\leq \|w_k(t_0) - w_k(t)\|_{X_3} + \|w_k(t) - w(t)\|_{X_3} + \|w(t) - w(t_0)\|_{X_3} \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\end{aligned} \tag{5.38}$$

Hence $w_k(t_0) \to w(t_0)$ in X_3.

6. Since the solutions w_k satisfy

$$\frac{1}{2}\|w_k(t)\|^2_X - \int_{\mathbb{R}^n} F(u_k)dx = E(w_k(t)) = E(g) \tag{5.39}$$

for all t we may take the lim inf of both sides to obtain

$$\frac{1}{2}\|w(t)\|^2_X - \int_{\mathbb{R}^n} F(u)dx \leq E(g) \tag{5.40}$$

for all t in I. Reversing the roles of t and s (see (4.63)) proves the opposite inequality, and hence the equality

$$\frac{1}{2}\|w(t)\|^2_X - \int_{\mathbb{R}^n} F(u)dx = E(g) \tag{5.41}$$

7. Since $w(t)$ is continuous with values in X_3 the function $\int_{\mathbb{R}^n} F(u(t))dx$ is continuous, and therefore, by conservation of energy, $\|w(t)\|_X$ is also continuous. Together with the weak continuity of $w(t)$ in X this proves that $w \in C(I, X)$. ■

6 Appendix: Proofs of Technical Lemmas

Lemma 6.1 *If $Re(\mu)$ is fixed, then*

$$|\tilde{J}_\mu(s)| \leq Ce^{\pi|Im(\mu)|} \tag{6.1}$$

for $|s| < 1$.

Proof. The series expansion of the Bessel function of order μ (valid for any μ) is

$$J_\mu(s) = \sum_{j=0}^{\infty} \frac{(-1)^j (s/2)^{2j+\mu}}{\Gamma(\mu + j + 1)} \tag{6.2}$$

If $Re(\mu)$ is not a negative integer then

$$J_\mu(s) = \frac{(s/2)^\mu}{\Gamma(\mu+1)} \sum_{j=0}^{\infty} \frac{(-1)^j (s/2)^{2j}}{\prod_{k=0}^{j}(\mu + k)} \tag{6.3}$$

Since $Re(\mu + k) \neq 0$ for any $1 \leq k \leq j$, it follows that there is a positive constant C depending only on $Re(\mu)$ so that $|\prod_{k=0}^{j}(\mu + k)| \geq C(1 + |Im(\mu)|)^j$, and therefore

$$|\tilde{J}_\mu(s)| \leq C|\Gamma(\mu + 1)|^{-1} e^{Cs^2/(1+|Im(\mu)|)} \tag{6.4}$$

The result then follows by using the estimate

$$\lim_{|y|\to\infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = (2\pi)^{1/2} \tag{6.5}$$

in [8] p13. If $Re(\mu)$ is a negative integer we break up the sum as

$$\sum_{j=0}^{-Re(\mu)-1} \frac{(-1)^j (s/2)^{2j+\mu}}{\Gamma(\mu + j + 1)} + \sum_{j=-Re(\mu)}^{\infty} \frac{(-1)^j (s/2)^{2j+\mu}}{\Gamma(\mu + j + 1)} \tag{6.6}$$

Since the Gamma function has simple poles at the negative integers, it follows that $|\Gamma(\mu+j+1)|^{-1} \leq C$ for $0 \leq j \leq -Re(\mu)-1$ and $|Im(\mu)| \leq 1$. By (6.5) we then have $|\Gamma(\mu+j+1)|^{-1} \leq Ce^{\pi|Im(\mu)|}$ for $0 \leq j \leq -Re(\mu) - 1$. We therefore have

$$\left| \sum_{j=0}^{-Re(\mu)-1} \frac{(-1)^j (s/2)^{2j+\mu}}{\Gamma(\mu + j + 1)} \right| \leq Ce^{\pi|Im(\mu)|} s^{Re(\mu)} \tag{6.7}$$

for $|s| < 1$. The second term is estimated as before to obtain

$$\sum_{j=-Re(\mu)}^{\infty} \frac{(-1)^j (s/2)^{2j+\mu}}{\Gamma(\mu + j + 1)} \leq Cs^{-Re(\mu)} e^{Cs^2/(c+|Im(\mu)|)} \tag{6.8}$$

This proves the lemma. ∎

Lemma 6.2 *If $Re(\mu)$ is fixed, then*

$$|\tilde{J}_\mu(s)| \leq Ce^{2\pi|Im(\mu)|}s^{-\mu-1/2} \tag{6.9}$$

for $|s| > 1$.

Proof. We can express J_μ in terms of the Hankel functions as

$$J_\mu = \frac{H_\mu^{(1)} + H_\mu^{(2)}}{2} \tag{6.10}$$

As shown in Watson p168, if $Re(\mu) > -1/2$ we have

$$\begin{aligned} H_\mu^{(1)}(s) &= \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{i(s-\frac{\pi}{2}\mu-\frac{\pi}{4})}}{\Gamma(\mu+1/2)} \int_\gamma e^{-u}u^{\mu-1/2}\left(1+\frac{iu}{2s}\right)^{\mu-1/2} du \\ H_\mu^{(2)}(s) &= \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{-i(s-\frac{\pi}{2}\mu-\frac{\pi}{4})}}{\Gamma(\mu+1/2)} \int_\gamma e^{-u}u^{\mu-1/2}\left(1-\frac{iu}{2s}\right)^{\mu-1/2} du \end{aligned} \tag{6.11}$$

