Math 375: Probability Theory
Spring 2018
Homework 6 Solutions

1. Suppose the gym you go to has a 30 minute limit on treadmills. For the purposes of this question, assume everyone who uses a treadmill stays on for exactly 30 minutes.

   (a) There is one particular treadmill you like to use. When you arrive, someone is already on it. You don't know when he started running on it, so his time remaining could be anywhere from 0 to 30 minutes. Assume his time remaining is uniformly distributed over $[0, 30]$. What is your expected wait time for the treadmill?

   Solution. Let Y be the expected wait time. By assumption it is uniformly distributed over $[0, 30]$, so its expected value is $\frac{1}{2}(0 + 30) = 15$ minutes.

   (b) Now suppose the gym has N treadmills, and you are willing to use any one of them. When you arrive at the gym, all the treadmills are in use. Under the same assumptions as in part (a), let Z be the time you need to wait to use a treadmill. Find the cdf and pdf of Z.

   Solution. Let X_j be the time the person on treadmill j has remaining. Then $Z \geq z$ if and only if $X_j \geq z$ for all $j = 1, \ldots, N$. Assuming independence of the start times of the people on each of the treadmills, this implies

   $$P(Z \geq z) = P(X_1 \geq z)P(X_2 \geq z) \cdots P(X_N \geq z)$$

   Now for each j,

   $$P(X_j \geq z) = \int_z^{30} \frac{1}{30}\, dx = 1 - \frac{z}{30}$$

   and thus $P(Z \geq z) = \left(1 - \frac{z}{30}\right)^N$. Therefore $F(z) = 1 - \left(1 - \frac{z}{30}\right)^N$ and $f(z) = F'(z) = \frac{N}{30}\left(1 - \frac{z}{30}\right)^{N-1}$.

   (c) Under the same assumptions as in part (b), find your expected wait time for a treadmill. How many treadmills should the gym have in order for the wait time to be 5 minutes or less?

   Solution. First, $E(Z) = \int_0^{30} z f(z)\, dz$. Integrating by parts with $u = z$ and $dv = f(z)\, dz$ gives $du = dz$ and $v = -\left(1 - \frac{z}{30}\right)^N$, so

   $$\begin{aligned} E(Z) &= -z\left(1 - \frac{z}{30}\right)^N \Bigg|_0^{30} + \int_0^{30} \left(1 - \frac{z}{30}\right)^N \\ &= -\frac{30}{N+1}\left(1 - \frac{z}{30}\right)^{N+1} \Bigg|_0^{30} \\ &= \frac{30}{N+1} \end{aligned}$$

   In order for $E(Z)$ to be 5, we need $N = 5$ treadmills.

   (d) Now assume that all N treadmills are in use and that one person is already waiting ahead of you. What is your expected wait time now?

   Solution. Whichever treadmill becomes available first will be taken by the person ahead of you, who will use it for 30 minutes. Thus, one of the other $N-1$ treadmills will become available before that person is done. So you are really just waiting for one of those $N-1$ treadmills to become available. Thus, by the previous result, your expected wait time is $\frac{30}{(N-1)+1} = \frac{30}{N}$, as long as $N > 1$. (If $N = 1$, there is no other treadmill to use, so you need to wait until the person ahead of you used it for 30 minutes. His expected wait time is 15 minutes, so yours is $30 + 15 = 45$ minutes!)

2. (a) Let Y be a continuous random variable with density $f_Y(y)$. Prove that the density for $Z = aY + b$ is $f_Z(z) = \frac{1}{a} f_Y\left(\frac{z-b}{a}\right)$. (Hint: First consider how the cumulative distribution functions are related.)

   Solution. The cumulative distribution functions are $F_Y(y) = P(Y \le y)$ and $F_Z(z) = P(Z \le z)$. Thus

$$F_Z(z) = P(aY + b \le z) = P(Y \le (z-b)/a) = F_Y((z-b)/a),$$

   so

$$f_Z(z) = F_Z'(z) = \frac{1}{a} F_Y'((z-b)/a) = \frac{1}{a} f_Y((z-b)/a).$$

   (b) Show that if Y has normal density $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}$, then $Z = \frac{Y-\mu}{\sigma}$ has standard normal density $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

   Solution. By part (a) with $a = 1/\sigma$ and $b = -\mu/\sigma$,

$$f_Z(z) = \sigma f_Y(\sigma(z + \mu/\sigma)) = \sigma f_Y(\sigma z + \mu) = \sigma \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(\sigma z + \mu - \mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

3. Let Y be a continuous random variable whose density $f(y)$ satisfies $f(y_0 - y) = f(y_0 + y)$ for all y. Prove that both the mean and median of Y are equal to y_0.