where $\gamma(\eta) = \eta e^{\frac{\pi}{4}i}$, $r \in [0, \infty)$. We repeat here the exposition in [21] Ch7.2, keeping careful track of the dependence on $Im(\mu)$. Using the binomial expansion

$$(1-y)^\beta = \sum_{j=0}^{m} \frac{(-\beta)_j y^j}{j!} + \frac{(-\beta)_{m+1}y^{m+1}}{m!} \int_0^1 (1-v)^m(1-yv)^{\beta-m-1}dv \tag{6.12}$$

with $\beta = \mu - 1/2$ and $y = \frac{u}{2is}$ and recalling that

$$\Gamma(\nu) = \int_\gamma e^{-u}u^{\nu-1}du \tag{6.13}$$

we obtain

$$H_\mu^{(1)}(s) = \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{i(s-\frac{\pi}{2}\mu-\frac{\pi}{4})}}{\Gamma(\mu+1/2)} \left(\sum_{j=0}^{m} \frac{(1/2-\mu)_j\Gamma(\mu+j+1/2)}{j!(2is)^j} + \frac{I_{m+1}(\mu,s)}{(2is)^{m+1}}\right) \tag{6.14}$$

where

$$I_{m+1}(\mu,s) = \frac{(1/2-\mu)_{m+1}}{m!} \int_\gamma \int_0^1 \frac{(1-v)^m}{e^u} u^{\mu+m-3/2}\left(1-\frac{uv}{2is}\right)^{\mu-m-3/2} dvdu \tag{6.15}$$

Since $arg(u) = \pi/4$ along γ and since s and v are real, it is easily verified that

$$\left|1-\frac{uv}{2is}\right| \geq \frac{1}{2} \qquad \text{and} \qquad 0 \leq arg\left(1-\frac{uv}{2is}\right) \leq \frac{3\pi}{4} \tag{6.16}$$

Thus if m is chosen so that $Re(\mu) - m - 3/2 < 0$ we have

$$\left|\int_0^1 (1-v)^m\left(1-\frac{uv}{2is}\right)^{\mu-m-3/2} dv\right| \leq Ce^{\frac{3\pi}{4}|Im(\mu)|} \tag{6.17}$$

where C does not depend on $Im(\mu)$. Also, since $arg(u) = \pi/4$ on γ it follows that

$$\left|\int_\gamma e^{-u}u^{\mu+m-3/2}du\right| \leq Ce^{\frac{\pi}{4}|Im(\mu)|} \tag{6.18}$$

Thus

$$|I_{m+1}(\mu,s)| \leq P_m(|Im(\mu)|)e^{\pi|Im(\mu)|} \tag{6.19}$$

For large $|Im(\mu)|$ we then have

$$H_\mu^{(1)}(s) = \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{i(s-\frac{\pi}{2}\mu-\frac{\pi}{4})}}{\Gamma(\mu+1/2)} \sum_{j=0}^{m} \frac{(1/2-\mu)_j\Gamma(\mu+j+1/2)}{j!(2is)^j} + O\left(\frac{e^{2\pi|Im(\mu)|}}{s^{m+1}}\right) \tag{6.20}$$

Similarly

$$H_\mu^{(2)}(s) = \left(\frac{2}{\pi s}\right)^{1/2} \frac{e^{-i(s-\frac{\pi}{2}\mu-\frac{\pi}{4})}}{\Gamma(\mu+1/2)} \sum_{j=0}^{m} \frac{(1/2-\mu)_j\Gamma(\mu+j+1/2)}{(-1)^j j!(2is)^j} + O\left(\frac{e^{2\pi|Im(\mu)|}}{s^{m+1}}\right) \tag{6.21}$$

Hence we can write

$$J_\mu(s) = \left(\frac{2}{\pi s}\right)^{1/2} \left(Q_m^1\cos(s-c_\mu) + Q_m^2\sin(s-c_\mu)\right) + O\left(\frac{e^{2\pi|Im(\mu)|}}{s^{m+1}}\right) \tag{6.22}$$

where $c_\mu = \frac{\pi}{2}(\mu+\frac{1}{2})$ and

$$Q_m^1(\mu,s) = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} \frac{(-1)^k(1/2-\mu)_{2k}\Gamma(\mu+2k+1/2)}{(2k)!(2s)^{2k}\Gamma(\mu+1/2)} \tag{6.23}$$

$$Q_m^2(\mu,s) = \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \frac{(-1)^k(1/2-\mu)_{2k+1}\Gamma(\mu+2k+3/2)}{(2k+1)!(2s)^{2k+1}\Gamma(\mu+1/2)} \tag{6.24}$$

Ignoring all but the first term of this expression gives

$$J_\mu(s) = \left(\frac{2}{\pi s}\right)^{1/2} \cos\left(s - \frac{\pi}{2}\mu - \frac{\pi}{4}\right) + O\left(e^{2\pi|Im(\mu)|}s^{-3/2}\right) \tag{6.25}$$

for $Re(\mu) > -1/2$. The same formula holds for $Re(\mu) \leq -1/2$ since

$$J_\mu = \frac{e^{-\pi\mu i}H_\mu^{(1)} + e^{\pi\mu i}H_\mu^{(2)}}{2} \tag{6.26}$$

This proves the lemma. ■

Together Lemma 6.1 and Lemma 6.2 imply

Lemma 6.3 *If $Re(\mu) \geq -1/2$ is fixed, then*

$$\|\tilde{J}_\mu\|_\infty = O\left(e^{2\pi|Im(\mu)|}\right) \tag{6.27}$$

as $|Im(\mu)| \to \infty$.

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