   Solution. First, if $g(z)$ is any even function, then for any $b > 0$ we have

$$\int_0^b g(z)\,dz = \int_0^b g(-z)\,dz$$

   Making the substitution $u = -z$ gives

$$\int_0^b g(-z)\,dz = \int_0^{-b} -g(u)\,du = \int_{-b}^0 g(u)\,du.$$

   Changing the variable of integration back to z in the last integral then shows

$$\int_0^b g(z)\,dz = \int_{-b}^0 g(z)\,dz,$$

   and taking the limit as b approaches infinity proves

$$\int_0^\infty g(z)\,dz = \int_{-\infty}^0 g(z)\,dz.$$

   The assumption that $f(y_0 - y) = f(y_0 + y)$ implies that $g(z) = f(y_0 + z)$ is even and thus

$$\int_0^\infty f(y_0 + z)\,dz = \int_{-\infty}^0 f(y_0 + z)\,dz.$$

Finally, substituting $y = y_0 + z$ in each integral gives

$$\int_{y_0}^{\infty} f(y)\,dy = \int_{-\infty}^{y_0} f(y)\,dy,$$

which proves the median is y_0.

By similar reasoning, if $h(z)$ is any odd function, then

$$\int_0^{\infty} h(z)\,dz = -\int_{-\infty}^{0} h(z)\,dz,$$

which implies

$$\int_{-\infty}^{\infty} h(z)\,dz = 0.$$

Thus, since $h(z) = zf(y_0 + z)$ is odd, it follows that

$$\int_{-\infty}^{\infty} zf(y_0 + z)\,dz = 0.$$

Letting $y = y_0 + z$ again, this becomes

$$\int_{-\infty}^{\infty} (y - y_0)f(y)\,dy = 0$$

But

$$\int_{-\infty}^{\infty} (y - y_0)f(y)\,dy = \int_{-\infty}^{\infty} yf(y)\,dy - \int_{-\infty}^{\infty} y_0 f(y)\,dy = \mu - y_0,$$

so $\mu = y_0$.

4. Define $I = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$. Prove that $I = \sqrt{2\pi}$, as follows. First write

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy \right) e^{-x^2/2}\,dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2}\,dy \right) dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2}\,dy\,dx
\end{aligned}$$

Use polar coordinates to calculate the double integral.

Solution. In polar coordinates, the integral becomes

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta &= 2\pi \int_0^{\infty} e^{-r^2/2} r\,dr \\
&= \lim_{b\to\infty} 2\pi \int_0^{b} e^{-r^2/2} r\,dr \\
&= \lim_{b\to\infty} -2\pi e^{-r^2/2} \Big|_0^b \\
&= \lim_{b\to\infty} -2\pi e^{-b^2/2} + 2\pi \\
&= 2\pi
\end{aligned}$$

and thus $I = \sqrt{2\pi}$.

5. Prove that $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, as follows. Write

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty y^{\beta-1}e^{-y}\,dy \int_0^\infty x^{\alpha-1}e^{-x}\,dx \\
&= \int_0^\infty \left(\int_0^\infty y^{\beta-1}e^{-y}\,dy\right) x^{\alpha-1}e^{-x}\,dx \\
&= \int_0^\infty \left(\int_0^\infty x^{\alpha-1}e^{-x}y^{\beta-1}e^{-y}\,dy\right)\,dx \\
&= \int_0^\infty \int_0^\infty x^{\alpha-1}e^{-x}y^{\beta-1}e^{-y}\,dy\,dx
\end{aligned}$$

In the last integral make the change of variable $x=uv$, $y=u(1-v)$ and show that it reduces to $B(\alpha,\beta)\Gamma(\alpha+\beta)$.

Solution. First notice that the transformation $x=uv$, $y=u(1-v)$ maps the domain $(0,\infty)\times(0,1)$ to $(0,\infty)\times(0,\infty)$. Its Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}=u.$$

Thus the last integral becomes

$$\begin{aligned}
\int_0^1\int_0^\infty (uv)^{\alpha-1}e^{-uv}(u(1-v))^{\beta-1}e^{-u(1-v)}u\,du\,dv &= \int_0^1\int_0^\infty u^{\alpha+\beta-1}v^{\alpha-1}e^{-u}(1-v)^{\beta-1}\,du\,dv \\
&= \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv \int_0^\infty u^{\alpha+\beta-1}e^{-u}\,du \\
&= B(\alpha,\beta)\Gamma(\alpha+\beta).
\end{aligned}$$